CDS 101/110a: Lecture 6-1
Transfer Functions

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Goals:

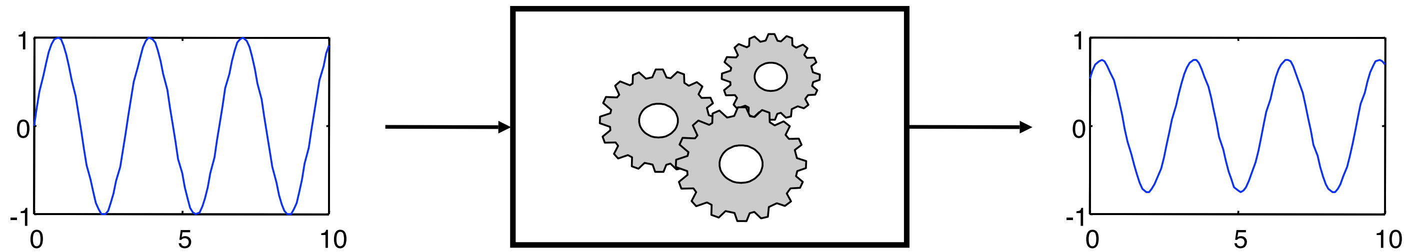
- Motivate and define the input/output transfer function of a linear system
- Understand the relationships among frequency response (Bode plot), transfer function, and state-space model
- Introduce block diagram algebra for transfer functions of interconnected systems

Reading:

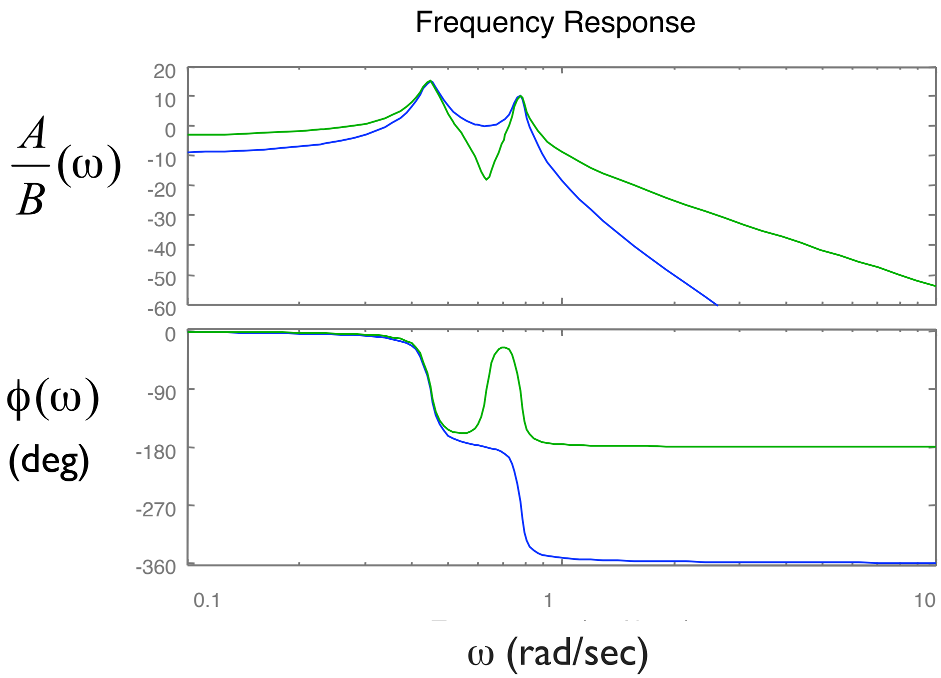
- Åström and Murray, *Feedback Systems,* Ch 8
- *Advanced*: Lewis, Chapters 3-4
- CDS 210: DFT, Chapter 2

Frequency Domain Modeling

Defn. The *frequency response* of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



