



CDS 110: Lecture 2-2

Modeling Using Differential Equations



Richard M. Murray

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Goals:

- Provide a more detailed description of the use of ODEs for modeling
- Provide examples of the type of analysis that can be done using ODEs

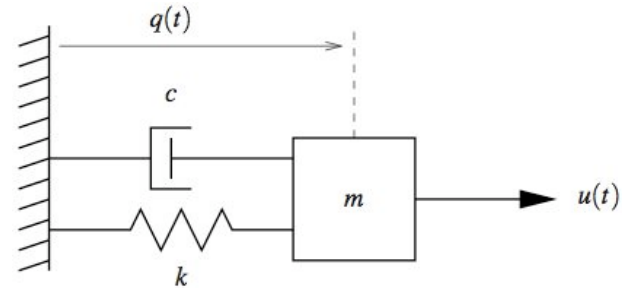
Reading:

- Åström and Murray, *Feedback Systems*, Ch 2 and Sec 3.1
- Advanced: Lewis, *A Mathematical Approach to Classical Control*, Ch 1

Review: Second Order Differential Equations (Ma 1)

Damped oscillator dynamics

$$m\ddot{q} + c\dot{q} + kq = u$$



Homogeneous solution: $u(t) = 0$

- Guess form of the solution: $q(t) = e^{\alpha t}(A \cos \omega t + B \sin \omega t)$
- Substitute into ODE and solve for the constants

$$0 = e^{\alpha t} \left((B(c + 2\alpha m)\omega + A(m\alpha^2 + c\alpha - m\omega^2 + k)) \cos \omega t \right. \\ \left. + (Bm\alpha^2 + Bc\alpha - c2A\omega\alpha - Bm\omega^2 + Bk - Ac\omega) \sin \omega t \right)$$

$$q_0 = A$$

$$v_0 = A\alpha + B\omega$$

Solve for A & B

Coefficients of sin/cos must be zero
Use to solve for α, ω

- Simplify the solution by pulling out common terms

$$q(t) = e^{-\zeta\omega_0 t} \left(q_0 \cos \omega_d t + \left(\frac{\zeta\omega_0}{\omega_d} q_0 + \frac{1}{\omega_d} v_0 \right) \sin \omega_d t \right)$$

$$\omega_0 = \sqrt{k/m}$$

$$\zeta = \frac{1}{2} \sqrt{c^2/km}$$

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

- Note: this solution holds when $\zeta < 1$

Second Order Differential Equations, ctd

$$m\ddot{q} + c\dot{q} + kq = u$$

Particular response: zero initial conditions

- $q(0) = 0, \dot{q}(0) = 0$
- Response to constant (step) input, $u(t) = F$

$$q(t) = \frac{F}{m\omega_0^2} \left(1 - e^{-\zeta\omega_0 t} \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_0 t} \sin \omega_d t \right)$$

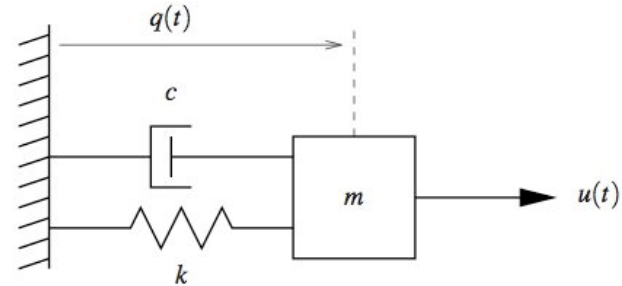
- Response to sinusoidal input, $u(t) = A \sin \omega t$

$$q(t) = M A \sin(\omega t + \theta) \quad M e^{i\omega} = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega}$$

- Form of the solution: sinusoid at same frequency, with shift in mag & phase
- Solving by hand is a mess; we will learn much better ways later

Complete solution: homogeneous + particular

- Warning: be careful to make sure the initial conditions are satisfied



More General Forms of Differential Equations

State space form

$$\frac{dx}{dt} = f(x, u)$$

$$y = h(x, u)$$

General form

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du$$

Linear system

$$x \in \mathbb{R}^n, u \in \mathbb{R}^p$$

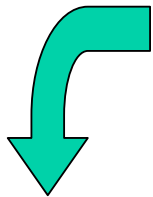
$$y \in \mathbb{R}^q$$

- x = state; n th order
- u = input; will usually set $p = 1$
- y = output; will usually set $q = 1$

Higher order, linear ODE

$$\frac{d^n q}{dt^n} + a_1 \frac{d^{n-1} q}{dt^{n-1}} + \cdots + a_n q = u$$

$$y = b_1 \frac{d^{n-1} q}{dt^{n-1}} + \cdots + b_{n-1} \dot{q} + b_n q$$



$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d^{n-1} q / dt^{n-1} \\ d^{n-2} q / dt^{n-2} \\ \vdots \\ dq / dt \\ q \end{bmatrix}$$

