Chapter 6

Transfer Functions

As a matter of idle curiosity, I once counted to find out what the order of the set of equations in an amplifier I had just designed would have been, if I had worked with the differential equations directly. It turned out to be 55.

Henrik Bode, 1960

This chapter introduces the concept of *transfer function* which is a compact description of the input-output relation for a linear system. Combining transfer functions with block diagrams gives a powerful method of dealing with complex systems. The relations between transfer functions and other system descriptions of dynamics is also discussed.

6.1 Introduction

The transfer function is a convenient representation of a linear time invariant dynamical system. Mathematically the transfer function is a function of complex variables. For finite dimensional systems the transfer function is simply a rational function of a complex variable. The transfer function can be obtained by inspection or by by simple algebraic manipulations of the differential equations that describe the systems. Transfer functions can describe systems of very high order, even infinite dimensional systems governed by partial differential equations. The transfer function of a system can be determined from experiments on a system.

6.2 The Transfer Function

An input-output description of a system is essentially a table of all possible input-output pairs. For linear systems the table can be characterized by one input pair only, for example the impulse response or the step response. In this section we will consider another interesting pairs of signals.

Transmission of Exponential Signals

Exponential signals play an important role in linear systems. They appear in the solution of the differential equation (6.5) and in the impulse response of a linear systems. Many signals can be represented as exponentials or as sum of exponentials. A constant signal is simply $e^{\alpha t}$ with $\alpha = 0$. Damped sine and cosine signals can be represented by

$$e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\sin \omega t + i \cos \omega t)$$

Many other signals can be represented by linear combination of exponentials. To investigate how a linear system responds to the exponential input $u(t) = e^{st}$ we consider the state space system

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du. \end{aligned} \tag{6.1}$$

Let the input signal be $u(t) = e^{st}$ and assume that $s \neq \lambda_i(A)$, $i = 1, \dots, n$, where $\lambda_i(A)$ is the ith eigenvalue of A. The state is then given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau}\, d\tau = e^{At}x(0) + e^{At} \int_0^t e^{(sI - A)\tau)} B\, d\tau$$

Since $s \neq \lambda(A)$ the integral can be evaluated and we get

$$\begin{aligned} x(t) &= e^{At}x(0) + e^{At}(sI - A)^{-1} \Big|_{\tau=0}^{t} e^{(sI-A)\tau))} B \\ &= e^{At}x(0) + e^{At}(sI - A)^{-1}\Big(e^{(sI-A)t} - I\Big)B \\ &= e^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + (sI - A)^{-1}Be^{st} \end{aligned}$$

The output of (6.1) is thus

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + [D + C(sI - A)^{-1}B]e^{st}, \end{aligned}$$

a linear combination of exponential functions with exponents e^{st} and $e^{\lambda_i t}$, where λ_i are the eigenvalues of A. One term of the output is proportional to the input $u(t) = e^{st}$. This term is called the *pure exponential response.* If the initial state is chosen as

$$x(0) = (sI - A)^{-1}B$$

the output only consists of the pure exponential response and both the state and the output are proportional to the input

$$\begin{aligned} x(t) &= (sI - A)^{-1}Be^{st} = (sI - A)^{-1}Bu(t) \\ y(t) &= [C(sI - A)^{-1}B + D]e^{st} = [C(sI - A)^{-1}B + D]u(t). \end{aligned}$$

The ratio of the output and the input

$$G(s) = C(sI - A)^{-1}B + D \tag{6.2}$$

is the *transfer function* of the system. (6.1) The function

$$G_{xu}(s) = C(sI - A)^{-1}$$

is the transfer function from input to state.

Using transfer functions the response of the system (6.1) to an exponential input is thus

$$y(t) = Cx(t) + Du(t) = Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + G(s)e^{st}, \tag{6.3}$$

Coordinate Changes and Invariants

The matrices A, B and C in (6.1) depend on the coordinate system but not the matrix D which directly relates inputs and outputs. Since transfer function relates input to outputs the transfer function should also be invariant to coordinate changes in the state space. To show this consider the model (6.1) and introduce new coordinates z by the transformation $z = Tx$, where T is a regular matrix. The system is then described by

$$\begin{aligned} \frac{dz}{dt} &= T(Ax + Bu) = TAT^{-1}z + TBu = \tilde{A}z + \tilde{B}u \\ y &= Cx + DU = CT^{-1}z + Du = \tilde{C}z + Du \end{aligned}$$

This system has the same form as (6.1) but the matrices A, B and C are different

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB, \qquad \tilde{C} = CT^{-1}, \qquad \tilde{D} = D \tag{6.4}$$

Computing the transfer function of the transformed model we get

$$\begin{aligned}\tilde{G}(s) &= \tilde{C}(sI-\tilde{A})^{-1}\tilde{B} = CT^{-1}T(sI-A)^{-1}T^{-1}TB \\ &= CT^{-1}(sI-TAT^{-1})^{-1}TB = C(sI-A)^{-1}B = G(s)\end{aligned}$$

which is identical to the transfer function (6.2) computed from the system description (6.1). The transfer function is thus invariant to changes of the coordinates in the state space.