$u = A\sin(\omega t)$ $\qquad$ $y = B\sin(\omega t + \phi)$



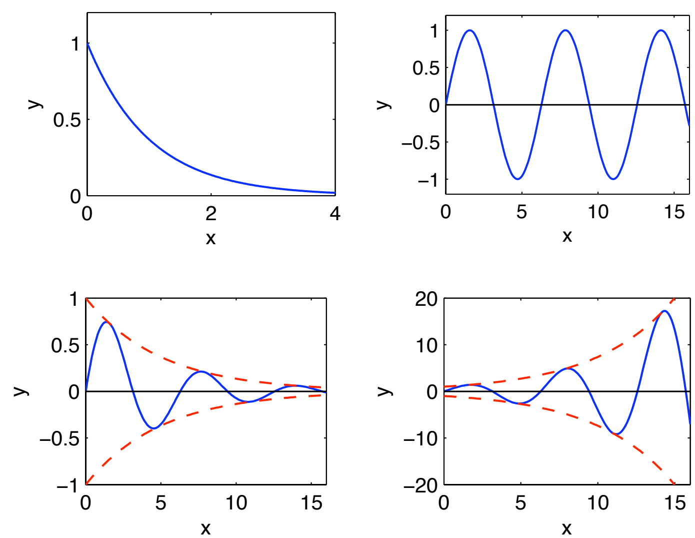
Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity ⇒ can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

Transmission of Exponential Signals

Exponential signal: $e^{st} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$

- Construct constant inputs + sines/cosines by linear combinations
  - Constant: $u(t) = c = ce^{0t}$
  - Sinusoid: $u(t) = A \sin(\omega t) = \frac{A}{2i}(e^{i\omega t} - e^{-i\omega t})$
- Exponential response can be computed via the convolution equation



$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau}\, d\tau \\
&= e^{At}x(0) + e^{At}(sI - A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\
&= e^{At}x(0) + e^{At}(sI - A)^{-1}\Big(e^{(sI-A)t} - I\Big) B \\
&= e^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + (sI - A)^{-1} B e^{st}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + \Big(C(sI - A)^{-1}B + D\Big)e^{st}
\end{aligned}
$$

Transfer Function and Frequency Response

Exponential response of a linear state space system

$$y(t) = Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + \Big(C(sI - A)^{-1}B + D\Big)e^{st}$$

transient steady state

Transfer function

- Steady state response is proportional to exponential input => look at input/output ratio
- $G(s) = C(sI - A)^{-1}B + D$ is the *transfer function* between input and output

Common transfer functions

$\dot{y} = u$	$\frac{1}{s}$
$y = \dot{u}$	s
$\dot{y} + ay = u$	$\frac{1}{s+a}$
$\ddot{y} = u$	$\frac{1}{s^2}$
$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
$y(t) = u(t - \tau)$	$e^{-\tau s}$

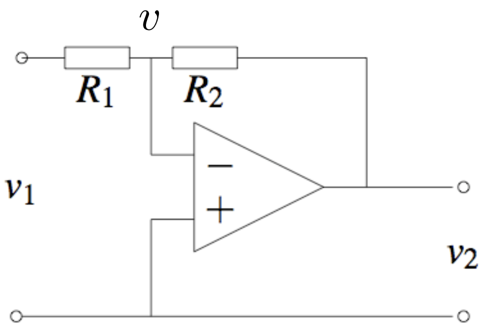
Frequency response

$$u(t) = A\sin\omega t = \frac{A}{2i}(e^{i\omega t} - e^{-i\omega t})$$

$$\begin{aligned} y_{ss}(t) &= \frac{A}{2i}\Big(G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\Big) \\ &= A \cdot |G(i\omega)| \sin(\omega t + \arg G(i\omega)) \end{aligned}$$

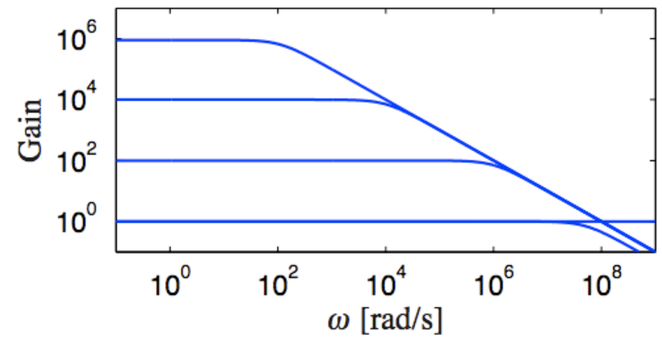
gain phase

Example: Electrical Circuits



Op amp dynamics:

