CDS 101: Lecture 9.1
Limits of Performance

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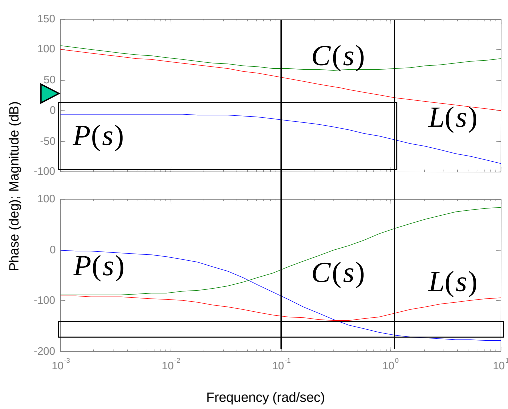
Goals:
- Describe limits of performance on feedback systems
- Introduce Bode's integral formula and the "waterbed" effect
- Show some of the limitations of feedback due to RHP poles and zeros

Reading:
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 9

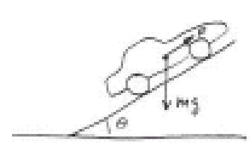
Review from Last Week

Loop Shaping for Stability & Performance
- Steady state error, bandwidth, tracking

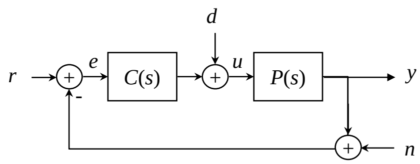
$$H_{ue}(s) = K_p + K_I \cdot \frac{1}{s} + K_D s$$



Main ideas
- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, PI, PID



"Gang of Four"



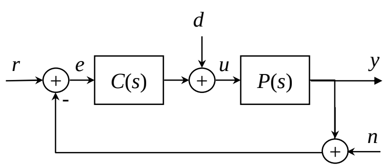
$$\begin{bmatrix} e \\ u \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC} \\ \frac{C}{1+PC} & \frac{PC}{1+PC} & \frac{C}{1+PC} \\ \frac{PC}{1+PC} & \frac{P}{1+PC} & \frac{PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Noise and disturbances
- d = process disturbances
- n = sensor noise
- Keep track of transfer functions between all possible inputs and outputs

Design represents a tradeoff between the quantities
- Keep L=PC large for good performance (H_{er} << 1)
- Keep L=PC small for good noise rejection (H_{yn} << 1)

Four unique transfer functions define performance
- Stability is always determined by 1/(1+PC)
- Numerator determined by forward path between input and output

Algebraic Constraints on Performance

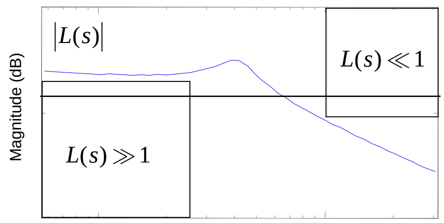


$$H_{er} = \frac{1}{1+PC} =: S$$ Sensitivity function

$$H_{yn} = \frac{PC}{1+PC} =: T$$ Complementary sensitivity function

Goal: keep S & T small
- S small $\Rightarrow$ low tracking error
- T small $\Rightarrow$ good noise rejection (and robustness [CDS 110b])

Problem: S + T = 1
- Can't make *both* S & T small at the same frequency
- Solution: keep S small at low frequency and T small at high frequency
- Loop again interpretation: keep L large at low frequency, and small at high frequency



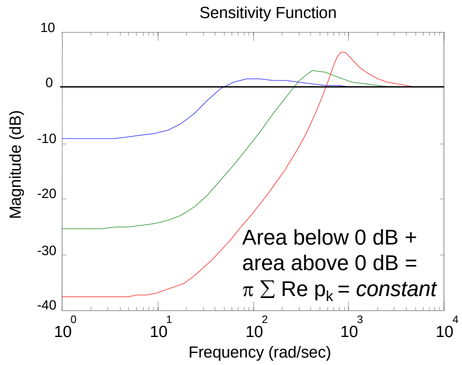
- Transition between large gain and small gain complicated by stability (phase margin)

