

CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101/110
Homework Set #4

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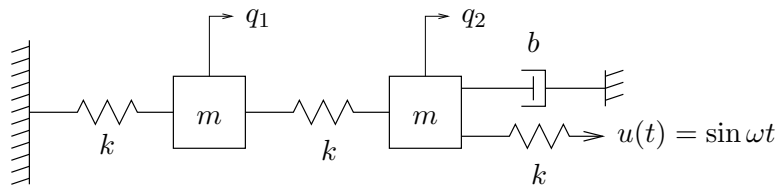
Note: In the upper left hand corner of the first page of your homework set, please put the class you are taking (CDS 101, CDS 110) and the number of hours that you spent on this homework set (including reading).

The following exercise should be completed only by students taking CDS 101:

0. The queue dynamics of a router can be modelled as $\dot{q}(t) = (y(t) - c)/c$, where $q(t)$ is the size of the queue, $y(t)$ is the rate at which data is sent to the router, and c is the capacity of the router. If $y(t)$ is always fixed at $c + \epsilon$, $\epsilon > 0$, what will happen to the queue? Explain in simple terms what congestion collapse is and how TCP congestion control algorithms solve the problem.

All students should complete the following problems:

1. For each of the following linear systems, determine whether the origin is asymptotically stable and, if so, plot the step response and frequency response for the system. If there are multiple inputs or outputs, plot the response for each pair of inputs and outputs.
 - (a) *Coupled mass spring system.* Consider the coupled mass spring system we saw in class, which has a damper on only one of the masses:



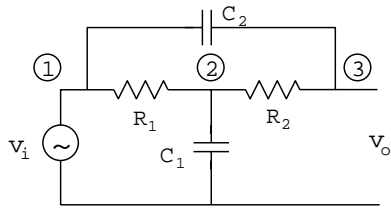
The equations of motion are given by

$$\frac{d}{dt} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k/m & k/m & 0 & 0 \\ k/m & -2k/m & 0 & -b/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k/m \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

Use $m = 250$, $k = 50$, $b = 10$ for the parameter values.

- (b) *Bridged Tee Circuit.* Consider the following electrical circuit, with input v_i and output v_o :

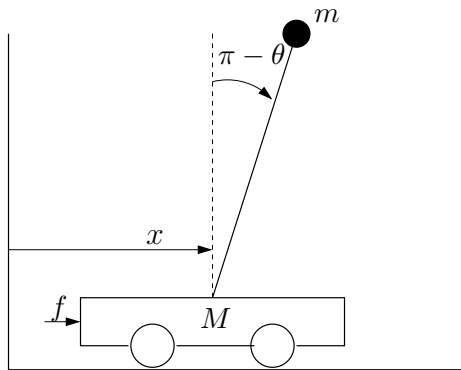


$$\frac{d}{dt} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \\ \frac{1}{C_2 R_2} \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + v_i$$

where v_{c1} and v_{c2} are the voltages across the two capacitors. Assume that $R_1 = 100\Omega$, $R_2 = 100\Omega$ and $C_1 = C_2 = 1 \times 10^{-6}\text{F}$.

2. (MATLAB/SIMULINK) Consider the inverted pendulum on a cart, as show in the figure below:



$$(M + m)\ddot{x} + ml \cos \theta \ddot{\theta} = -b\dot{x} + ml \sin \theta \dot{\theta}^2 + F$$

$$(J + ml^2)\ddot{\theta} + ml \cos \theta \ddot{x} = -mgl \sin \theta$$

$$M = 0.5 \text{ kg} \quad m = 0.2 \text{ kg}$$

$$b = 0.1 \text{ N/m/sec} \quad l = 0.3 \text{ m}$$

$$J = 0.006 \text{ kg m}^2$$

This system has been modeled in SIMULINK in the file `hw4cartpend.mdl`, available from the course web page.

The linearization of the system was given in class:

$$\frac{d}{dt} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 gl^2}{J(M+m) + Mml^2} & \frac{-(J + ml^2)b}{J(M+m) + Mml^2} & 0 \\ 0 & \frac{mgl(M+m)}{J(M+m) + Mml^2} & \frac{-mlb}{J(M+m) + Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J + ml^2}{J(M+m) + Mml^2} \\ \frac{ml}{J(M+m) + Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Note: in the SIMULINK model, the output has been set to include all of the states ($y = x$). You will need this for part (c) below.

- (a) Use the MATLAB `linmod` command to numerically compute the linearization of the original nonlinear system at the equilibrium point $(x, \theta, \dot{x}, \dot{\theta}) = (0, \pi, 0, 0)$. Compare the eigenvalues of the analytical linearization to those of the one you obtained with `linmod` and verify they agree.
- (b) We can design a stabilizing control law for this system using “state feedback”, which is a control law of the form $u = -Kx$ (we will learn about this more next week). The closed loop system under state feedback has the form

$$\dot{x} = (A - BK)x.$$

Show that the following state feedback stabilizes the linearization of the inverted pendulum on a cart: $K = (-1, 18.7, -1.7, 3.5)$.

