


CDS 101: Lecture 5.1

Reachability and State Space Feedback



Richard M. Murray and Hideo Mabuchi
 25 October 2004

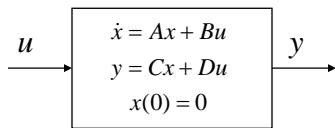
Goals:

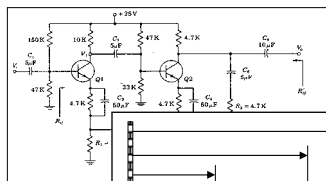
- Define reachability of a control system
- Give tests for reachability of linear systems and apply to examples
- Describe the design of state feedback controllers for linear systems

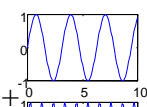
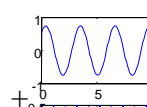
Reading:

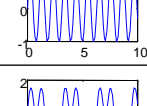
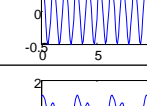
- Åström and Murray, *Analysis and Design of Feedback Systems*, Ch 5

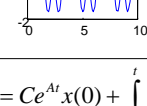
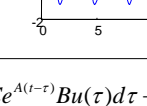
Review from Last Week





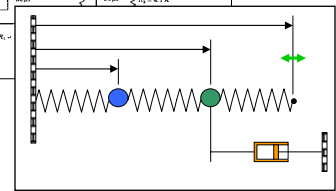



$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Properties of linear systems

- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems



Control Design Concepts

System description: single input, single output system (MIMO also OK)

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, x(0) \text{ given}$$

$$y = h(x, u) \quad u \in \mathbb{R}, y \in \mathbb{R}$$

Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" $u = \alpha(x)$ such that

$$\lim_{t \rightarrow \infty} x(t) = x_e \text{ for all } x(0) \in \mathbb{R}^n$$

Reachability: steer the system between two points

- Given $x_0, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that

$$\dot{x} = f(x, u(t)) \text{ takes } x(t_0) = x_0 \rightarrow x(T) = x_f$$

Tracking: track a given output trajectory

- Given $y_d(t)$, find $u = \alpha(x, t)$ such that

$$\lim_{t \rightarrow \infty} (y(t) - y_d(t)) = 0 \text{ for all } x(0) \in \mathbb{R}^n$$

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3

Reachability of Input/Output Systems

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, x(0) \text{ given}$$

$$y = h(x, u) \quad u \in \mathbb{R}, y \in \mathbb{R}$$

Defn An input/output system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ and any time $T > 0$ there exists an input $u: [0, T] \rightarrow \mathbb{R}$ such that the solution of the dynamics starting from $x(0) = x_0$ and applying input $u(t)$ gives $x(T) = x_f$.

Remarks

- In the definition, x_0 and x_f do not have to be equilibrium points \Rightarrow we don't necessarily stay at x_f after time T .
- Reachability is defined in terms of states \Rightarrow doesn't depend on output
- For *linear systems*, can characterize reachability by looking at the general solution:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad x(T) = e^{AT} x_0 + \int_{\tau=0}^T e^{A(T-\tau)} Bu(\tau) d\tau$$

If integral is "surjective" (as a linear operator), then we can find an input to achieve any desired final state.

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Tests for Reachability

$$\begin{aligned} \dot{x} &= Ax + Bu \quad x \in \mathbb{R}^n, x(0) \text{ given} \\ y &= Cx + Du \quad u \in \mathbb{R}, y \in \mathbb{R} \end{aligned} \quad x(T) = e^{AT}x_0 + \int_{\tau=0}^T e^{A(T-\tau)}Bu(\tau)d\tau$$

Thm A linear system is reachable if and only if the $n \times n$ reachability matrix

$$\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

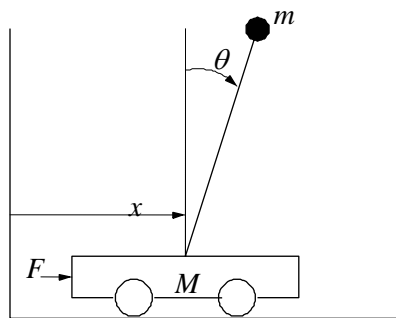
is full rank.

