Goals:
- Describe canonical control design problem and standard performance measures
- Show how to use “loop shaping” to achieve a performance specification
- Work through a detailed example of a control design problem

Reading:
- Åström and Murray, *Feedback Systems*, Ch 11
- *Advanced*: Lewis, Chapter 12
- CDS 210: DFT, Chapters 4 and 6
Frequency Domain Performance Specifications

Specify bounds on the loop transfer function to guarantee desired performance

- **Steady state error:**
  \[ H_{er} = \frac{1}{1 + L} \]
  \[ H_{yr} = \frac{L}{1 + L} \]

- Bandwidth: assuming ~90° phase margin
  \[ \frac{L}{1 + L} (j\omega_c) \approx \frac{1}{1 + j} = \frac{1}{\sqrt{2}} \]
  \[ \Rightarrow \text{sets crossover freq} \]

- Tracking: \( X\% \) error up to frequency \( \omega_t \) \( \Rightarrow \) determines gain bound \((1 + PC > 100/X)\)
Relative Stability

Relative stability: how stable is system to disturbances at certain frequencies?

- System can be stable but still have bad response at certain frequencies
- Typically occurs if system has low phase margin ⇒ get resonant peak in closed loop ($M_r$) + poor step response
- Solution: specify minimum phase margin. Typically 45° or more
Canonical Control Design Problem

Noise and disturbances
- $d =$ process disturbances
- $n =$ sensor noise
- Keep track of transfer functions between all possible inputs and outputs

Design represents a tradeoff between the quantities
- Keep $L=PC$ large for good performance ($H_{er} \ll 1$)
- Keep $L=PC$ small for good noise rejection ($H_{yn} \ll 1$)

$F = 1$: Four unique transfer functions define performance ("Gang of Four")
- Stability is always determined by $1/(1+PC)$ assuming stable process & controller
- Numerator determined by forward path between input and output

More generally: 6 primary transfer functions; simultaneous design of each

$\begin{bmatrix} \eta \\ y \\ u \end{bmatrix} = \begin{bmatrix} \frac{P}{1+PC} & -\frac{PC}{1+PC} & \frac{PCF}{1+PC} \\ \frac{P}{1+PC} & \frac{1}{1+PC} & \frac{PCF}{1+PC} \\ -\frac{PC}{1+PC} & -\frac{C}{1+PC} & \frac{CF}{1+PC} \end{bmatrix} \begin{bmatrix} d \\ n \\ r \end{bmatrix}$
Two Degree of Freedom Design

Sensitivity Function

\[ S = \frac{1}{1 + PC} \]

Complementary sensitivity

\[ T = \frac{PC}{1 + PC} \]

Load sensitivity

\[ PS = \frac{P}{1 + PC} \]

Noise sensitivity

Typical design procedure

- Design \( C \) to provide good load/noise response
- Design \( F \) to provide good response to reference
Algebraic Constraints on Performance

Goal: keep $S$ & $T$ small
- $S$ small $\Rightarrow$ low tracking error
- $T$ small $\Rightarrow$ good noise rejection (and robustness [CDS 110b])

Problem: $S + T = 1$
- Can’t make both $S$ & $T$ small at the same frequency
- Solution: keep $S$ small at low frequency and $T$ small at high frequency
- Loop gain interpretation: keep $L$ large at low frequency, and small at high frequency

$H_{er} = \frac{1}{1 + PC} =: S$
$H_{yn} = \frac{PC}{1 + PC} =: T$

- Transition between large gain and small gain complicated by stability (phase margin)
Loop Shaping Revisited

**Disturbance rejection** \( H_{ed} = \frac{P}{1 + L} \)

- Would like \( H_{ed} \) to be small \( \Rightarrow \) large \( L(s) \)
- Typically require this in low frequency range

**High frequency measurement noise** \( H_{un} = \frac{L}{P(1 + L)} \)

- Want to make sure that \( H_{un} \) is small (avoid amplifying noise) \( \Rightarrow \) small \( L(s) \)
- Typically generates constraints in high frequency range

**Robustness: gain and phase margin**

- Focus on gain crossover region: make sure the slope is “gentle” at gain crossover
- Fundamental tradeoff: transition from high gain to low gain through crossover
Lead compensation

Use to increase phase in frequency band

- Effect: lifts phase by increasing gain at high frequency
- Very useful controller; increases PM
- Bode: add phase between zero and pole
- Nyquist: increase phase margin

\[
K \frac{a}{b} \quad C(s) \quad P(s) \quad L(s)
\]

\[
\omega_z = a \quad \omega_p = b
\]

\[
r \rightarrow e \rightarrow \frac{K(s + a)}{s + b} \rightarrow u \rightarrow P(s) \rightarrow y
\]

\[
a < b \quad K > 0
\]
Process Inversion

Simple trick: invert out process

- Write all performance specs in terms of the desired loop transfer function
- Choose \( L(s) \) that satisfies specifications
- Choose controller by inverting \( P(s) \)

\[
C(s) = \frac{L(s)}{P(s)}
\]

Pros

- Very easy design process
- \( L(s) = \frac{1}{s} \) often works very well
- Can be used as a first cut, with additional shaping to tune design

Cons

- High order controllers (at least same order as the process you are controlling)
- Requires “perfect” model of your process (since you are inverting it)
- *Does not work if you have right half plane poles or zeros* (get internal instability)

\[
S = \frac{1}{1 + PC} \quad T = \frac{PC}{1 + PC} \quad PS = \frac{P}{1 + PC} \quad CS = \frac{C}{1 + PC}
\]
Example: Control of Vectored Thrust Aircraft

System description
- Vector thrust engine attached to wing
- Inputs: fan thrust, thrust angle (vectored)
- Outputs: position and orientation
- States: $x$, $y$, $\theta$ + derivatives
- Dynamics: flight aerodynamics

Control approach
- Design “inner loop” control law to regulate pitch ($\theta$) using thrust vectoring
- Second “outer loop” controller regulates the position and altitude by commanding the pitch and thrust
- Basically the same approach as aircraft control laws
Performance Specification and Design Approach

Performance Specification
- ≤ 1% steady state error
  - Zero frequency gain > 100
- ≤ 10% tracking error up to 10 rad/sec
  - Gain > 10 from 0-10 rad/sec
- ≥ 45° phase margin
  - Gives good relative stability
  - Provides robustness to uncertainty

Design approach
- Open loop plant has poor phase margin
- Add phase lead in 5-50 rad/sec range
- Increase the gain to achieve steady state and tracking performance specs
- Avoid integrator to minimize phase

\[
P(s) = \frac{r}{Js^2 + ds + mg} \\
C(s) = K \frac{s + a}{s + b} \\
\]

\[
a = 25 \quad b = 300 \quad K = 15 \times 300
\]
Select parameters to satisfy specs

- Place phase lead in desired crossover region (given by desired BW)
- Phase lead peaks at 10X of zero location
- Place pole sufficiently far out to insure that phase does not decrease too soon
- Set gain as needed for tracking + BW
- Verify controller using Nyquist plot, etc
Control Verification: Gang of 4

Remarks
• Check each transfer function to look for peaks, large magnitude, etc
• Example: Noise sensitivity function \((CS)\) has very high gain; step response verifies poor step response
• Implication: controller amplifies noise at high frequency \(\Rightarrow\) will generate \textit{lots} of motion of control actuators (flaps)
• Fix: roll off the loop transfer function faster (high frequency pole)
Summary: Loop Shaping

Loop Shaping for Stability & Performance
- Steady state error, bandwidth, tracking

Main ideas
- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, lead, PI