$$\frac{v_{\text{out}}}{v} = -\frac{ak}{s+a} =: G(s)$$



Circuit dynamics (Kirchoff's laws):

$$\frac{v_1 - v}{R_1} = \frac{v - v_2}{R_2} \qquad \text{and} \qquad v_2 = G(s)v$$

$$\frac{v_2}{v_1} = \frac{R_2 G(s)}{R_1 + R_2 + R_1 G(s)} = \frac{R_2 ak}{R_1 ak + (R_1 + R_2)(s+a)}.$$

- Algebraic manipulation can be used as long as we assume exponential signals and all of the components (blocks) are linear
- Transfer function between input and output hows gain-bandwidth tradeoff
- Homework: derive transfer function for a PI controller using an op amp

Transfer Function Properties

Thm. The *transfer function* for a linear system Σ=(*A*,*B*,*C*,*D*) is given by

$$G(s) = C(sI - A)^{-1}B + D \qquad s \in \mathbb{C}$$

Thm. The transfer function *G*(*s*) corresponding to Σ=(*A*,*B*,*C*,*D*) has the following properties:

- *H*(*s*) is a ratio of polynomials *n*(*s*)/*d*(*s*) where *d*(*s*) is the *characteristic equation* for the matrix *A* and *n*(*s*) has order less than or equal to *d*(*s*).
- The *steady state* frequency response of Σ has gain |*G*(*j*ω)| and phase arg *G*(*j*ω):

$$\begin{aligned} u &= A\sin(\omega t) \\ y &= |G(i\omega)| A \sin(\omega t + \arg G(i\omega)) + \text{transients} \end{aligned}$$

Remarks

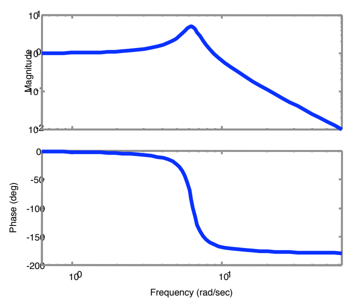
- Formally, can show that *G*(*s*) is the *Laplace transform* of the impulse response of Σ
- "*y*=*G*(*s*)*u*" is formally *Y*(*s*)=*G*(*s*)*U*(*s*) where *Y*(*s*) and *U*(*s*) are the Laplace transforms of *y*(*t*) and *u*(*t*). (Multiplication in the Laplace domain corresponds to convolution.)

Series Interconnections

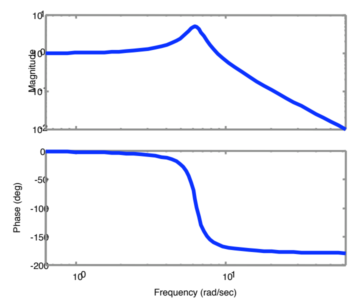
Q: what happens when we connect two systems together *in series?*

$$\xrightarrow{u_1} \boxed{\begin{aligned}\dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ y_1 &= C_1 x_1 + D_1 u_1\end{aligned}} \xrightarrow{y_1 \qquad u_2} \boxed{\begin{aligned}\dot{x}_2 &= A_2 x_2 + B_2 u_2 \\ y_2 &= C_2 x_2 + D_2 u_2\end{aligned}} \xrightarrow{y_2}$$

