CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110a

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This lecture describes how to derive and use transfer functions for linear systems, including a description of how to sketch a Bode plot by hand. If there is time, I will also talk about the relationship between the transfer function and the Laplace transform.

Reading:

- Åström and Murray, Chapter 8

1 Transfer Functions for Linear Systems

Transfer function for state space systems

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu,\\ y &= Cx + Du\end{aligned} \qquad \Longrightarrow \qquad y(t) = Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + \underbrace{\Big(C(sI - A)^{-1}B + D\Big)}_{H_{yu}(s)} e^{st},$$

- Transfer function only captures the *steady state* response
- If you know about Laplace transforms, this is the same formula you get by taking the Laplace transform of all signals

Transfer function for linear differential equations Consider an input/output differential equation of the form

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u.$$

We can compute the transfer function using an exponential signal $u = e^{st}$ and searching for an output of the form $y = Ye^{st}$:

$$(s^n + a_1 s^{n-1} + \cdots + a_n) y_0 e^{st} = (b_0 s^m + b_1 s^{m-1} \cdots + b_m) e^{-st}.$$

The *steady state* output is thus given by

$$y(t) = Ye^{st} = \underbrace{\frac{b(s)}{a(s)}}_{H_{yu}(s)} e^{st}, \qquad \begin{aligned} a(s) &= s^n + a_1 s^{n-1} + \cdots + a_n,\\ b(s) &= b_0 s^m + b_1 s^{m-1} + \cdots + b_m. \end{aligned} \tag{1}$$

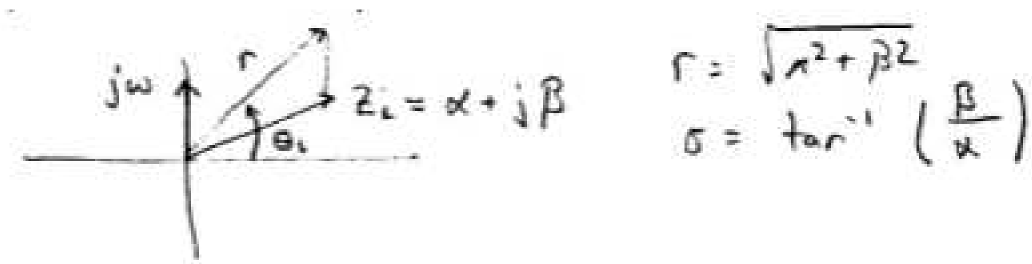
2 Bode Plots

Given a transfer function $G(s) = b(s)/a(s)$, we can sketch a Bode plot by looking at poles and zeros:

$$G(s) = k\frac{(s+z_1)\dots(s+z_m)}{(s+p_1)\dots(s+p_n)}, \qquad z_i, p_i \in \mathbb{C}$$

$$G(j\omega) = k\frac{(j\omega+z_1)\dots(j\omega+z_m)}{(j\omega+p_1)\dots(j\omega+p_n)}.$$

Each term can be written as $r_j e^{i\theta_j}$, where r_j = magnitude, $\theta_j = phase$.



$$\begin{aligned} H(i\omega) &= k\frac{r_1^z e^{i\theta_1^z}\dots r_m^z e^{i\theta_m^z}}{r_1^p e^{i\theta_1^p}\dots r_n^p e^{i\theta_n^p}} \\ &= k r_1^z e^{i\theta_1^z}\dots r_m^z e^{i\theta_m^z} r_1^p e^{-i\theta_1^p}\dots r_n^p e^{-i\theta_n^p} \\ &= r_1^z \dots r_m^z r_1^p \dots r_m^p e^{\theta_1^z+\dots+\theta_m^z-\theta_1^p-\dots-\theta_m^p} \end{aligned}$$

Bode plot

- log amplitude $\implies$ *add* gain of poles and zeros (plus k)
- linear phase $\implies$ *add* phase of zeros, subtract phase of poles

This allows us to sketch Bode plots in an easy way

Examples (to be sketched in lecture)

- $H(s) = s^k$, $k = \dots, -2, -1, 0, 1, 2, \dots$
- $H(s) = \dfrac{1}{s+a}$
- $H(s) = \dfrac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
- $H(s) = \dfrac{s+b}{s+a}$, a ¡ b
- $H(s) = \dfrac{k(s+b)}{(s+a)s^2 + 2\zeta\omega_0 s + \omega_0^2}$, $a < \omega_0 < b$

3 Laplace Transforms

Definition:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad \operatorname{Re} s > s_0.$$

- Notation: $F = \mathcal{L}(f)$
- Linear operator: $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$

Laplace transform of a derivative and integral:

$$\mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(0) = sF(s) - f(0), \qquad \mathcal{L}\int_0^t f(\tau)\, d\tau = \frac{1}{s}\mathcal{L}f = \frac{1}{s}F(s).$$

Laplace transform of a convolution equation. Given two function $h(t)$ and $u(t)$,

$$y(t) = (h * u)(t) = \int_0^\infty h(t-\tau)u(\tau)\, d\tau \qquad \Longrightarrow \qquad Y(s) = H(s)U(s)$$