Bode's Integral Formula and the Waterbed Effect

Bode's integral formula for S = 1/(1+PC) = 1/(1+L):

- Let p_k be the *unstable* poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
- **Theorem**: the area under the sensitivity function is a conserved quantity:

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re}\, p_k$$



Waterbed effect:

- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in ω; Bode plots are logarithmic

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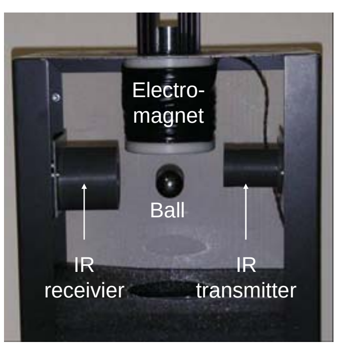
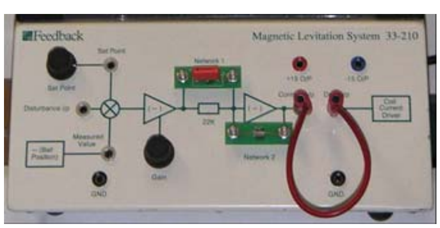
Example: Magnetic Levitation



System description

- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States: z, $\dot{z}$
- Dynamics: *F* = *ma*, *F* = magnetic force generated by wire coil
- See MATLAB handout for details



Controller circuit

- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current

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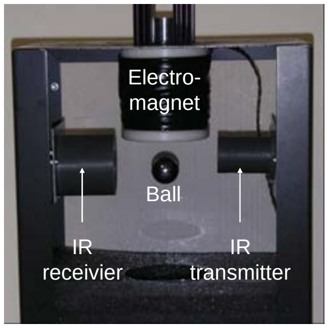
Equations of Motion



Process: actuation, sensing, dynamics

$$m\ddot{z} = k_m(k_A u)^2/z^2 - mg$$

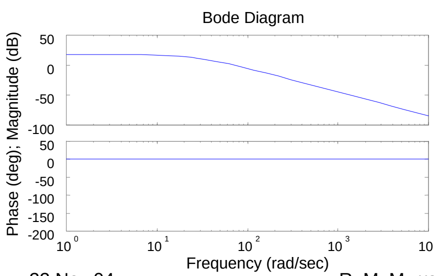
$$v_{ir} = k_T z + v_0$$

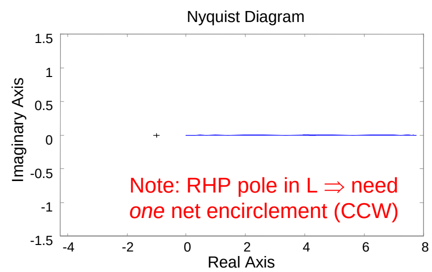
- u = current to electromagnet
- v_{ir} = voltage from IR sensor

Linearization:

$$P(s) = \frac{k}{s^2 - r^2} \qquad k, r > 0$$

- Poles at $s = \pm r \Rightarrow$ open loop unstable





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Control Design

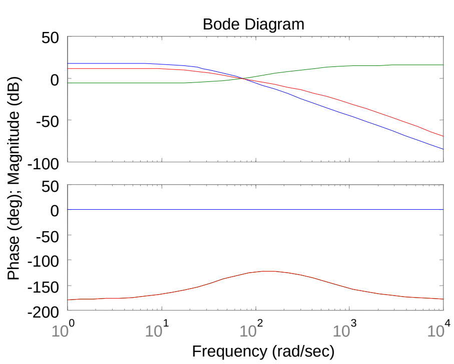
Need to *create* encirclement

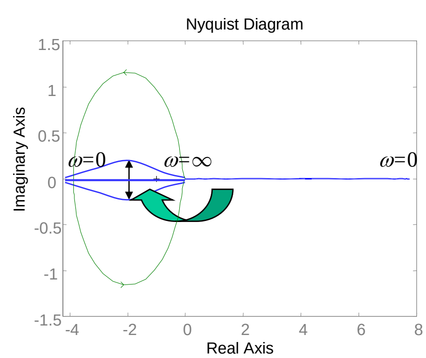
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator

- Produce phase lead at crossover
- Generates loop in Nyquist plot

$$C(s) = -k\frac{s+a}{s+b}$$





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Performance Limits

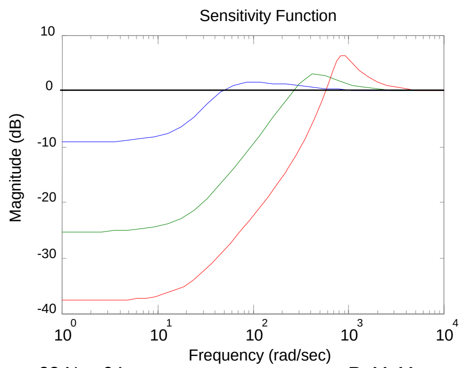
Nominal design gives low perf

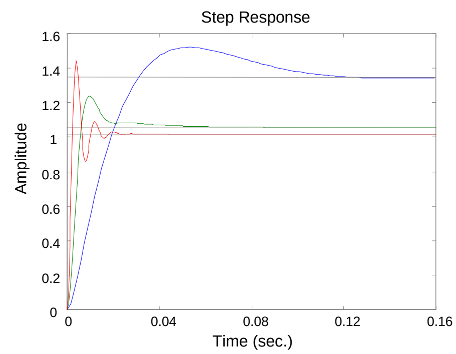
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi r$$

- Must increase sensitivity at some point





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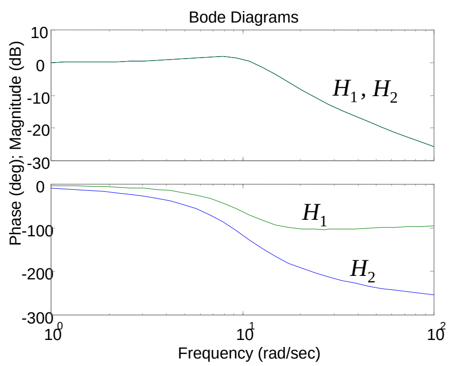
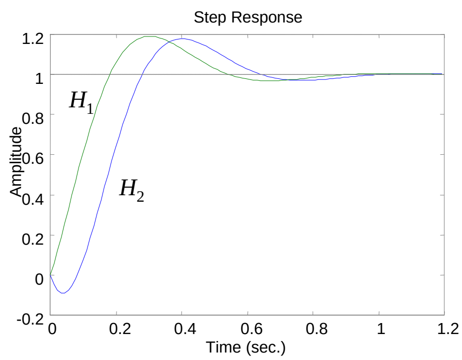
Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior

- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move *opposite* from input for a short period of time

Example: $H_1(s) = \dfrac{s+a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ **vs** $H_2(s) = \dfrac{s-a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$





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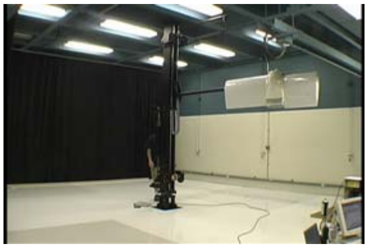
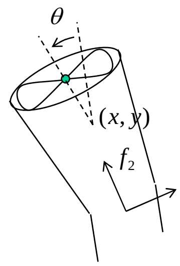
Example: Lateral Control of the Ducted Fan



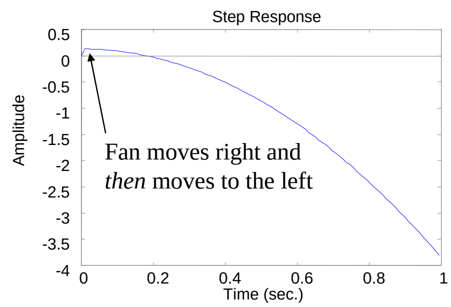


$$H_{xf_1}(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: 0, 0, $-\sigma \pm j\,\omega_d$
- Zeros: $\pm\sqrt{mgl}$