$u = A\sin(\omega t)$



$y = g_1 A\sin(\omega t + \phi_1)$



$y_2 = g_1 g_2 A \times \sin(\omega t + \phi_1 + \phi_2)$

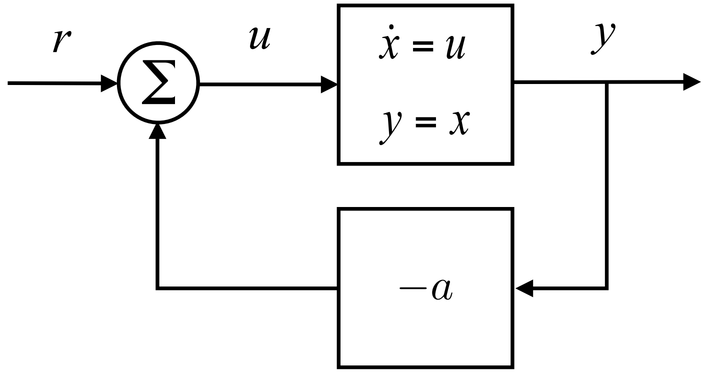
A: Transfer functions *multiply*

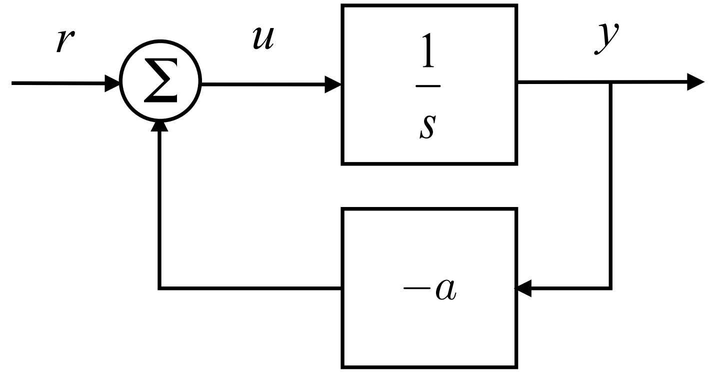
- Gains multiply
- Phases add
- Generally: transfer functions well formulated for frequency domain interconnections

$$\xrightarrow{u_1} \boxed{G_1(s)} \xrightarrow{u_2 = y_1} \boxed{G_2(s)} \xrightarrow{y_2}$$

$$\xrightarrow{u_1} \boxed{G_2(s)G_1(s)} \longrightarrow \quad y_2 = G_2(s)G_1(s)u_1$$

Feedback Interconnection





State space derivation

$$\dot{x} = u = r - ay = -ax + r$$
$$y = x$$

Frequency response $r = A \sin(\omega t)$

$$y = \left| \frac{1}{\sqrt{a^2 + \omega^2}} \right| \sin\left(\omega t - \tan^{-1}(\frac{\omega}{a})\right)$$

Transfer function derivation

$$y = \frac{u}{s} = \frac{r - ay}{s}$$
$$y = \frac{r}{s + a} = G(s) r$$

Frequency response

$$y = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

Poles and Zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$H(s) = \frac{n(s)}{d(s)}$$
$$d(s) = \det(sI - A)$$

- Roots of $d(s)$ are called *poles* of $H(s)$
- Roots of $n(s)$ are called *zeros* of $H(s)$

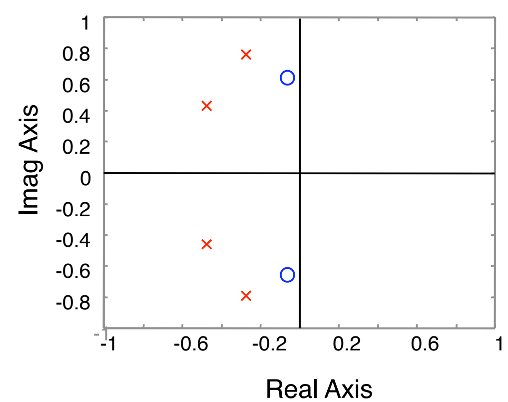
Poles of $H(s)$ determine the stability of the (closed loop) system

- Denominator of transfer function = characteristic polynomial of state space system
- Provides easy method for computing stability of systems
- Right half plane (RHP) poles (Re > 0) correspond to unstable systems

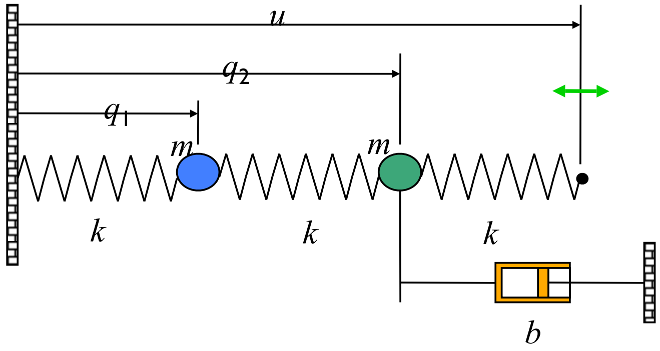
Zeros of $H(s)$ related to frequency ranges with limited transmission

