CDS 101/110a: Lecture 2.1
Dynamic Behavior

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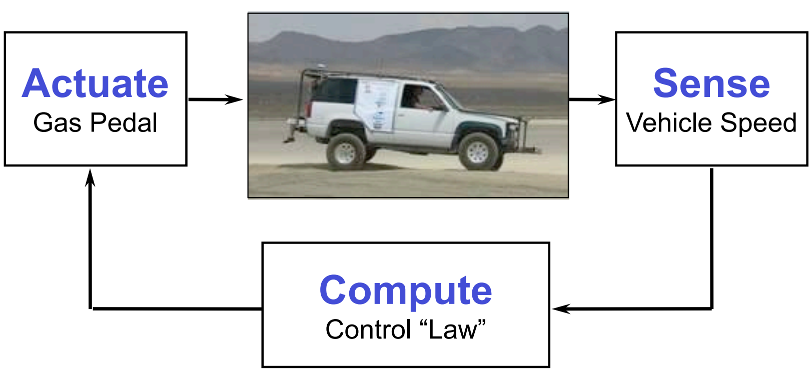
Goals:
- Learn to use phase portraits to visualize behavior of dynamical systems
- Understand different types of stability for an equilibrium point
- Know the difference between local/global stability and related concepts

Reading:
- Åström and Murray, *Feedback Systems,* Chapter 4 [90 minutes]
- Advanced: S. H. Strogatz, *Nonlinear Dynamics and Chaos,* Chapter 6

Dynamic Behavior (and Stability)

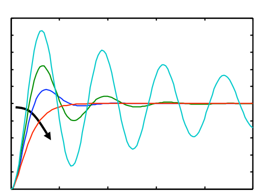


Goal #1: Stability
- Check if closed loop response is stable

$\dot{x} = f(x,u) \qquad u = k(x)$

control law
system input

Goal #2: Performance
- Look at how the closed loop system behaves, in a dynamic context



Response depends on choice of control (all are stable)

Goal #3: Robustness (later)

Phase Portraits (2D systems only)

Phase plane plots show 2D dynamics as *vector fields* & *stream functions*

- $\dot{x} = f(x, u(x)) = F(x)$
- Plot $F(x)$ as a vector on the plane; stream lines follow the flow of the arrows

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$

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phaseplot('dosc', ...
  [-1 1 10], [-1 1 10], 0.1, ...
  boxgrid([-1 1 10], [-1 1 10]));
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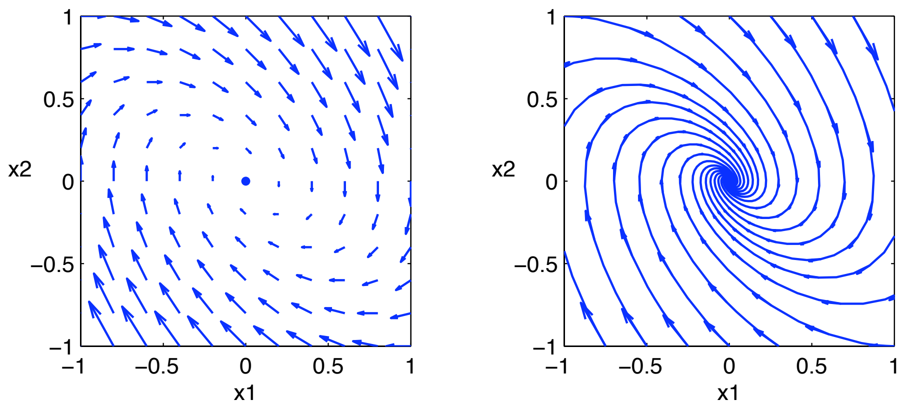
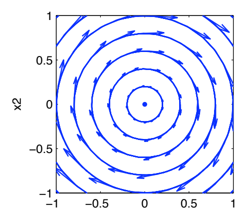


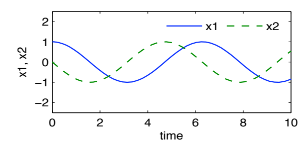
Stability of Equilibrium Points

An equilibrium point is:

Stable if initial conditions that start near the equilibrium point, stay near

- Also called "stable in the sense of Lyapunov"

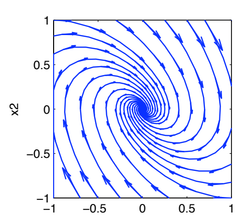


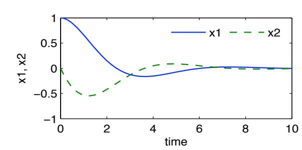


$$\|x(0) - x_e\| < \delta \implies \|x(t) - x_e\| < \epsilon$$

Asymptotically stable if all nearby initial conditions converge to the equilibrium point