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = [b_1 \quad b_2 \quad \cdots \quad b_n] x$$

Analytical Solutions of ODEs

Scalar systems

$$\frac{dx}{dt} = ax + u$$

$$y = x$$

$$x_h(t) = e^{at}x_0$$

$$u = A \sin \omega t$$

$$y = A \frac{\omega e^{at} - \omega \cos \omega t - a \sin \omega t}{a^2 + \omega^2}$$

Decoupled systems

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u$$

$$y = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n] x + Du$$

$$\dot{x}_i = \lambda_i x_i + \beta_i u$$

$$x_i(t) = e^{\lambda_i t} x(0) +$$

$$\int_0^t e^{\lambda_i(t-\tau)} \beta_i u(\tau) d\tau$$

- Effect of input modeled by “convolution integral”



General solutions

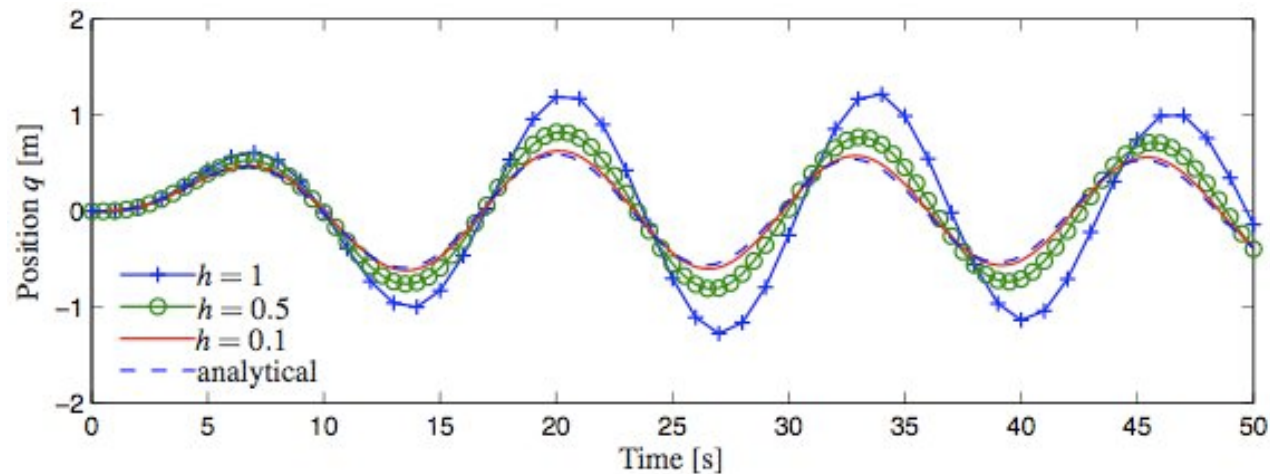
- Linear systems: use Jordan canonical form and “matrix exponential” (more later)
- Nonlinear system: generally no closed form solutions, expect in special cases

Numerical Solution of ODEs

Numerical simulation: Euler integration

$$\frac{dx}{dt} = \lim_{\epsilon \rightarrow 0} \frac{x(t + \epsilon) - x(t)}{\epsilon} \implies x(t + \epsilon) \approx x(t) + \epsilon f(x(t), u(t))$$

- If ϵ chosen sufficiently small, get good approximation analytical solution
- Solution is in the form of a difference equation (with step size ϵ)



- More accurate algorithms: build better approximation to the derivative
- Faster algorithms: choose the step size based on how quickly solution is changing
- Example: Runge Kutta (ode45)

Analyzing Models using ODEs: Frequency Response

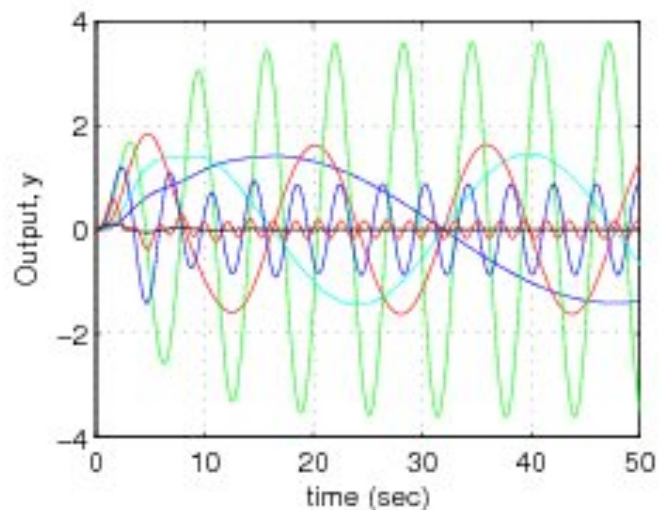
How does linear system respond to sinusoidal inputs?