- A pure imaginary zero at $s=j\omega_z$ blocks any output at that frequency ($G(j\omega_z) = 0$)
- Zeros provide limits on performance, especially RHP zeros (more on this later)

$$H(s) = k\frac{s^2 + b_1 s + b_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \xrightarrow{\text{pzmap}}$$



Example: Coupled Masses



$$H_{q_1 f} = \frac{0.04}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

$$H_{q_2 f} = \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

Poles (H_{q1f} and H_{q2f})

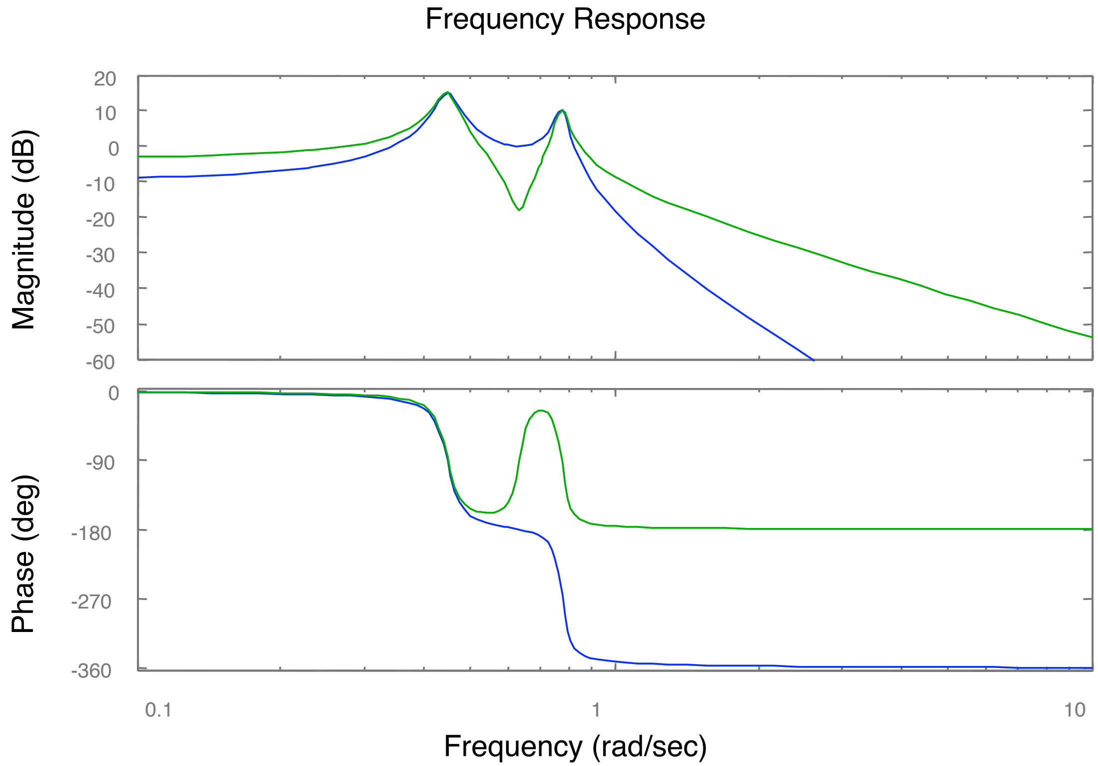
- $-0.0200 \pm 0.7743j$
- $-0.0200 \pm 0.4468j$

Zeros (H_{q2f})