Remarks

- Very simple test to apply. In MATLAB, use `ctrb(A,B)` and check rank w/ `det()`
- If this test is satisfied, we say “the pair (A,B) is reachable”
- Some insight into the proof can be seen by expanding the matrix exponential

$$\begin{aligned} e^{A(T-\tau)}B &= \left(I + A(T-\tau) + \frac{1}{2}A^2(T-\tau)^2 + \dots + \frac{1}{(n-1)!}A^{n-1}(T-\tau)^{n-1} + \dots \right) B \\ &= B + AB(T-\tau) + \frac{1}{2}A^2B(T-\tau)^2 + \dots + \frac{1}{(n-1)!}A^{n-1}B(T-\tau)^{n-1} + \dots \end{aligned}$$

Example #1: Linearized pendulum on a cart



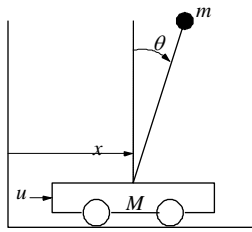
Question: can we locally control the position of the cart by proper choice of input?

Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2gl^2}{J(M+m) + Mml^2} & \frac{-(J + ml^2)b}{J(M+m) + Mml^2} & 0 \\ 0 & \frac{mgl(M+m)}{J(M+m) + Mml^2} & \frac{-mlb}{J(M+m) + Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J + ml^2}{J(M+m) + Mml^2} \\ \frac{ml}{J(M+m) + Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 g l^2}{q} & -\frac{(J + ml^2)b}{q} & 0 \\ 0 & \frac{mgl(M+m)}{q} & \frac{-mb}{q} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J + ml^2}{q} \\ \frac{ml}{q} \end{bmatrix} u$$

$q = J(M + m) + Mml^2$

• Simplify by setting $b = 0$

Reachability matrix

$$M_c = \begin{bmatrix} 0 & \frac{J + ml^2}{q} & 0 & \frac{ml(J + ml^2)}{q^2} \\ 0 & \frac{ml}{q} & 0 & \frac{m^2 g l^2 (M + m)}{q^2} \\ \frac{J + ml^2}{q} & 0 & \frac{ml(J + ml^2)}{q^2} & 0 \\ \frac{ml}{q} & 0 & \frac{m^2 g l^2 (M + m)}{q^2} & 0 \end{bmatrix}$$

$B \quad AB \quad A^2B \quad A^3B$

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- \Rightarrow reachable as long as $g(M+m) \neq 1$
- \Rightarrow can "steer" linearization between points by proper choice of input

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7

Control Design Concepts

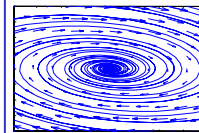
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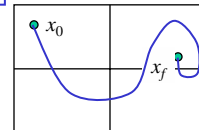
Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" $u = \alpha(x)$ such that $\lim_{t \rightarrow \infty} x(t) = x_e$ for all $x(0) \in \mathbb{R}^n$



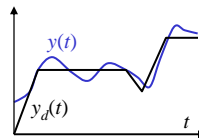
✓ Reachability: steer the system between two points

- Given $x_0, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that $\dot{x} = f(x, u(t))$ takes $x(t_0) = x_0 \rightarrow x(t_f) = x_f$



Tracking: track a given output trajectory

- Given $y_d(t)$, find $u = \alpha(x, t)$ such that $\lim_{t \rightarrow \infty} (y(t) - y_d(t)) = 0$ for all $x(0) \in \mathbb{R}^n$



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8

State space controller design for linear systems

$$\begin{aligned} \dot{x} &= Ax + Bu & x \in \mathbb{R}^n, x(0) \text{ given} & & x(T) &= e^{AT}x_0 + \int_{\tau=0}^T e^{A(T-\tau)}Bu(\tau)d\tau \\ y &= Cx + Du & u \in \mathbb{R}, y \in \mathbb{R} & & & \end{aligned}$$

Goal: find a linear control law $u=Kx$ such that the closed loop system

$$\dot{x} = Ax + BKx = (A + BK)x$$

is stable at $x_e=0$.

Remarks

- Stability based on eigenvalues \Rightarrow use K to make eigenvalues of $(A+BK)$ stable
- Can also link eigenvalues to *performance* (eg, initial condition response)
- Question: when can we place the eigenvalues anyplace that we want?

Theorem The eigenvalues of $(A+BK)$ can be set to arbitrary values if and only if the pair (A,B) is reachable.