$$m\ddot{q} + c\dot{q} + kq = u(t)$$

$$u(t) = A \sin \omega t$$

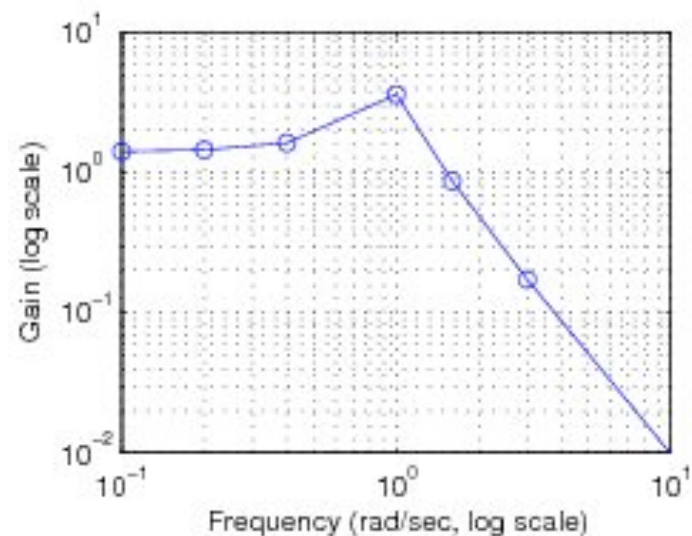
$$q(t) = M(\omega) \sin(\omega t + \theta(\omega))$$

magnitude $\nearrow$ phase $\nearrow$



General properties

- Linear systems: sinusoidal input at frequency $\omega \Rightarrow$ sinusoidal output at frequency ω
- Gain = $\frac{\text{output magnitude}}{\text{input magnitude}} = \frac{M(\omega)}{A}$
- Phase: shift in input sinusoid versus output sinusoid



Analyzing Models Using ODEs: Stability

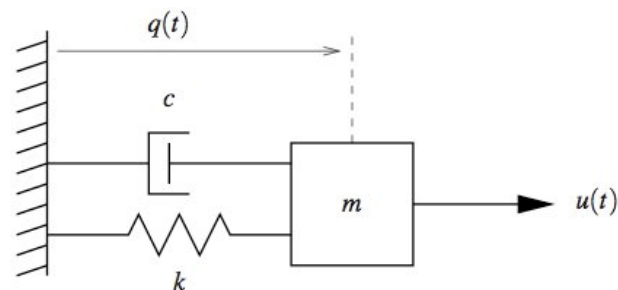
ODEs can also be used to prove stability of a systems

- Try to reason about the long term behavior of *all* solutions
- Stability $\approx$ all solutions return to equilibrium point (more precise defn later)

$$m\ddot{q} + c\dot{q} + kq = 0$$

Example: spring mass system

- Can we show that all solutions return to rest w/out explicitly solving ODE?
- Idea: look at how energy evolves in time



- Start by writing equations in state space form
 - Compute energy and its derivative
- $$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix} \quad \begin{matrix} x_1 = q \\ x_2 = \dot{q} \end{matrix}$$

$$\begin{aligned} V(x) &= \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 & \frac{dV}{dt} &= kx_1\dot{x}_1 + mx_2\dot{x}_2 \\ & & &= kx_1x_2 + mx_2\left(-\frac{c}{m}x_2 - \frac{k}{m}x_1\right) = -cx_2^2 \end{aligned}$$

- Energy is positive $\Rightarrow x_2$ must eventually go to zero
- If x_2 goes to zero, can show that x_1 must also approach zero (Lasalle, W3)