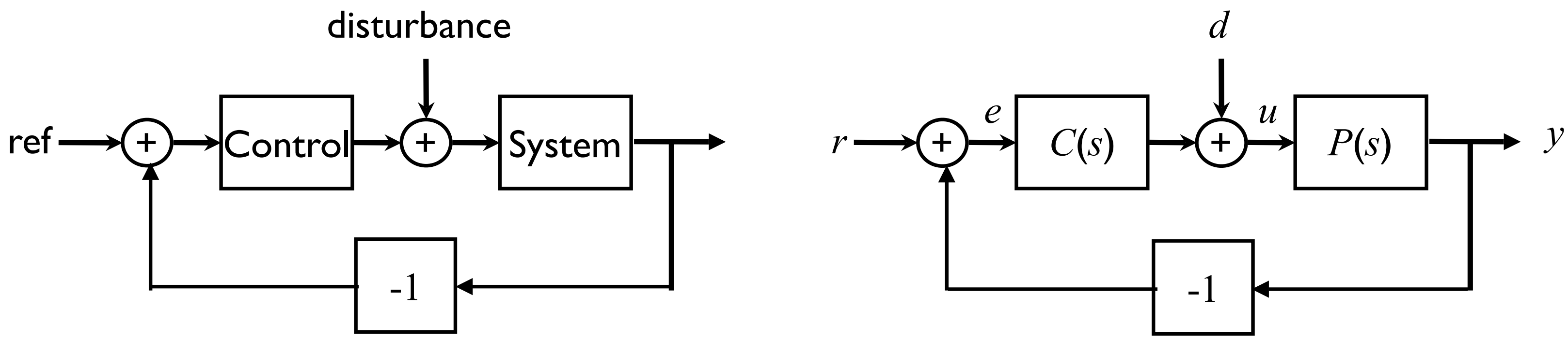
- $-0.0200 \pm 0.6321j$

Interpretation

- Zeros in H_{q2f} give low response at $\omega \approx 0.6321$



Control Analysis and Design Using Transfer Functions



Transfer functions provide a method for “block diagram algebra”

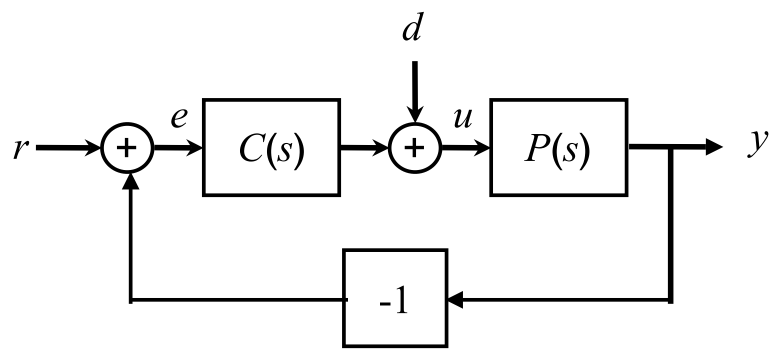
- Easy to compute transfer functions between various inputs and outputs
  - $H_{er}(s)$ is the transfer function between the reference and the error
  - $H_{ed}(s)$ is the transfer function between the disturbance and the error

Transfer functions provide a method for performance specification

- Since transfer functions provide frequency response directly, it is convenient to work in the “frequency domain”
  - $H_{er}(s)$ should be small in the frequency range 0 to 10 Hz (good tracking)

Block Diagram Algebra

Basic idea: treat transfer functions as multiplication, write down equations



$$y = P(s)u$$
$$u = d + C(s)e$$
$$e = r - y$$

Manipulate equations to compute desired signals

$$\begin{aligned} e &= r - y \\ &= r - P(s)u \\ &= r - P(s)(d + C(s)e) \end{aligned}$$

$$(1 + P(s)C(s))e = r - P(s)d$$

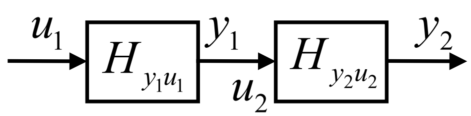
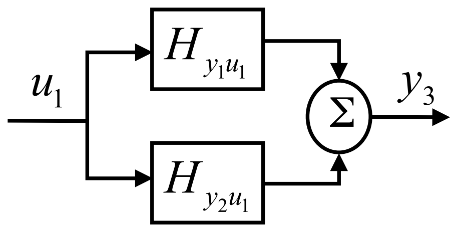
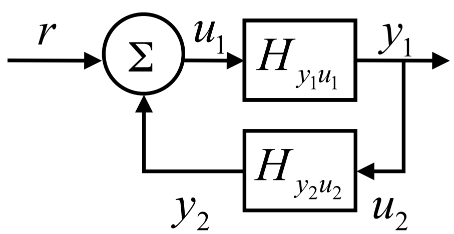
$$e = \underbrace{\frac{1}{1 + P(s)C(s)}}_{H_{er}} r - \underbrace{\frac{P(s)}{1 + P(s)C(s)}}_{H_{ed}} d$$

