This lecture describes how to derive and use transfer functions for linear systems, including a description of how to sketch a Bode plot by hand. If there is time, I will also talk about the relationship between the transfer function and the Laplace transform.

Reading:

- Åström and Murray, Chapter 8

1 Transfer Functions for Linear Systems

Transfer function for state space systems

\[
\begin{align*}
\frac{dx}{dt} &= Ax + Bu, \\
y &= Cx + Du \\
y(t) &= Ce^{At} \left( x(0) - (sI - A)^{-1}B \right) + \frac{C(sI - A)^{-1}B + D}{H_{yu}(s)} e^{st},
\end{align*}
\]

- Transfer function only captures the steady state response
- If you know about Laplace transforms, this is the same formula you get by taking the Laplace transform of all signals

Transfer function for linear differential equations

Consider an input/output differential equation of the form

\[
\frac{d^ny}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^mu}{dt^m} + b_1 \frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_m u.
\]

We can compute the transfer function using an exponential signal \( u = e^{st} \) and searching for an output of the form \( y = Ye^{st} \):

\[
(s^n + a_1 s^{n-1} + \cdots + a_n)y_0 e^{st} = (b_0 s^m + b_1 s^{m-1} + \cdots + b_m)e^{-st}.
\]

The steady state output is thus given by

\[
y(t) = Ye^{st} = \frac{b(s)}{a(s)} e^{st}, \quad a(s) = s^n + a_1 s^{n-1} + \cdots + a_n, \quad b(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_m.
\]

(1)
2 Bode Plots

Given a transfer function \( G(s) = \frac{b(s)}{a(s)} \), we can sketch a Bode plot by looking at poles and zeros:

\[
G(s) = \frac{k(s + z_1) \cdots (s + z_m)}{(s + p_1) \cdots (s + p_n)}, \quad z_i, p_i \in \mathbb{C}
\]

\[
G(j\omega) = \frac{k(j\omega + z_1) \cdots (j\omega + z_m)}{(j\omega + p_1) \cdots (j\omega + p_n)}.
\]

Each term can be written as \( r_j e^{i\theta_j} \), where \( r_j = \text{magnitude}, \theta_j = \text{phase} \).

\[
H(i\omega) = k \frac{r_1 e^{i\theta_1} \cdots r_m e^{i\theta_m}}{r_1 e^{-i\theta_1} \cdots r_n e^{-i\theta_n}}
\]

\[
= k r_1^z e^{i\theta_1} \cdots r_m^z e^{i\theta_m} r_1^{-z} e^{-i\theta_1} \cdots r_n^{-z} e^{-i\theta_n}
\]

\[
= r_1^z \cdots r_m^z r_1^{-z} \cdots r_n^{-z} e^{i\theta_1 - \theta_2 - \cdots - \theta_m - \cdots - \theta_n}
\]

Bode plot

- log amplitude \( \implies \) add gain of poles and zeros (plus \( k \))
- linear phase \( \implies \) add phase of zeros, subtract phase of poles

This allows us to sketch Bode plots in an easy way

Examples (to be sketched in lecture)

- \( H(s) = s^k, k = \ldots, -2, -1, 0, 1, 2, \ldots \)
- \( H(s) = \frac{1}{s + a} \)
- \( H(s) = \frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \)
- \( H(s) = \frac{s + b}{s + a}, a \not= b \)
- \( H(s) = \frac{k(s + b)}{(s + a)s^2 + 2\zeta\omega_0 s + \omega_0^2}, a < \omega_0 < b \)
3 Laplace Transforms

Definition:
\[ F(s) = \int_0^\infty e^{-st} f(t) \, dt, \quad \text{Re} \, s > s_0. \]

- Notation: \( F = \mathcal{L}(f) \)
- Linear operator: \( \mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g) \)

Laplace transform of a derivative and integral:
\[ \mathcal{L}\left(\frac{df}{dt}\right) = sF - f(0) = sF(s) - f(0), \quad \mathcal{L}\left(\int_0^t f(\tau) \, d\tau\right) = \frac{1}{s} \mathcal{L}f = \frac{1}{s} F(s). \]

Laplace transform of a convolution equation. Given two functions \( h(t) \) and \( u(t) \),
\[ y(t) = (h * u)(t) = \int_0^\infty h(t - \tau)u(\tau) \, d\tau \quad \implies \quad Y(s) = H(s)U(s) \]