Modeling from Experiments

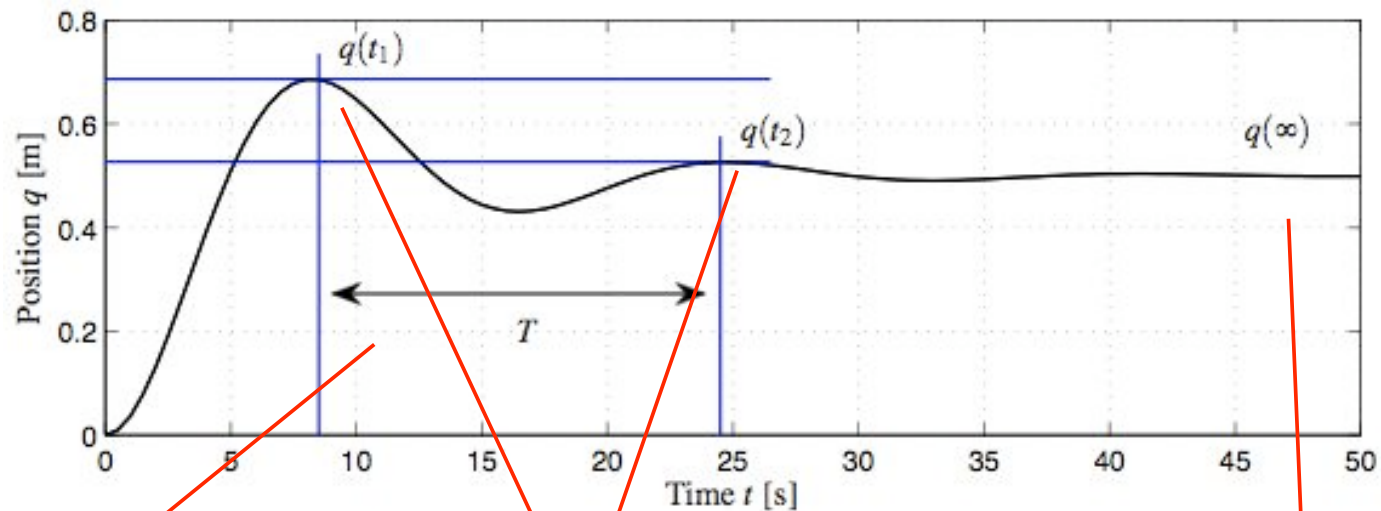
Example: spring mass system

- Measure response of system to a step input

$$q(t) = \frac{F_0}{k} \left(1 - \exp\left(-\frac{ct}{2m}\right) \sin(\omega_d t + \phi) \right)$$

$$\omega_d = \frac{\sqrt{4km - c^2}}{2m}$$

$$\phi = \tan^{-1} \left(\sqrt{4km - c^2} \right).$$



$$\frac{2\pi}{T} = \frac{\sqrt{4km - c^2}}{2m}$$

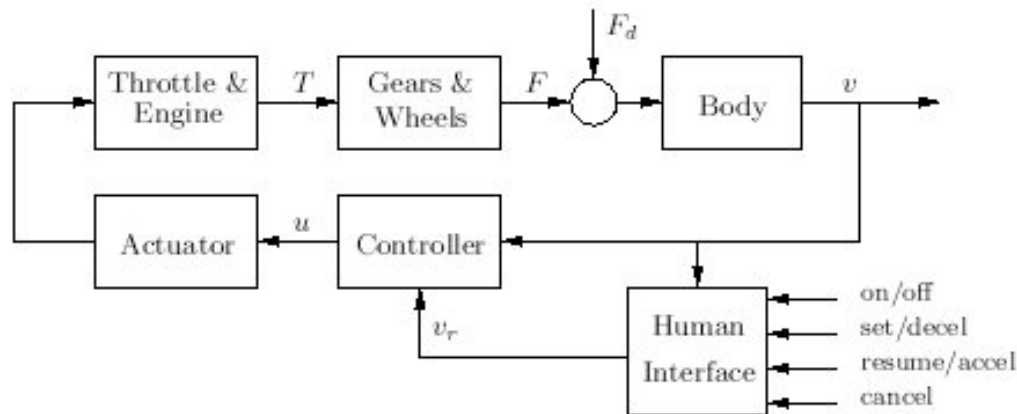
$$\log(q(t_1) - F_0/k) -$$

$$\log(q(t_2) - F_0/k) = \frac{c}{2m}(t_2 - t_1)$$

$$q(\infty) = \frac{F_0}{k}$$

Block Diagrams

Block diagrams separate components of a system into manageable units



Example: cruise control

- Each block corresponds to a portion of the overall dynamics
- Write out the individual blocks as input/output systems

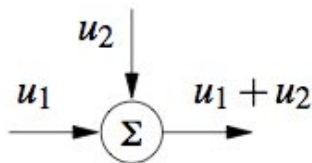
Body

- Dynamics: $m \frac{dv}{dt} = F - F_d$
- State: v - velocity of vehicle
- Inputs: F, F_d - force from wheels, external disturbances (wind, hills, etc)
- Output: v - velocity of vehicle