Note: linearity gives superposition of terms

Algebra works because we are working in frequency domain

- Time domain (ODE) representations are not as easy to work with
- Formally, all of this works because of Laplace transforms (see book)

Block Diagram Algebra

Type	Diagram	Transfer function
Series		$H_{y_2u_1} = H_{y_2u_2} H_{y_1u_1} = \dfrac{n_1n_2}{d_1d_2}$
Parallel		$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \dfrac{n_1d_2 + n_2d_1}{d_1d_2}$
Feedback		$H_{y_1r} = \dfrac{H_{y_1u_1}}{1 + H_{y_1u_1} H_{y_2u_2}} = \dfrac{n_1d_2}{n_1n_2 + d_1d_2}$

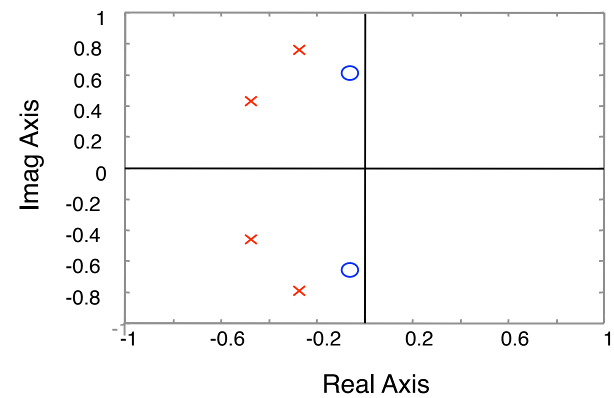
- These are the basic manipulations needed; some others are possible
- Formally, could work all of this out using the original ODEs ($\Rightarrow$ nothing *really* new)

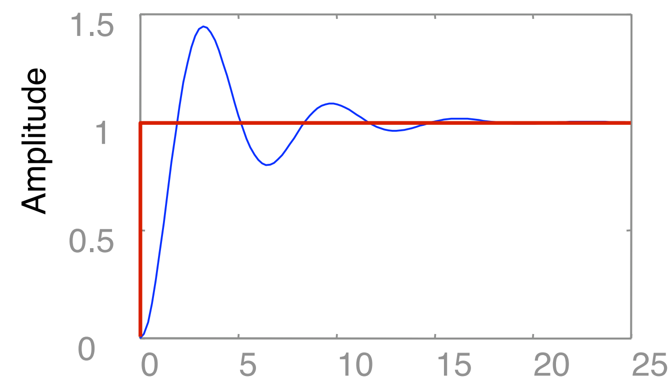
MATLAB manipulation of transfer functions

Creating transfer functions

- [num, den] = ss2tf(A, B, C, D)
- sys = tf(num, den)
- num, den = [1 a b] $\rightarrow s^2 + as + b$

```
» tf(sys)
     Transfer function:
               1
      ---------------
      s^2 + 0.2 s + 1
```