- Stable + converging

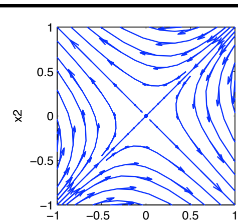


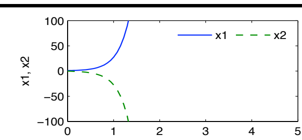


$$\lim_{t\to\infty} x(t) = x_e \quad \forall \|x(0) - x_e\| < \epsilon$$

Unstable if some initial conditions diverge from the equilibrium point

- May still be some initial conditions that converge



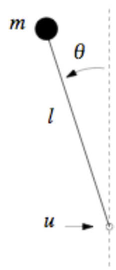


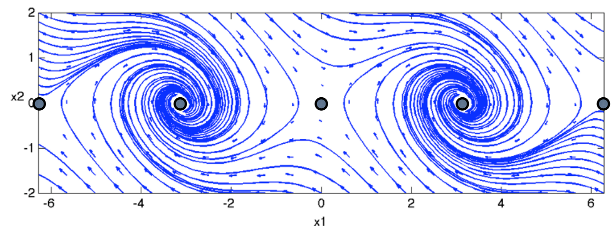
Equilibrium Points

Equilibrium points represent stationary conditions for the dynamics

The *equilibria* of the system $\dot{x} = F(x)$ are the points x_e such that $f(x_e) = 0$.

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix} \qquad \Longrightarrow \qquad x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$

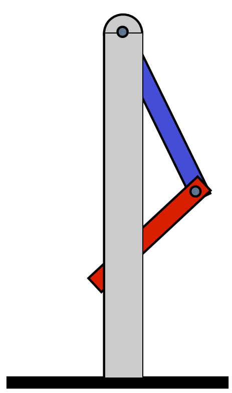


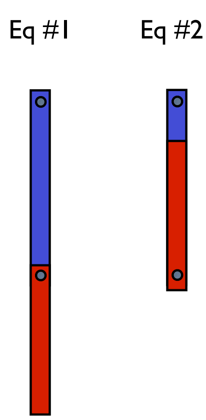


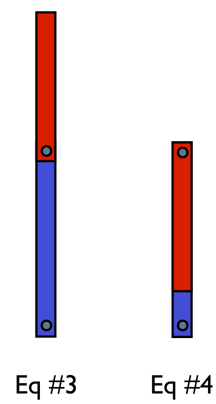
Example #1: Double Inverted Pendulum

Two series coupled pendula

- States: pendulum angles (2), velocities (2)
- Dynamics: *F* = *ma* (balance of forces)
- Dynamics are very nonlinear







Stability of equilibria

- Eq #1 is stable
- Eq #3 is unstable
- Eq #2 and #4 are unstable, but with some stable "modes"

Stability of Linear Systems

Linear dynamical system with state $x \in \mathbb{R}^n$

$$\frac{dx}{dt} = Ax \qquad x(0) = x_0,$$

Stability determined by the eigenvalues $\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$

- Simplest case: diagonal *A* matrix (all eigenvalues are real)

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x$$

$$\dot{x}_i = \lambda_i x_i$$

$$x_i(t) = e^{\lambda_i t} x(0)$$

  - System is asy stable if $\lambda_i < 0$

- Block diagonal case (complex eigenvalues)

$$\frac{dx}{dt} = \begin{bmatrix} \sigma_1 & \omega_1 & & 0 & 0 \\ -\omega_1 & \sigma_1 & & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & & \sigma_m & \omega_m \\ 0 & 0 & & -\omega_m & \sigma_m \end{bmatrix} x$$

$$x_{2j-1}(t) = e^{\sigma_j t}\big(x_i(0)\cos\omega_j t + x_{i+1}(0)\sin\omega_j t\big)$$
$$x_{2j}(t) = e^{\sigma_j t}\big(x_i(0)\sin\omega_j t - x_{i+1}(0)\cos\omega_j t\big)$$

  - System is asy stable if $\operatorname{Re}\lambda_i = \sigma_i < 0$

- **Theorem** linear system is asymptotically stable if and only if $\operatorname{Re}\lambda_i < 0 \quad \forall\, \lambda_i \in \lambda(A)$