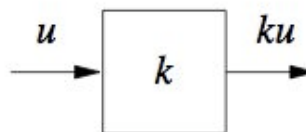
Dynamic versus state blocks

- Some blocks represent static relationships (no states); eg, gears and wheels

Standard Block Diagram Notation



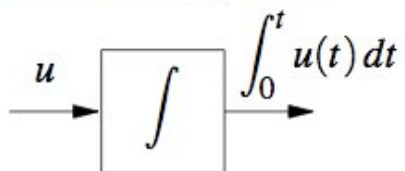
(a) Summing junction



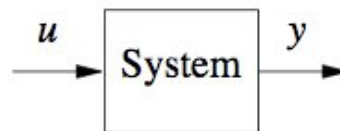
(b) Gain block



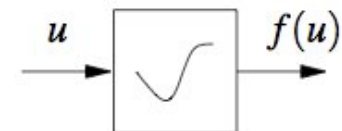
(c) Saturation



(d) Integrator



(e) Input/output system

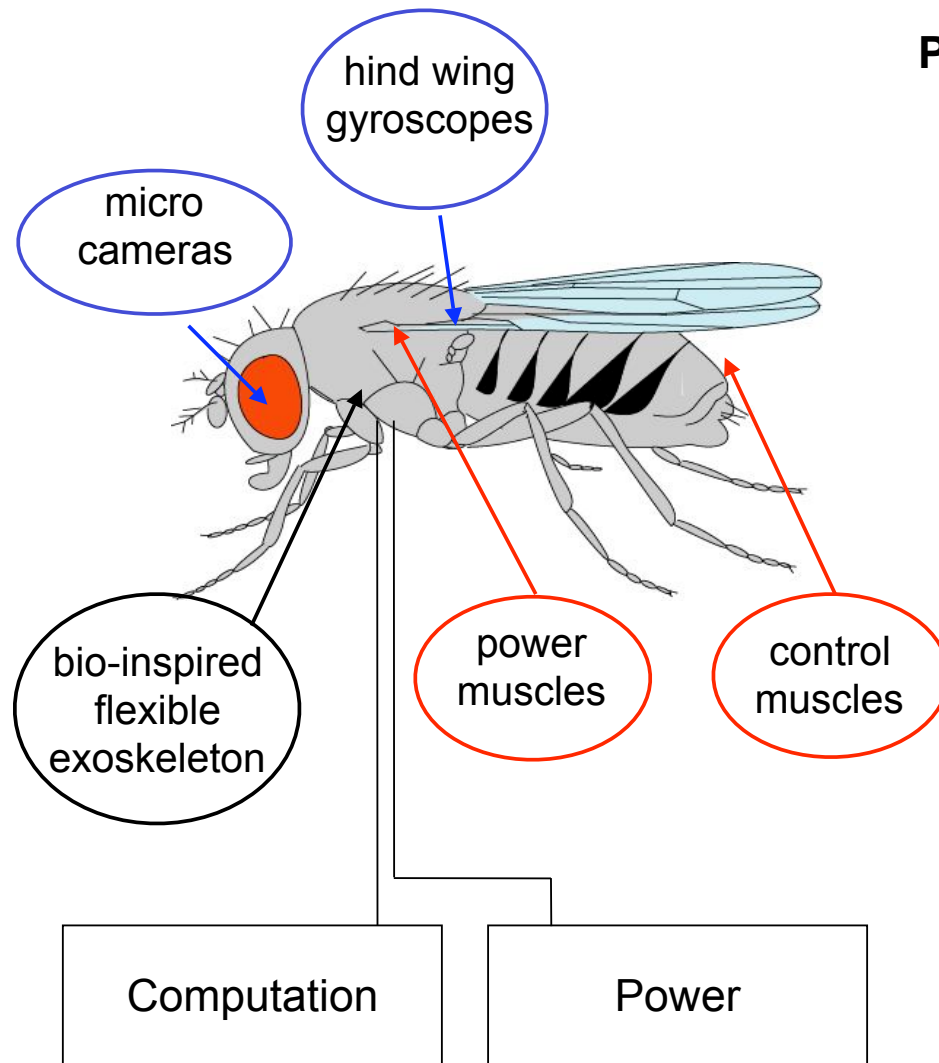


(f) Nonlinear map

Remarks

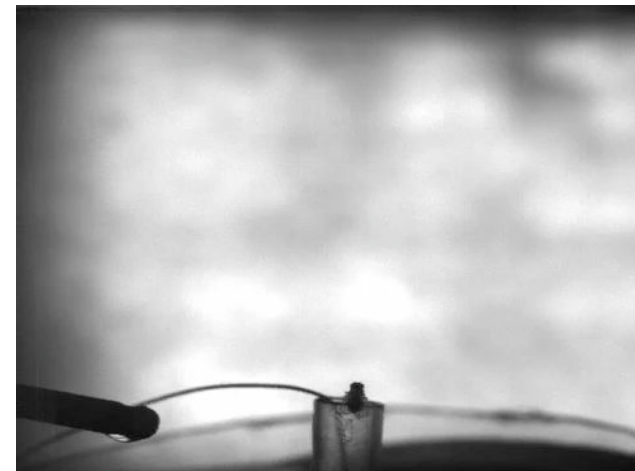
- SIMULINK uses slightly different symbols in a few places (eg, gain block)