Interconnecting blocks

- sys= series(sys1, sys2), parallel, feedback

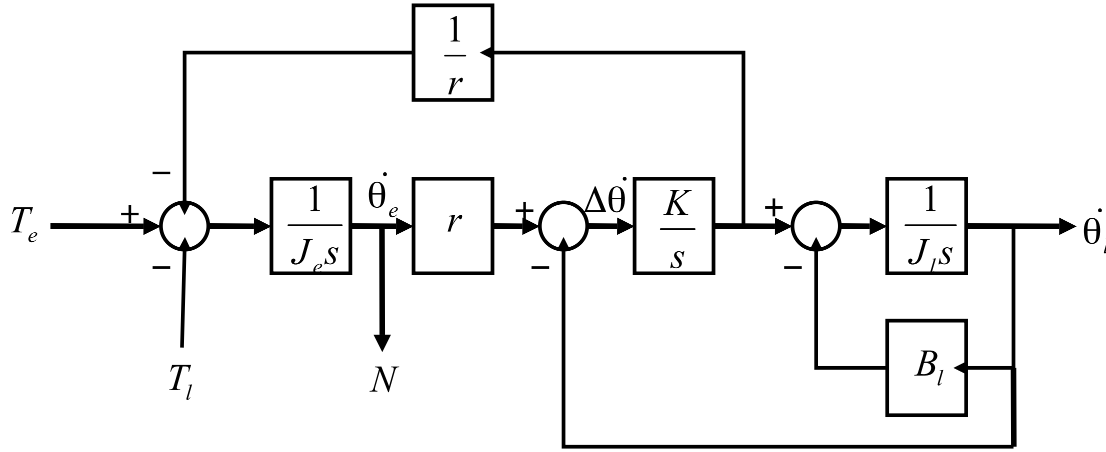
Computing poles and zeros

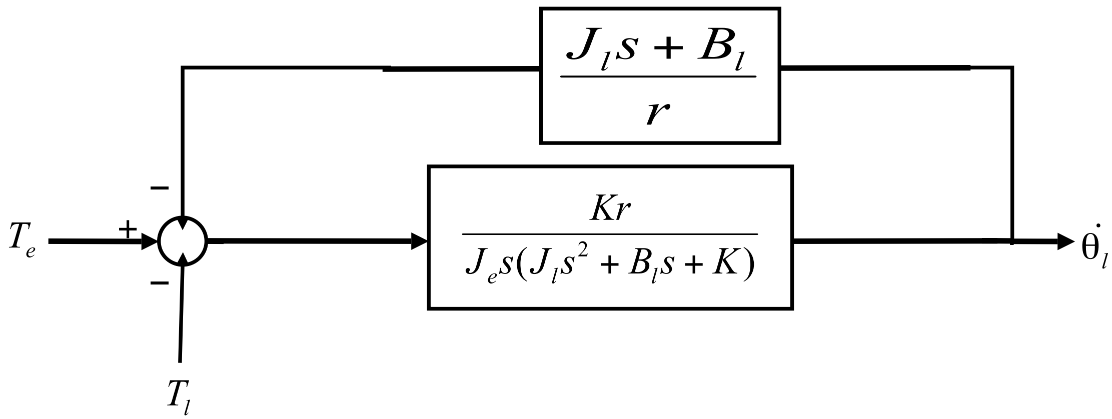
- pole(sys), zero(sys)
- pzmap(sys)

I/O response

- step(sys), bode(sys)

Example: Engine Control of a GM Astro

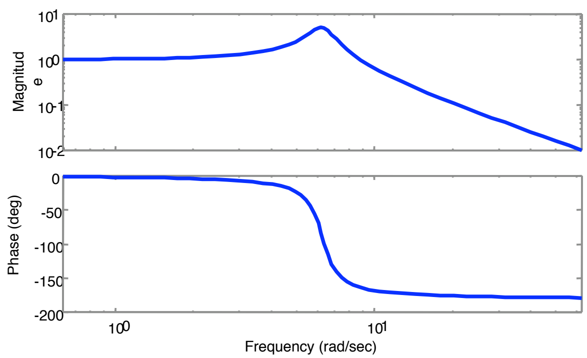




$$H_{\theta_l T_e}(s) = \frac{Kr}{J_e J_l s^3 + J_e B_l s^2 + (J_e K + K J_l)s + K B_l}$$

Summary: Frequency Response & Transfer Functions

$$u = A\sin(\omega t) \longrightarrow \boxed{\begin{gathered}\dot{x} = Ax + Bu\\ y = Cx + Du\\ x(0) = 0\end{gathered}} \longrightarrow y_{ss} = A\cdot|G(i\omega)|\times \sin(\omega t + \arg G(i\omega))$$



$$G(s) = C(sI - A)^{-1}B + D$$

$$G_{y_2 u_1} = G_{y_2 u_2} G_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$