CDS 101: Lecture 8.1
Frequency Domain Design using PID

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Goals:
- Describe the use of frequency domain performance specifications
- Show how to use “loop shaping” using PID to achieve a performance specification

Reading:
- Åström and Murray, Analysis and Design of Feedback Systems, 7.6ff and Ch 8

Review from Last Week

Thm (Nyquist).
\[ P \] # RHP poles of \( L(s) \)
\[ N \] # CW encirclements
\[ Z \] # RHP zeros
\[ Z = N + P \]
**Frequency Domain Performance Specifications**

Specify bounds on the loop transfer function to guarantee desired performance.

\[ L(s) = P(s)C(s) \]

\[ H_{\omega} = \frac{1}{1 + L} \quad H_{fr} = \frac{L}{1 + L} \]

- Steady state error:
  \[ H_{\omega}(0) = 1/(1 + L(0)) \approx 1/L(0) \]
  \[ \Rightarrow \text{zero frequency ("DC") gain} \]
- Bandwidth: assuming \( \sim 90^\circ \) phase margin
  \[ \frac{L}{1 + L}(j\omega_c) \approx \frac{1}{1 + j} = \frac{1}{\sqrt{2}} \]
  \[ \Rightarrow \text{sets crossover freq} \]
- Tracking: \( \times \% \) error up to frequency \( \omega_t \)
  \[ \Rightarrow \text{determines gain bound} (1 + PC > 100/X) \]

**Relative Stability**

Relative stability: how stable is system to disturbances at certain frequencies?

- System can be stable but still have bad response at certain frequencies
- Typically occurs if system has low phase margin \( \Rightarrow \) get resonant peak in closed loop \( (M_r) + \) poor step response
- Solution: specify minimum phase margin. Typically 45° or more

\[ H_{fr} = \frac{L}{1 + L} \]
Overview of Loop Shaping

Performance specification
- Steady state error
- Tracking error
- Bandwidth
- Relative stability

Approach: “shape” loop transfer function using \( C(s) \)
- \( P(s) + \) specifications given
- \( L(s) = P(s) C(s) \)
  - Use \( C(s) \) to choose desired shape for \( L(s) \)
- Important: can’t set gain and phase independently

Gain/phase relationships

Gain and phase for transfer function w/ real coeffs are not independent
- Given a given shape for the gain, there is a unique “minimum phase” transfer function that achieves that gain at the specified frequencies
- Basic idea: slope of the gain determines the phase
- Implication: you have to tradeoff gain versus phase in control design
Overview: PID control

\[ u = K_p e + K_i \int e + K_d \dot{e} \]

Intuition
- Proportional term: provides inputs that correct for “current” errors
- Integral term: insures steady state error goes to zero
- Derivative term: provides “anticipation” of upcoming changes

A bit of history on “three term control”
- First appeared in 1922 paper by Minorsky: “Directional stability of automatically steered bodies” under the name “three term control”
- Also realized that “small deviations” (linearization) could be used to understand the (nonlinear) system dynamics under control

Utility of PID
- PID control is most common feedback structure in engineering systems
- For many systems, only need PI or PD (special case)
- Many tools for tuning PID loops and designing gains (see reading)

Proportional Feedback

Simplest controller choice: \( u = K_p e \)
- Effect: lifts gain with no change in phase
- Good for plants with low phase up to desired bandwidth
- Bode: shift gain up by factor of \( K_p \)
- Nyquist: scale Nyquist contour

\[ K_p > 0 \]
Proportional + Integral Compensation

Use to eliminate steady state error
- Effect: lifts gain at low frequency
- Gives zero steady state error
- Bode: infinite SS gain + phase lag
- Nyquist: no easy interpretation
- Note: this example is unstable

\[ E(s) = K_p e + K_i \int e + K_d \frac{de}{dt} \]

Proportional + Integral + Derivative (PID)

Transfer function for PID controller
\[ u = K_p e + K_i \int e + K_d \frac{de}{dt} \]
\[ H_w(s) = K_p + K_i \frac{1}{s} + K_d s \]

- Idea: gives high gain at low frequency plus phase lead at high frequency
- Place \( \omega_1 \) and \( \omega_2 \) below desired crossover freq
Example: Cruise Control using PID - Specification

**Performance Specification**
- $\leq 1\%$ steady state error
  - Zero frequency gain $> 100$
- $\leq 10\%$ tracking error up to 10 rad/sec
  - Gain $> 10$ from 0-10 rad/sec
- $\geq 45^\circ$ phase margin
  - Gives good relative stability
  - Provides robustness to uncertainty

**Observations**
- Purely proportional gain won’t work: to get gain above desired level will not leave adequate phase margin
- Need to increase the phase from $\sim 0.5$ to 2 rad/sec and increase gain as well

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Example: Cruise Control using PID - Design

**Approach**
- Use integral gain to make steady state error small (zero, in fact)
- Use derivative action to increase phase lead in the crossover region
- Use proportional gain to give desired bandwidth

**Controller**

$$C(s) = \frac{2000s^2 + 1.1s + 0.1}{s} = 2200 + \frac{200}{s} + 2000s$$

**Closed loop system**
- Very high steady state gain
- Adequate tracking @ 1 rad/sec
- $\sim 80^\circ$ phase margin
Example: Cruise Control using PID - Verification

\[ P(s) = \frac{1/m}{s + b/m} \frac{r}{s + a} \]
\[ C(s) = 2000 \frac{s^2 + 1.1s + 0.1}{s} \]

Summary: Frequency Domain Design using PID

Loop Shaping for Stability & Performance
- Steady state error, bandwidth, tracking
  \[ H_{\text{ss}}(s) = K_p + K_i \cdot \frac{1}{s} + K_d s \]

Main ideas
- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, PI, PID