Example: Hovering Mesoscale Robot (HOMER)



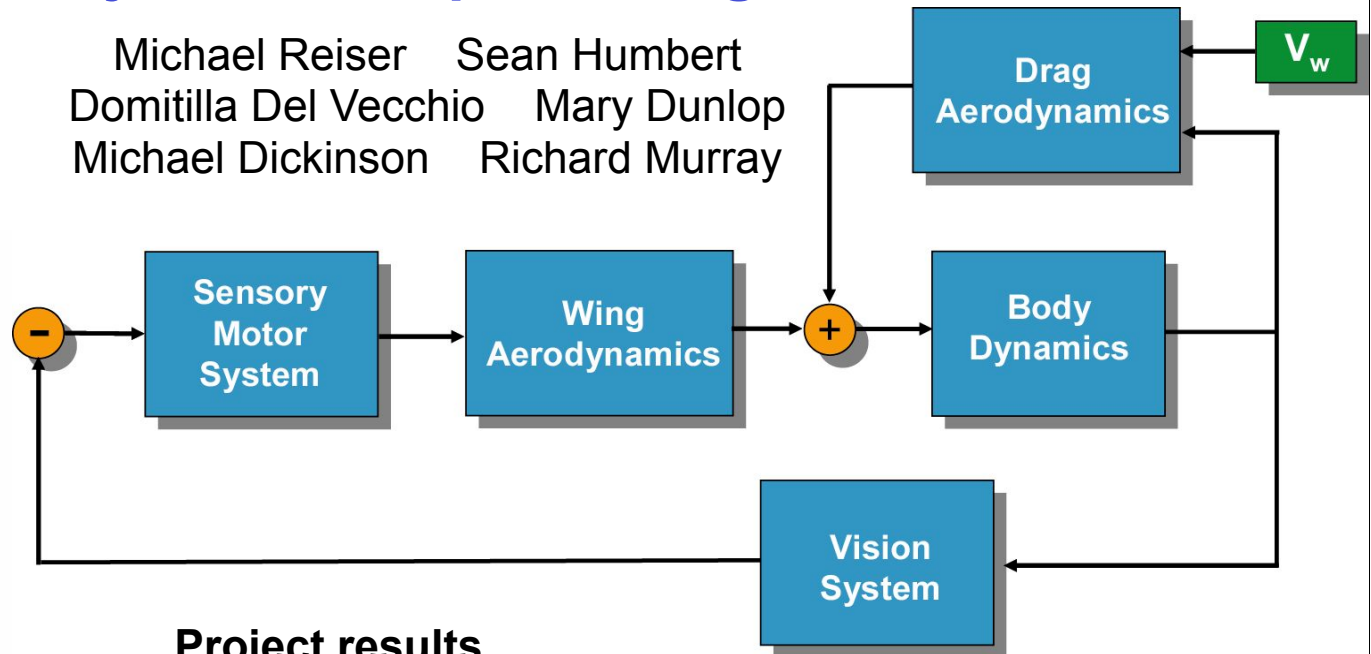
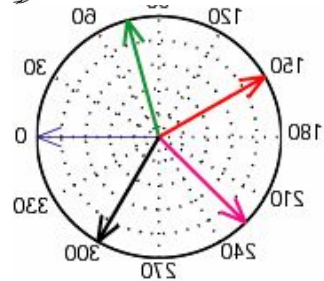
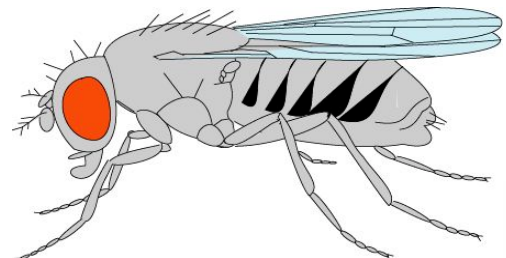
Project Goals

- Characterize and reverse engineer the sensory-motor control system of the fly
- Apply salient features to the design of micro air vehicles and other autonomous systems
- Experimentation and modeling key components of flight control system: (1) take-off, (2) robustness to wing gust, (3) chemical tracking, and (4) sensory fusion (visual, gyro)



Vision as a Compensatory Mechanism for Disturbance Rejection in Upwind Flight

Michael Reiser Sean Humbert
Domitilla Del Vecchio Mary Dunlop
Michael Dickinson Richard Murray



Project results

- Interconnected simplified models that provide bio-realistic behavior for upwind flight