Source of non-minimum phase behavior

- To move left, need to make $\theta > 0$
- To generate positive θ, need $f_1 > 0$
- Positive f_1 causes fan to move *right* initially
- Fan starts to move left after short time (as fan rotates)



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Stability in the Presence of Zeros

Loop gain limitations

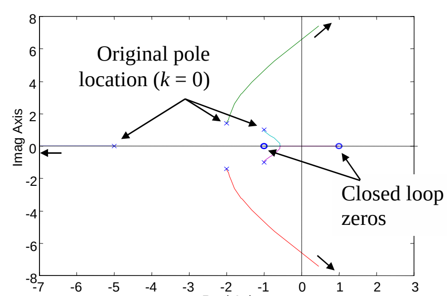
- Poles of closed loop = poles of 1 + L. Suppose $C = k\, n_c/d_c$, where k is the gain of the controller

$$1 + L = 1 + k\frac{n_c n_p}{d_c d_p} = \frac{d_c d_p + k n_c n_p}{d_c d_p}$$

- For large k, closed loop poles approach open loop zeros
- RHP zeros limit maximum gain ⇒ serious design constraint!

Root locus interpretation

- Plot location of eigenvalues as a function of the loop gain k
- Can show that closed loop poles go from open loop poles (k = 0) to open loop zeros (k = \infty)



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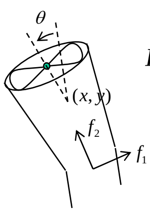
Additional performance limits due to RHP zeros

Another waterbed-like effect: look at maximum of H_{er} over frequency range:

$$M_1 = \max_{\omega_1 \le \omega \le \omega_2} |H_{er}(j\omega)| \qquad M_2 = \max_{0 \le \omega \le \infty} |H_{er}(j\omega)|$$

Thm: Suppose that *P* has a RHP zero at *z*. Then there exist constants c_1 and c_2 (depending on ω_1, ω_2, *z*) such that $c_1 \log M_1 + c_2 \log M_2 \ge 0$.

- M_1 typically << 1 $\Rightarrow$ M_2 must be larger than 1 (since sum is positive)
- If we increase performance in active range (make M_1 and H_{er} smaller), we must lose performance (H_{er} increases) some place else
- Note that this affects *peaks* not integrals (different from RHP poles)



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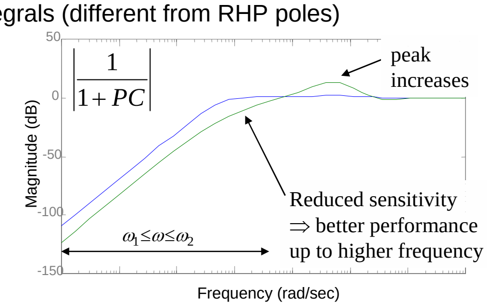

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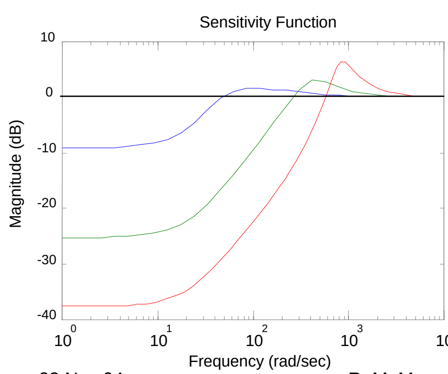
Summary: Limits of Performance

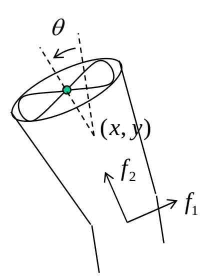
Many limits to performance

- Algebraic: S + T = 1
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of S

Main message: try to avoid RHP poles and zeros when-ever possible (eg, re-design)

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \mathrm{Re}\, p_k$$





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