CDS 101/110a: Lecture 8-2
Limits on Performance

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Goals:
• Describe limits of performance on feedback systems
• Introduce Bode’s integral formula and the “waterbed” effect
• Show some of the limitations of feedback due to RHP poles and zeros

Reading:
• Åström and Murray, Feedback Systems, Ch 11
• Advanced: Lewis, Chapter 9
• CDS 210: DFT, Ch 6

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Algebraic Constraints on Performance

Goal: keep S & T small
- S small ⇒ low tracking error
- T small ⇒ good noise rejection (and robustness [CDS 110b])

Problem: S + T = 1
- Can’t make both S & T small at the same frequency
- Solution: keep S small at low frequency and T small at high frequency
- Loop gain interpretation: keep L large at low frequency, and small at high frequency

\[
S = \frac{1}{1 + PC}, \quad T = \frac{PC}{1 + PC}
\]

\[
|L(s)| > 1 \quad L(s) < 1
\]
Bode’s Integral Formula and the Waterbed Effect

Bode’s integral formula for $S = 1/(1+PC) = 1/(1+L)$:

- Let $p_k$ be the unstable poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
- **Theorem**: the area under the sensitivity function is a conserved quantity:

$$\int_0^{\infty} \log_e |S(j\omega)| d\omega = \int_0^{\infty} \log_e \frac{1}{1 + L(j\omega)} d\omega = \pi \sum \text{Re } p_k$$

**Waterbed effect:**

- Making sensitivity smaller over some frequency range requires *increase* in sensitivity somewhere else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in $\omega$; Bode plots are logarithmic

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**Example: Magnetic Levitation**

**System description**

- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States: $z$, $\dot{z}$
- Dynamics: $F = ma$, $F$ = magnetic force generated by wire coil
- See MATLAB handout for details

**Controller circuit**

- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current
Equations of Motion

Process: actuation, sensing, dynamics
\[ m\ddot{z} = mg - ka(ka)^2/z^2 \]
\[ v_{ir} = k_Tz + v_0 \]
- \( u \) = current to electromagnet
- \( v_{ir} \) = voltage from IR sensor

Linearization:
\[ P(s) = \frac{-k}{s^2 - r^2} \quad k, r > 0 \]
- Poles at \( s = \pm r \Rightarrow \) open loop unstable

Control Design

Need to create encirclement
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator
- Produce phase lead at crossover
- Generates loop in Nyquist plot
\[ C(s) = -k\frac{s + a}{s + b} \]
Performance Limits

Nominal design gives low perf
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement
\[ \int_{0}^{\infty} \log |S(j\omega)|d\omega = \pi r \]
- Must increase sensitivity at some point

Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior
- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move opposite from input for a short period of time

Example:
\[ H_1(s) = \frac{s + a}{s^2 + 2\xi\omega_n s + \omega_n^2} \]
\[ H_2(s) = \frac{s - a}{s^2 + 2\xi\omega_n s + \omega_n^2} \]
**Example: Lateral Control of the Ducted Fan**

Source of non-minimum phase behavior
- To move left, need to make $\theta > 0$
- To generate positive $\theta$, need $f_1 > 0$
- Positive $f_1$ causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)

$$H_{sf}(s) = \frac{(s^2 - mg\ell)}{s^3 (J s^2 + ds + mg\ell)}$$
- Poles: 0, 0, $-\alpha \pm j \omega_d$
- Zeros: $\pm \sqrt{mg\ell}$

**Stability in the Presence of Zeros**

Loop gain limitations
- Poles of closed loop = poles of $1 + L$. Suppose $C = k \frac{n_c}{d_c}$, where $k$ is the gain of the controller
  $$1 + L = 1 + k \frac{n_c n_p}{d_c d_p} = \frac{d_c d_p}{d_c d_p} + kn_c n_p$$
- For large $k$, closed loop poles approach open loop zeros
- RHP zeros limit maximum gain $\Rightarrow$ serious design constraint!

Root locus interpretation
- Plot location of eigenvalues as a function of the loop gain $k$
- Can show that closed loop poles go from open loop poles ($k = 0$) to open loop zeros ($k = \infty$)
Additional performance limits due to RHP zeros

Another waterbed-like effect: look at maximum of $H_{er}$ over frequency range:

$$M_1 = \max_{\omega_1<\omega<\omega_2} |H_{er}(j\omega)| \quad M_2 = \max_{\text{loc max}} |H_{er}(j\omega)|$$

**Thm:** Suppose that $P$ has a RHP zero at $z$. Then there exist constants $c_1$ and $c_2$ (depending on $\omega_1$, $\omega_2$, $z$) such that

$$c_1 \log M_1 + c_2 \log M_2 \geq 0$$

- $M_1$ typically $<< 1 \Rightarrow M_2$ must be larger than 1 (since sum is positive)
- If we increase performance in active range (make $M_1$ and $H_{er}$ smaller), we must lose performance ($H_{er}$ increases) some place else
- Note that this affects peaks not integrals (different from RHP poles)

\[H(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}\]

- Poles: 0, 0, $-\alpha \pm j \omega_d$
- Zeros: $\pm \sqrt{mgl}$

Summary: Limits of Performance

Many limits to performance

- Algebraic: $S + T = 1$
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of $S$

Main message: try to avoid RHP poles and zeros whenever possible (eg, re-design)
% L9_1_maglev.m - MATLAB code for lecture 9.1
%文件名：L9_1_maglev.m - MATLAB代码
%文件名：L9_1_maglev.m
%% Magnetic Levitation System
%% z1 = position of ball from centerline (down is positive)
%% z2 = velocity of ball
%% Vy = output (position sensor) voltage
%% Vu = input (electromagnet) voltage
%% The dynamics are of the form xdot = f(x, u), y = h(x):
%% f = [ g-km/m*(kA*Vu)^2/z1^2 ];
%% h = kT*z1+v0
%% Parameter values for system
kT = 613.65; % gain between position and voltage
v0 = -16.18; % voltage offset at zero position
m = 0.2; % mass of ball, kg
g = 9.81; % gravitational constant
kA = 1; % electromagnet conductance
km = 3.13e-3 * m/2 / kA^2; % gain on magnetic attractive force
%% Equilibrium point calculation
z10 = -v0/kT;
z20 = 0;
Vu0 = 1/kA*sqrt(z10^2*m*g/km);
%% Create a MATLAB LTI system from this representation
magP = ss(A, B, C, D);

% Maglev controller circuit
% Note: these values are only approximately equivalent to what exists in the circuit.
kl = 0.5; % gain set by gain pot
R1 = 22000; % Internal resistor
R2 = 22000; % Resistor plug-in
R = 2000; C = 1e-6; % RC plug-in

% Controller based on analog circuit
magC1 = -tf([(R1+R)*C 1], [R*C 1])*kl*R2/R1;

% Increased gain
magC2 = magC1*5;

% Loop transfer function
magS2 = feedback(1, magP*magC2); % sensitivity function
magT2 = feedback(magP*magC2, 1); % closed loop response

% Increased gain
magC3 = magC1*20;

% Loop transfer function
magS3 = feedback(1, magP*magC3); % sensitivity function
magT3 = feedback(magP*magC3, 1); % closed loop response

% Step response
figure(3); step(magT1, magT2, magT3)

% Closed loop Bode plot
figure(4); bode(magS1, magS2, magS3)