Insights

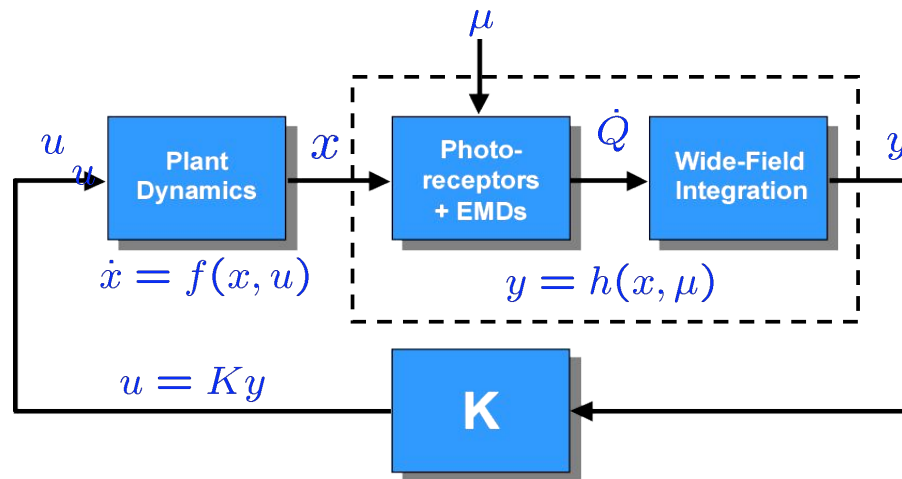
- Low level (fast!) vision and sensory motor processing capable of generated complex behaviors that achieve desired response

Vision-Based Navigation Using Wide-Field Integration

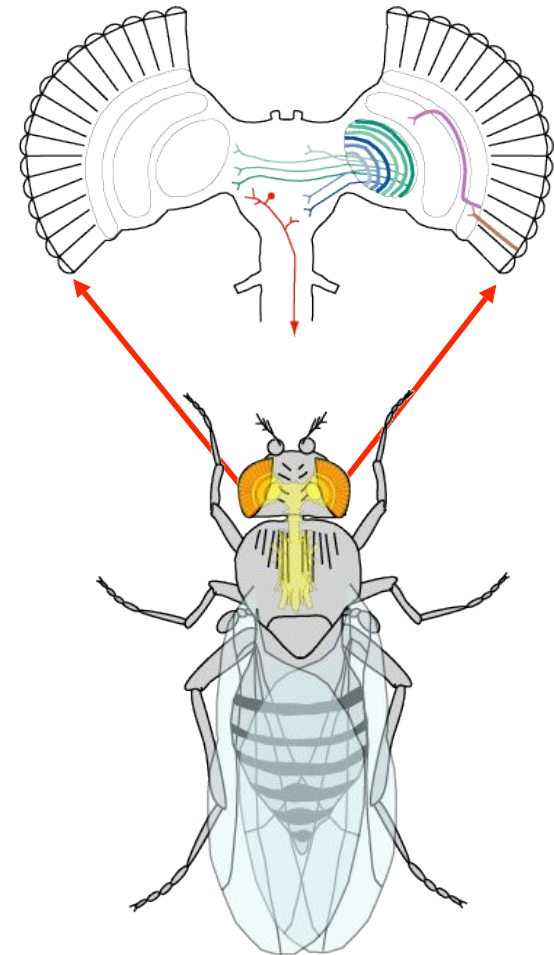
Sean Humbert (U. Maryland)