MATLAB: $K = \text{place}(A, B, \text{eigs})$

Example #2: Predator prey

Natural dynamics

$$\begin{aligned} \dot{x}_1 &= b_r x_1 - a x_1 x_2 \\ \dot{x}_2 &= a x_1 x_2 - d_f x_2 \end{aligned}$$



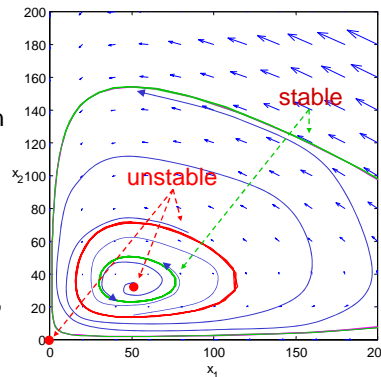
Controlled dynamics: modulate food supply

$$\begin{aligned} \dot{x}_1 &= b_r(1+u)x_1 - a x_1 x_2 \\ \dot{x}_2 &= a x_1 x_2 - d_f x_2 \end{aligned}$$

Q1: can we move from some initial population of foxes and rabbits to a specified one in time T by modulation if the food supply?

Q2: can we *stabilize* the population around the desired equilibrium point

Approach: try to answer this question *locally*, around the natural equilibrium point



Example #2: Problem setup

Equilibrium point calculation

$$\dot{x}_1 = b_r(1+u)x_1 - ax_1x_2$$

$$\dot{x}_2 = ax_1x_2 - d_f x_2$$

- $x_e = (50, 35)$

Linearization

- Compute linearization around equilibrium point, x_e :

$$A = \left. \frac{\partial f}{\partial x} \right|_{(x_e, u_e)} \quad B = \left. \frac{\partial f}{\partial u} \right|_{(x_e, u_e)}$$

- Redefine local variables: $z = x - x_e, v = u - u_e$

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -b_r - ax_{2,e} & -ax_{1,e} \\ ax_{2,e} & -d_f + ax_{1,e} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} b_r x_{1,e} \\ 0 \end{bmatrix} v$$

- Reachable? YES, if $b_r, a \neq 0$ (check $[B \ AB]$) \Rightarrow can locally steer to any point

```
% Compute the equil point
% predprey.m contains dynamics
f = inline('predprey(0,x)');
xeq = fsolve(f, [50,50]);

% Compute linearization
A = [
    br - a*xeq(2) - a*xeq(1);
    a*xeq(2), -df + a*xeq(1)
];
B = [br*xeq(1); 0];
```

Example #2: Stabilization via eigenvalue assignment

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -b_r - ax_{2,e} & -ax_{1,e} \\ ax_{2,e} & -d_f + ax_{1,e} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} b_r x_{1,e} \\ 0 \end{bmatrix} v$$

Control design:

$$v = -Kz = -K(x - x_e)$$

$$u = u_e + v = u_e - K(x - x_e)$$

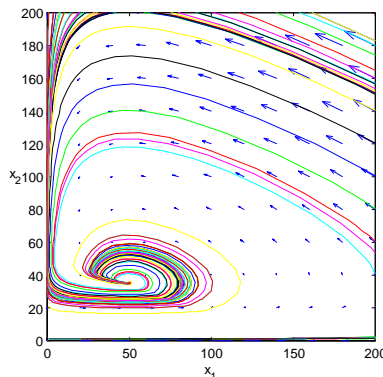
Place poles at stable values

- Choose $\lambda = -1, -2$
- $K = \text{place}(A, B, [-1; -2]);$

Modify dynamics to include control

$$\dot{x}_1 = b_r(1 - K(x - x_e))x_1 - ax_1x_2$$

$$\dot{x}_2 = ax_1x_2 - d_f x_2$$



Implementation Details

Eigenvalues determine performance

- For each eigenvalue $\lambda_i = \sigma_i + j\omega_i$, get contribution of the form

$$y_i(t) = e^{-\sigma t} (a \sin(\omega t) + b \cos(\omega t))$$
- Repeated eigenvalues can give additional terms of the form $t^k e^{\sigma + j\omega}$

Pole-zero map

Step Response
From: 0(1)

Use estimator to determine the current state if you can't measure it

- Estimator looks at inputs and outputs of plant and estimates the current state
- Can show that if a system is *observable* then you can construct an estimator
- Use the *estimated* state as the feedback

$$u = K\hat{x}$$
- Kalman* filter is an example of an estimator

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Summary: Reachability and State Space Feedback

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

$$u = u_e - K(x - x_e)$$

Key concepts

- Reachability: find u s.t. $x_0 \rightarrow x_f$
- Reachability rank test for linear systems
- State feedback to assign eigenvalues

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14