CDS 101: Lecture 9.1
Limits of Performance

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Goals:
- Describe limits of performance on feedback systems
- Introduce Bode’s integral formula and the "waterbed" effect
- Show some of the limitations of feedback due to RHP poles and zeros

Reading:
- Åström and Murray, Analysis and Design of Feedback Systems, Ch 9

Review from Last Week
Loop Shaping for Stability & Performance
- Steady state error, bandwidth, tracking

\[ H_m(s) = K_p + K_f \cdot \frac{1}{s} + K_d s \]

Main ideas
- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, PI, PID
"Gang of Four"

Noise and disturbances
- \( d \) = process disturbances
- \( n \) = sensor noise
- Keep track of transfer functions between all possible inputs and outputs

Design represents a tradeoff between the quantities
- Keep \( L=PC \) large for good performance \((H_{tr} << 1)\)
- Keep \( L=PC \) small for good noise rejection \((H_{yn} << 1)\)

Four unique transfer functions define performance
- Stability is always determined by \( 1/(1+PC) \)
- Numerator determined by forward path between input and output

Algebraic Constraints on Performance

Goal: keep \( S \) & \( T \) small
- \( S \) small \(\Rightarrow\) low tracking error
- \( T \) small \(\Rightarrow\) good noise rejection (and robustness [CDS 110b])

Problem: \( S + T = 1 \)
- Can’t make both \( S \) & \( T \) small at the same frequency
- Solution: keep \( S \) small at low frequency and \( T \) small at high frequency
- Loop again interpretation: keep \( L \) large at low frequency, and small at high frequency

Sensitivity function
\[ H_{tr} = \frac{1}{1 + PC} =: S \]

Complementary sensitivity function
\[ H_{yn} = \frac{PC}{1 + PC} =: T \]
Bode’s Integral Formula and the Waterbed Effect

Bode’s integral formula for $S = 1/(1+PC) = 1/(1+L)$:
- Let $p_k$ be the unstable poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
- **Theorem**: the area under the sensitivity function is a conserved quantity:

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re } p_k$$

**Waterbed effect**:
- Making sensitivity smaller over some frequency range requires increase in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in $\omega$; Bode plots are logarithmic

Example: Magnetic Levitation

**System description**
- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States: $z, \dot{z}$
- Dynamics: $F = ma, F = \text{magnetic force generated by wire coil}$
- See MATLAB handout for details

**Controller circuit**
- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current
Equations of Motion

Process: actuation, sensing, dynamics
\[ m\ddot{z} = k_m (k_A u)^2 / z^2 - m_0 \]
\[ v_{ir} = k_T \dot{z} + v_0 \]
- \( u \) = current to electromagnet
- \( v_{ir} \) = voltage from IR sensor

Linearization:
\[ \frac{k}{s^2 - r^2} \]
\( k, r > 0 \)
- Poles at \( s = \pm r \Rightarrow \) open loop unstable

Control Design

Need to create encirclement
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator
- Produce phase lead at crossover
- Generates loop in Nyquist plot
\[ C(s) = \frac{s + a}{s + b} \]

Note: RHP pole in \( L \Rightarrow \) need one net encirclement (CCW)
Performance Limits

Nominal design gives low perf
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement
\[ \int_0^\infty \log |S(j\omega)| \, d\omega = \pi \tau \]
- Must increase sensitivity at some point

Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior
- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move opposite from input for a short period of time

Example:
\[ H_1(s) = \frac{s + a}{s^2 + 2\zeta \omega_n s + \omega_n^2} \quad \text{vs} \quad H_2(s) = \frac{s - a}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
Example: Lateral Control of the Ducted Fan

Source of non-minimum phase behavior
- To move left, need to make $\theta > 0$
- To generate positive $\theta$, need $f_1 > 0$
- Positive $f_1$ causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)

$H_w(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$
- Poles: 0, 0, $-\sigma \pm j \omega_d$
- Zeros: $\pm \sqrt{mgl}$

Stability in the Presence of Zeros

Loop gain limitations
- Poles of closed loop = poles of $1 + L$. Suppose $C = k \frac{n_c}{d_c}$, where $k$ is the gain of the controller
  \[ 1 + L = 1 + k \frac{n_cnp}{dcdp} = \frac{dcdp + kn_cnp}{dcdp} \]
- For large $k$, closed loop poles approach open loop zeros
- RHP zeros limit maximum gain $\Rightarrow$ serious design constraint!

Root locus interpretation
- Plot location of eigenvalues as a function of the loop gain $k$
- Can show that closed loop poles go from open loop poles ($k = 0$) to open loop zeros ($k = \infty$)
**Additional performance limits due to RHP zeros**

**Another waterbed-like effect:** look at maximum of $H_s$ over frequency range:

$$M_1 = \max_{\omega_1 \leq \omega \leq \omega_2} |H_s(j\omega)| \quad M_2 = \max_{\omega_1 \leq \omega \leq \omega_2} |H_{sr}(j\omega)|$$

**Thm:** Suppose that $P$ has a RHP zero at $z$. Then there exist constants $c_1$ and $c_2$ (depending on $\omega_1$, $\omega_2$, $z$) such that $c_1 \log M_1 + c_2 \log M_2 \geq 0$.

- $M_1$ typically $<< 1 \Rightarrow M_2$ must be larger than 1 (since sum is positive)
- If we increase performance in active range (make $M_1$ and $H_{sr}$ smaller), we must lose performance ($H_s$ increases) some place else
- Note that this affects peaks not integrals (different from RHP poles)

**Summary: Limits of Performance**

- Many limits to performance
  - Algebraic: $S + T = 1$
  - RHP poles: Bode integral formula
  - RHP zeros: Waterbed effect on peak of $S$

- Main message: try to avoid RHP poles and zeros whenever possible (eg, re-design)