- **Approach**

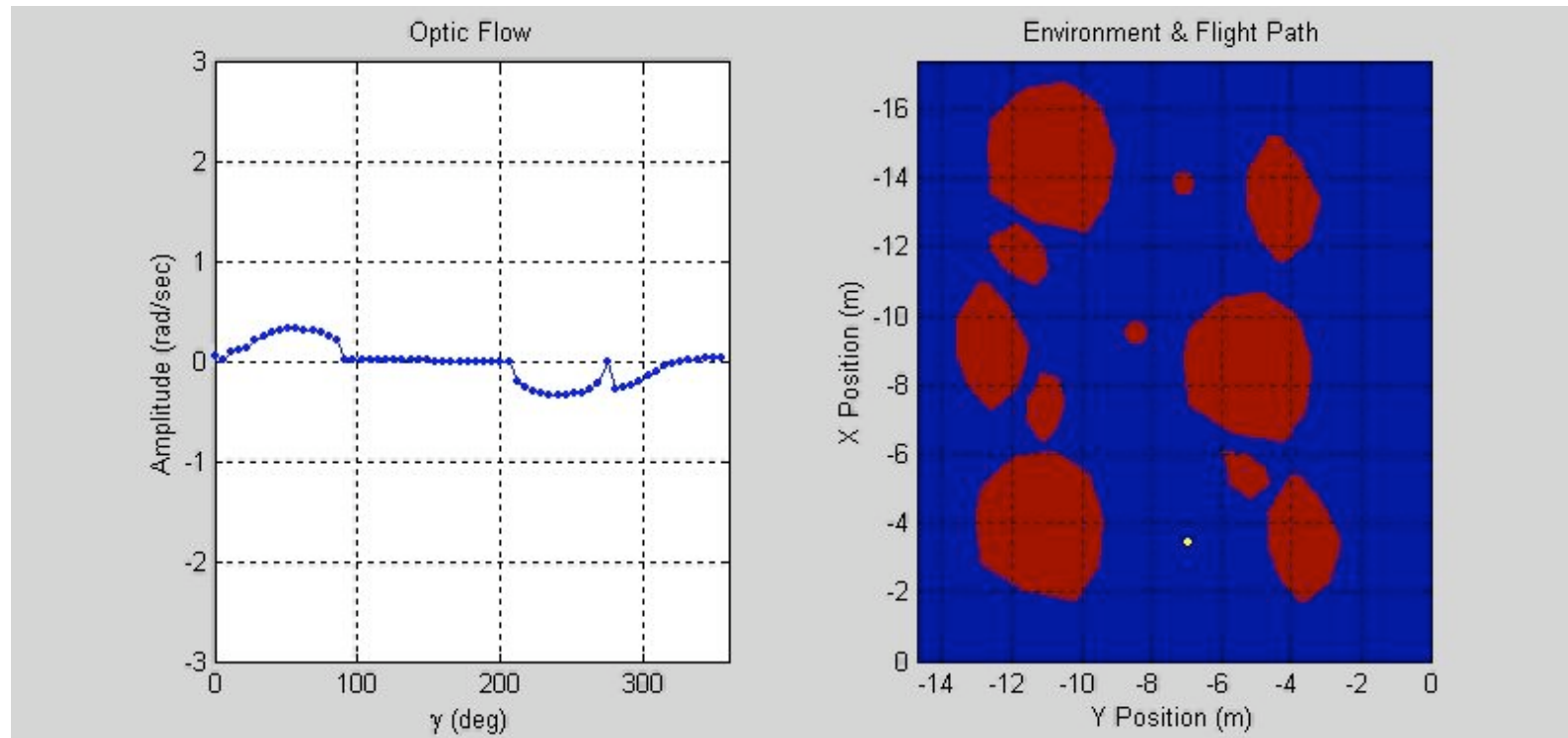
- Understand & characterize wide field integration processing in *Drosophila*
- Near 360° optical flow processing
- Very fast coupling to flight actuation



Flight Stabilization and Obstacle Avoidance

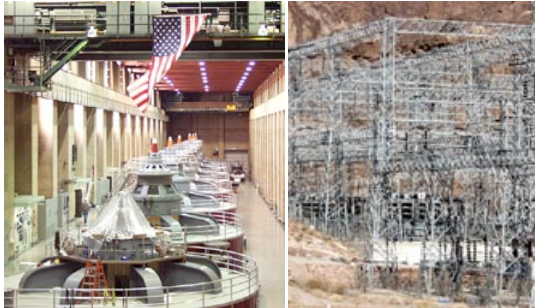


Engineering Applications in Vision-Based Navigation



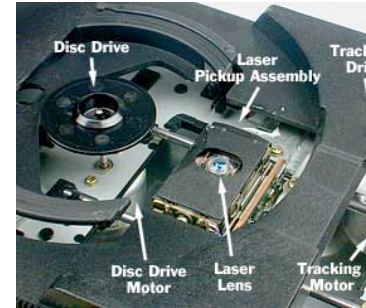
Summary: System Modeling

Model = state, inputs, outputs, dynamics



$$\frac{dx}{dt} = f(x, u)$$

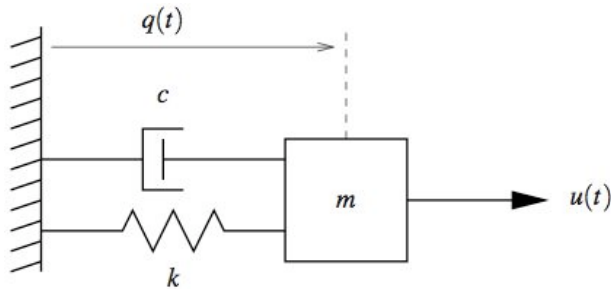
$$y = h(x)$$



$$x_{k+1} = f(x_k, u_k)$$

$$y_{k+1} = h(x_{k+1})$$

Models can be used to predict how a system will behave



$$q(t) = e^{-\zeta\omega_0 t} \left(q_0 \cos \omega_d t + \left(\frac{\zeta\omega_0}{\omega_d} q_0 + \frac{1}{\omega_d} v_0 \right) \sin \omega_d t \right)$$

$$q(t) = \frac{F}{m\omega_0^2} \left(1 - e^{-\zeta\omega_0 t} \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin \omega_d t \right)$$