- (c) Now build a simulation for the closed loop, *nonlinear* system in SIMULINK. Use the file `hw4cartpend.mdl` for the nonlinear equations of motion in it (you should look in the file and try to understand how it works). Simulate several different initial conditions and show that the controller *locally* asymptotically stabilizes the system to x_e from these initial conditions. Include plots of a representative simulation for an initial condition that is in the region of attraction of the controller and one that is outside the region of attraction. (Hint: remember that the equilibrium point that we linearized about was not zero. You will need to account for this in your controller by implementing $u = -K(x - x_e)$ for the nonlinear system.)
- (d) *Optional.* Use MATLAB to write an animation of your results and post them on the web.

Only CDS 110a students need to complete the following additional problems:

3. Consider the following discrete time system

$$\begin{aligned} z[k+1] &= Az[k] + Bu[k] \\ y[k+1] &= Cz[k+1]. \end{aligned}$$

In this problem we will derive the stability conditions, step response, and “convolution equation” for discrete time systems. (Don’t worry, they are much easier than the ODE versions.)

- (a) Assume that the matrix A has a full basis of eigenvectors $\{v_i\}$, so that any initial condition can be written as a linear combination of these eigenvectors. Let λ_i be the associated eigenvalues and show the discrete time system is asymptotically stable if and only if $|\lambda_i| < 1$.
- (b) Derive a formula for the transient response to an initial condition z_0 (the analog of $e^{AT}x_0$ for continuous time systems). Your answer should be in the form $y[k] = C\Phi[k]z_0$ where $\Phi[k]$ depends on the matrix A .
- (c) Derive a formula for the output response of the system to a general input u_k . Your result should be expressed as a sum involving terms of the form $\Phi[k-j]$ (similar to the terms $e^{A(t-\tau)}$ for continuous time systems).

Note: you can find the answer to this in many books on linear control systems, including those on reserve in the library. You are encouraged to look at them, but make sure you *understand* the answer you write down and include enough detail for the TAs to follow your derivation.

4. A simple model for congestion control between N computers connected by a router is given by the differential equation

$$\begin{aligned} \dot{x}_i &= -b\frac{x_i^2}{2} + (b_{\max} - b) \\ \dot{b} &= \sum_{i=1}^N x_i - c \end{aligned}$$

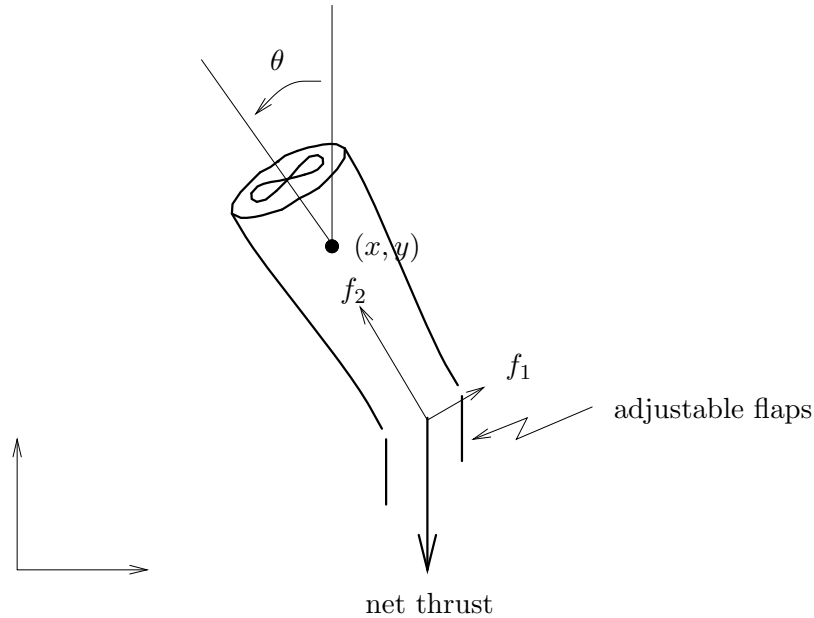
where $x_i \in \mathbb{R}$, $i = 1, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\max} > 0$ is the maximum buffer size, and $c > 0$ is the capacity of

the link connecting the router to the computers. The \dot{x}_i equation represents the control law that the individual computers use to determine how fast to send data across the network (this version is motivated by a protocol called “Reno”) and the \dot{b} equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1 , x_2 , and b).

- (a) Compute the equilibrium point(s) for the system.
- (b) For each equilibrium point, compute the linearization at the equilibrium point.
- (c) Using the parameters $b_{\max} = 1$ Mb and $c = 2$ Mb/sec, determine whether the equilibrium point(s) are (locally) stable, asymptotically stable, or unstable.

Supplemental problems: If you like, you may do any one problems of the following problems in place of Problem 4. These problems make use of domain-specific knowledge and so you should only do them if you are comfortable with that problem area. In addition, these problems are experimental in nature and you should ask questions quickly if you get stuck (it might not be your fault!).

