CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

Problem Set #1

R. M. Murray | Issued: 29 Sep 08
Fall 2008 | Due: 6 Oct 08

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Åström and Murray, Exercise 1.3
2. Åström and Murray, Exercise 1.4
3. Åström and Murray, Exercise 2.6, parts (a) and (b)

CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110a

R. M. Murray | **Problem Set #1** | Issued: 29 Sep 08
Fall 2008 | | Due: 6 Oct 08

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Åström and Murray, Exercise 1.4
2. Åström and Murray, Exercise 2.1
3. Consider a damped spring–mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = F.$$

Let $\omega_0 = \sqrt{k/m}$ be the natural frequency and $\zeta = c/(2\sqrt{km})$ be the damping ratio.

(a) Show that by rescaling the equations, we can write the dynamics in the form

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2 u, \tag{S1.1}$$

where $u = F/k$. This form of the dynamics is that of a linear oscillator with natural frequency ω_0 and damping ratio ζ.

(b) Show that the system can be further normalized and written in the form

$$\frac{dz_1}{d\tau} = z_2, \qquad \frac{dz_2}{d\tau} = -z_1 - 2\zeta z_2 + v. \tag{S1.2}$$

The essential dynamics of the system are governed by a single damping parameter ζ. The *Q-value* defined as $Q = 1/2\zeta$ is sometimes used instead of ζ.

(c) Show that the solution for the unforced system ($v = 0$) with no damping ($\zeta = 0$) is given by

$$z_1(\tau) = z_1(0)\cos\tau + z_2(0)\sin\tau, \qquad z_2(\tau) = -z_1(0)\sin\tau + z_2(0)\cos\tau.$$

Invert the scaling relations to find the form of the solution $q(t)$ in terms of $q(0)$, $\dot{q}(0)$ and ω_0.

(d) Consider the case where $\zeta = 0$ and $u(t) = \sin\omega t$, $\omega > \omega_0$. Solve for $z_1(\tau)$, the normalized output of the oscillator, with initial conditions $z_1(0) = z_2(0) = 0$ and use this result to find the solution for $q(t)$.

4. Consider the queuing system described in Example 2.10. The long delays created by temporary overloads can be reduced by rejecting requests when the queue gets large. This allows requests that are accepted to be serviced quickly and requests that cannot be accommodated to receive a rejection quickly so that they can try another server. Consider an admission control system described by

$$\frac{dx}{dt} = \lambda u - \mu_{\max}\frac{x}{x+1}, \qquad u = \mathrm{sat}_{(0,1)}(k(r - x)), \tag{S1.3}$$

where the controller is a simple proportional control with saturation ($\text{sat}_{(a,b)}$ is defined in equation (3.9)) and r is the desired (reference) queue length. Use a simulation to show that this controller reduces the rush-hour effect and explain how the choice of r affects the system dynamics. You should choose the parameters of your simulation to match those in Example 2.10: $\mu_{\max} = 1$, $\lambda = 0.5$ at time 0, increasing to $\lambda = 4$ at time 20 and returning to $\lambda = 0.5$ at time 25. Test your controller using $r = 2$ and $r = 5$ and explore several different values for k. Your solution should include the MATLAB code that you used plus plots for the final value of k you chose (and the two values of r). Make sure to label your plots and describe how your controller reduces the rush hour effect.

CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 210

R. M. Murray
Fall 2008

Problem Set #1

Issued: 29 Sep 08
Due: 6 Oct 08

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Åström and Murray, Exercise 1.5

2. Download the file "`cruise_ctrl.mdl`" from the companion web site. It contains a SIMULINK model of a simple cruise controller, similar to the one described in Section 1.4. Figure out how to run the example and plot the vehicle's speed as a function of time.

   (a) Leaving the control gains at their default values, plot the response of the system to a step input and measure the time it takes for the system error to settle to within 5% of commanded change in speed (i.e., 0.5 m/s).

   (b) By manually tuning the control gains, design a controller that settles at least 50% faster than the default controller. Include the gains you used, a plot of the closed loop response, and describe any undesirable features in the solution you obtain.

   All plots should included a title, labeled axes (with units), and reasonable axis limits.

3. [Contributed by Mary Dunlop, 2006] The motion of an ideal pendulum is described by

$$\ddot{\theta} + g \sin \theta = 0,$$

   where θ is the angle between the pendulum's position and vertical, and g is the gravitational acceleration.

   (a) Using the small angle approximation $\sin \theta \approx \theta$, solve for an expression for $\theta(t)$, written in terms of the initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = \omega_0$ and the parameter g.

   (b) Plot the pendulum's motion in three different environments: Earth ($g = 9.8$ m/s^2), the moon ($g = 1.6$ m/s^2), and on Temple I—the comet that the Deep Impact mission collided with on in July 2005 ($g = 0.00004$ m/s^2). Assume that the pendulum is given a small initial starting angle $\theta(0) = 0.05$ radians (about 3 degrees) and then released with no initial velocity ($\dot{\theta}(0) = 0$). Note that this is an idealized equation of motion and damping is not included, so there are no frictional forces to slow the pendulum down over time.

   (c) If the pendulum is pushed with a force $u(t)$, the equation of motion becomes

$$\ddot{\theta} + g \sin \theta = u(t).$$

   Apply the small angle approximation and assume $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. Solve for $\theta(t)$ when $u(t) = \sin t$.

4. Åström and Murray, Exercise 2.3

5. Consider the consensus problem described in Example 2.12 with N nodes and a connected graph describing the sensor network. Show that the quantity

$$W[k] = \frac{1}{N}\sum_{i=1}^{n} x_i[k]$$

is constant under the consensus protocol and use this fact to show that if the consensus protocol converges, then it converges to the average of the initial values of each node. (In computer programming, qualities such as W are called *invariants,* and the use of invariants is an important technique for verifying the correctness of computer programs.)