Local Stability of Nonlinear Systems

Asymptotic stability of the linearization implies *local* asymptotic stability of equilibrium point

- Linearization around equilibrium point captures "tangent" dynamics

$$\dot{x} = \cancelto{0}{F(x_e)} + \left.\frac{\partial F}{\partial x}\right|_{x_e} (x - x_e) + \text{higher order terms} \xrightarrow{\text{approx}} \begin{array}{l} z = x - x_e \\ \dot{z} = Az \end{array}$$

- If linearization is *unstable,* can conclude that nonlinear system is locally unstable
- If linearization is *stable* but not *asymptotically stable,* can't conclude anything about nonlinear system:

$$\dot{x} = \pm x^3 \xrightarrow{\text{linearize}} \dot{x} = 0$$

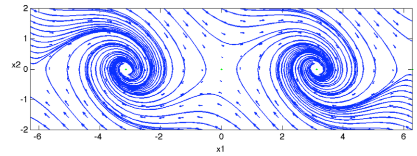
  - linearization is stable (but not asy stable)
  - nonlinear system can be asy stable or unstable

Local approximation particularly appropriate for control systems design

- Control often used to *ensure* system stays near desired equilibrium point
- If dynamics are well-approximated by linearization near equilibrium point, can use this to design the controller that keeps you there (!)

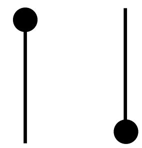
Example: Stability Analysis of Inverted Pendulum

System dynamics



$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix},$$

Upward equilibrium:



- $\theta = x_1 \ll 1 \quad \Longrightarrow \quad \sin x_1 \approx x_1$

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ x_1 - \gamma x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix} x$$

- Eigenvalues: $-\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{4+\gamma^2}$

Downward equilibrium:

- Linearize around $x_1 = \pi + z_1$: $\sin(\pi + z_1) = -\sin z_1 \approx -z_1$

$$\begin{aligned} z_1 &= x_1 - \pi \\ z_2 &= x_2 \end{aligned} \quad \Longrightarrow \quad \frac{dz}{dt} = \begin{bmatrix} z_2 \\ -z_1 - \gamma\, z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix} z$$

- Eigenvalues: $-\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{-4+\gamma^2}$

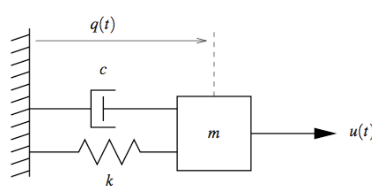
Reasoning about Stability using Lyapunov Functions

Basic idea: capture the behavior of a system by tracking "energy" in system

- Find a single function that captures distance of system from equilibrium
- Try to reason about the long term behavior of *all* solutions

Example: spring mass system

- Can we show that all solutions return to rest w/out explicitly solving ODE?
- Idea: look at how energy evolves in time



$$m\ddot{q} + c\dot{q} + kq = 0$$

- Start by writing equations in state space form
- Compute energy and its derivative

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix} \qquad \begin{aligned} x_1 &= q \\ x_2 &= \dot{q} \end{aligned}$$

$$V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 \qquad \begin{aligned} \frac{dV}{dt} &= kx_1\dot{x}_1 + mx_2\dot{x}_2 \\ &= kx_1x_2 + mx_2\left(-\frac{c}{m}x_2 - \frac{k}{m}x_1\right) = -cx_2^2 \end{aligned}$$

- Energy is positive $\Rightarrow$ x_2 must eventually go to zero
- If x_2 goes to zero, can show that x_1 must also approach zero (Krasovskii-Lasalle)

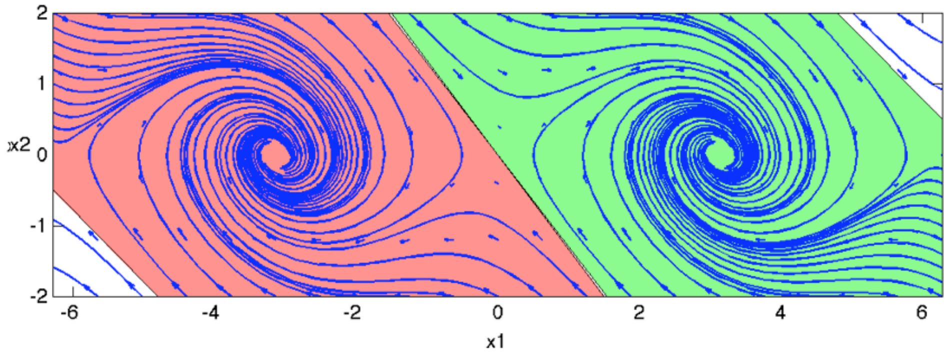
Local versus Global Behavior

Stability is a *local* concept

- Equilibrium points define the local behavior of the dynamical system
- Single dynamical system can have stable *and* unstable equilibrium points