5. Consider the motion of a small model aircraft powered by a vectored thrust engine, as shown below.



Let (x, y, θ) denote the position and orientation of the center of mass of the fan. We assume that the forces acting on the fan consist of a force f_1 perpendicular to the axis of the fan acting at a distance r and a force f_2 parallel to the axis of the fan. Let m be the mass of the fan, J the moment of inertia, γ the gravitational constant, and D the damping coefficient. Then the equations of motion for the fan are given by:

$$\begin{aligned} m\ddot{x} &= f_1 \cos \theta - f_2 \sin \theta - d\dot{x} \\ m\ddot{y} &= f_1 \sin \theta + f_2 \cos \theta - m\gamma - d\dot{y} \\ J\ddot{\theta} &= r f_1. \end{aligned}$$

It is convenient to redefine the inputs so that the origin is an equilibrium point of the system with zero input. If we let $u_1 = f_1$ and $u_2 = f_2 - m\gamma$ then the equations become

$$\begin{aligned} m\ddot{x} &= -m\gamma \sin \theta - d\dot{x} + u_1 \cos \theta - u_2 \sin \theta \\ m\ddot{y} &= m\gamma(\cos \theta - 1) - d\dot{y} + u_1 \sin \theta + u_2 \cos \theta \\ J\ddot{\theta} &= r u_1. \end{aligned} \tag{1}$$

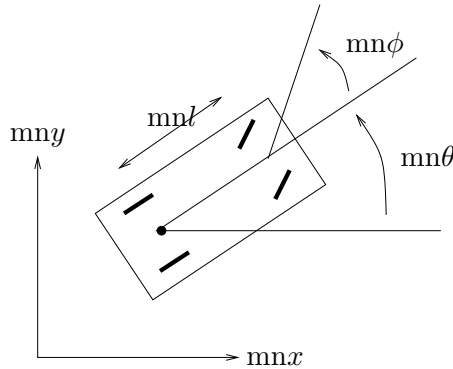
These equations are referred to as the *planar ducted fan equations* (this is an experiment that we use for CDS 111).

Use the following values for the parameters of the system:

$$\begin{aligned} \gamma &= 0.52 \text{ m/sec}^2 & m &= 4.25 \text{ kg} & d &= 0.1 \text{ kg/sec} \\ r &= 26 \text{ cm} & J &= 0.0475 \text{ kg m}^2 \end{aligned}$$

The reason that gravity γ is not 9.8 m/sec^2 is because of the presence of a counterweight to offset the weight of the fan.

- (a) Rewrite the equations of motion in state space form (still nonlinear, still symbolic). Choose x and y as the outputs.
 - (b) Compute the linearization of the system around the “hover” state: $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = (0, 0, 0, 0, 0, 0)$. Your result should be in terms of the symbolic parameters (don’t plug in the numbers, yet).
 - (c) Using the parameters above, determine if the linearization is stable, asymptotically stable, or unstable.
 - (d) Plot the step and frequency responses of the system from the two inputs to the two outputs (you should have eight plots total). (Hint: you might want to use the MATLAB `subplot` command to save some paper.)
6. Consider a simple kinematic model for an automobile with front and rear tires, as shown below.



The rear tires are aligned with the car, while the front tires are allowed to spin about the vertical axes. To simplify the derivation, we model the front and rear pairs of wheels as single wheels at the midpoints of the axles. The dynamics of the system arise by allowing the wheels to roll and spin, but not slip.

Let (x, y, θ, ϕ) denote the configuration of the car, parameterized by the xy location of the rear wheel(s), the angle of the car body with respect to the horizontal, θ , and the steering angle with respect to the car body, ϕ . The dynamics of the vehicle are given by

$$\begin{aligned} \dot{x} &= \cos \theta u_1 \\ \dot{y} &= \sin \theta u_1 \\ \dot{\theta} &= \frac{1}{l} \tan \phi u_1 \\ \dot{\phi} &= u_2 \end{aligned}$$

where u_1 corresponds to the forward velocity of the rear wheels of the car and u_2 corresponds to the velocity of the angle of the steering wheel. The parameter l is the wheel base of the vehicle and we will take this to be $l = 1$ m (roughly corresponding to Homer).

We wish to construct a model for the lateral (side-to-side) motion of the car, assuming a nominal trajectory that involves driving in a straight line at a fixed speed. To do this, we will assume that the car is travelling at speed V_0 and that we are only interested in the motion of the car in the y direction. The resulting equations of motion are given by

$$\begin{aligned}\dot{y} &= \sin \theta V_0 \\ \dot{\theta} &= \tan \phi V_0 \\ \dot{\phi} &= u\end{aligned}$$

where u is the velocity of the steering wheel.

- (a) Compute the equilibrium point(s) for the system.
- (b) For each equilibrium point, compute the linearization of the equilibrium point and determine whether the equilibrium points(s) are locally stable, asymptotically stable, or unstable.
- (c) Let $V_0 = 10$ m/s and suppose that we apply a control law of the form

$$u = k_1 y + k_2 \theta + k_3 \phi.$$

Show that you can choose the constants k_i such that the linearized system is asymptotically stable for straight line motion with $y = 0$. Using a SIMULINK model for the full nonlinear system, plot the response of your controller to an initial error of 1 meter, showing the state trajectory and the inputs as a function of time.