Region of attraction

- Set of initial conditions that converge to a given equilibrium point



Example #2: Predator Prey (ODE version)

Continuous time (ODE) version of predator prey dynamics:

$$\frac{dH}{dt} = rH\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H} \qquad H \geq 0$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL \qquad L \geq 0.$$

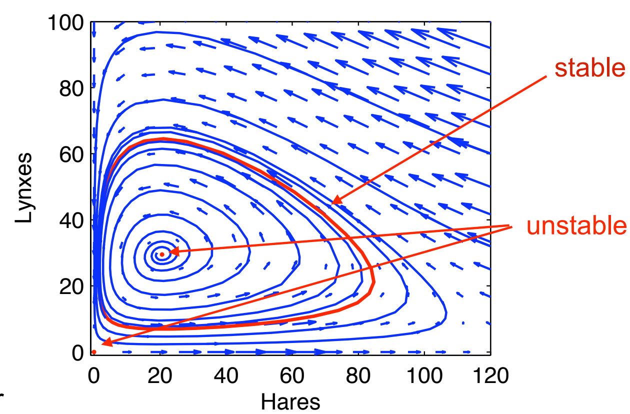
- Continuous time (ODE) model
- MATLAB: predprey.m (from web page)

Equilibrium points (2)

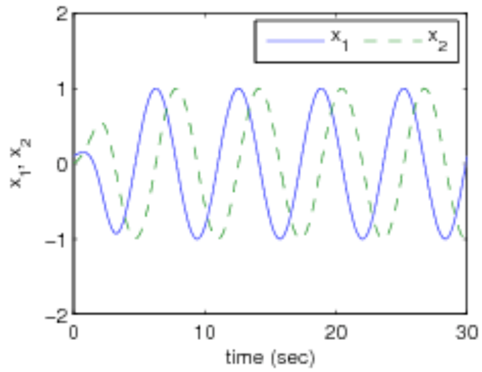
- ~(20.5, 29.5): unstable
- (0, 0): unstable

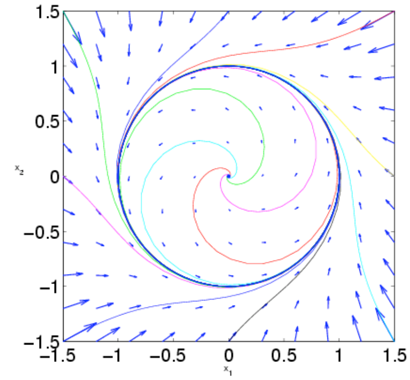
Limit cycle

- Population of each species oscillates over time
- Limit cycle is stable (nearby solutions converge to limit cycle)
- This is a *global* feature of the dynamics (not local to an equilibr point)



Simpler Example of a Limit Cycle





Dynamics:

$$\frac{dx_1}{dt} = -x_2 - x_1(1 - x_1^2 - x_2^2)$$
$$\frac{dx_2}{dt} = x_1 - x_2(1 - x_1^2 - x_2^2).$$

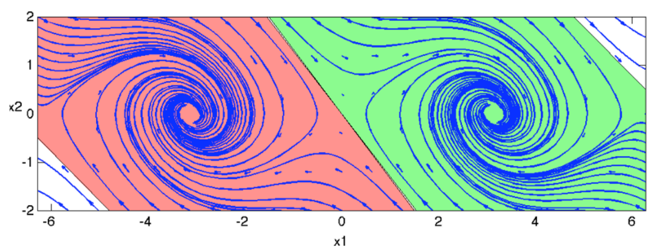
- Note that $\|x\| = 1$ is an *invariant set*
- From simulation, x(t+T) = x(t)

Stability of invariant set

$$V(x) = \frac{1}{4}(1 - x_1^2 - x_2^2)^2$$

$$\begin{aligned}\dot V(x) &= (x_1\dot x_1 + x_2\dot x_2)(1 - x_1^2 - x_2^2)\\ &= \cdots\\ &= -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2)^2\end{aligned}$$

Summary: Stability and Performance



Key topics for this lecture

- Stability of equilibrium points
- Eigenvalues determine stability for linear systems
- Local versus global behavior