Transfer Function of a Linear ODE

Consider a linear input/output system described by the differential equation

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_ny = b_0\frac{d^mu}{dt^m} + b_1\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_mu, \tag{6.5}$$

where u is the input and y is the output. Note that here we have generalized our previous system description to allow both the input and its derivatives to appear. The differential equation is completely described by two polynomials

$$\begin{aligned}a(s) &= s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n \\ b(s) &= b_0s^m + b_1s^{m-1} + \ldots + b_{m-1}s + b_m,\end{aligned} \tag{6.6}$$

where the polynomial $a(s)$ is the characteristic polynomial of the system.

To determine the transfer function of the system (6.5), let the input be $u(t) = e^{st}$. Then there is an output of the system that also is an exponential function $y(t) = y_0e^{st}$. Inserting the signals in (6.5) we find

$$(s^n + a_1s^{n-1} + \cdots + a_n)y_0e^{st} = (b_0s^m + b_1s^{m-1}\cdots + b_m)e^{-st}$$

If $a(\alpha) \neq 0$ it follows that

$$y(t) = y_0e^{st} = \frac{b(s)}{a(s)}e^{st} = G(s)u(t). \tag{6.7}$$

The transfer function of the system (6.5) is thus the rational function

$$G(s) = \frac{b(s)}{a(s)}, \tag{6.8}$$

where the polynomials $a(s)$ and $b(s)$ are given by (6.6). Notice that the transfer function for the system (6.5) can be obtained by inspection.

Example 6.1 *(Transfer functions of integrator and differentiator).* The transfer function $G(s) = 1/s$ corresponds to the diffrential equation

$$\frac{dy}{dt} = u,$$

which represents an integrator and a differentiator which has the transfer function $G(s) = s$ corresponds to the differential equation

$$y = \frac{du}{dt}.$$

Example 6.2 *(Transfer Function of a Time Delay).* Time delays appear in many systems, typical examples are delays in nerve propagation, communication and mass transport. A system with a time delay has the input output relation

$$y(t) = u(t - T) \tag{6.9}$$

Let the input be $u(t) = e^{st}$. Assuming that there is an output of the form $y(t) = y_0 e^{st}$ and inserting into (6.9) we get

$$y(t) = y_0 e^{st} = e^{s(t-T)} = e^{-sT} e^{st} = e^{-sT} u(t)$$

The transfer function of a time delay is thus $G(s) = e^{-sT}$ which is not a rational function.

Steady State Gain

The transfer function has many useful physical interpretations. The *steady state gain* of a system is simply the ratio of the output and the input in steady state. Assuming that the the input and the output of the system (6.5) are constants y_0 and u_0 we find that $a_n y_0 = b_n u_0$. The *steady state gain* is

$$\frac{y_0}{u_0} = \frac{b_n}{a_n} = G(0). \tag{6.10}$$

The result can also be obtained by observing that a unit step input can be represented as $u(t) = e^{st}$ with $s = 0$ and the above relation then follows from Equation (6.7).

Poles and Zeros

Consider a linear system with the rational transfer function

$$G(s) = \frac{b(s)}{a(s)}$$

The roots of the polynomial $a(s)$ are called *poles* of the system and the roots of $b(s)$ are called the *zeros* of the system. If p is a pole it follows that $y(t) = e^{pt}$ is a solution to the (6.5) with $u = 0$ (the homogeneous equation). The function e^{pt} is called a *mode* of the system. The free motion of the system after an arbitrary excitation is a weighted sum of the modes. Since the pure exponential output corresponding to the input $u(t) = e^{st}$ with $a(s) \neq 0$ is $G(s)e^{st}$ it follows that the pure exponential output is zero if $b(s) = 0$. Zeros of the transfer function thus blocks the transmission of the corresponding exponential signals.

The poles of the transfer function are the eigenvalues of the system matrix A in the state space model. They depend only on the the dynamics matrix A, which represents the intrinsic dynamics of the system. The zeros depend on how inputs and outputs are coupled to the states. The zeros thus depend on all matrices A, B, C and D in the state space description.

To find the zeros of the state space system (6.1) we observe that the zeros are complex numbers s such that the input $u(t) = e^{st}$ gives zero output. Inserting the pure exponential response $x(t) = x_0e^{st}$ and $y(t) = 0$ in (6.1) gives.

$$\begin{aligned} se^{st}x_0 &= Ax_0e^{st} + Bu_0e^{st} \\ 0 &= Ce^{st}x_0 + De^{st}u_0, \end{aligned}$$

which can be written as

$$\begin{bmatrix} sI - A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0.$$

This equation has a solution with nonzero x_0, u_0 only if the matrix on the left does not have full rank. The zeros are thus the values s such that

$$\det \begin{bmatrix} sI - A & B \\ C & D \end{bmatrix} = 0 \tag{6.11}$$

Notice in particular that if the matrix B has full rank the matrix has n linearly independent rows for all values of s. Similarly there are n linearly independent columns if the matrix C has full rank. This implies that systems where the matrices B or C are of full rank do not have zeros. In particular it means that a system has no zeros if it is fully actuated or of the full state is measured.

Example 6.3 *(Transfer Function for Heat Propagation).* Consider the one dimensional heat propagation in a semi-infinite metal rod. Assume that the input is the temperature at one end and that the output is the temperature at a point on the rod. Let θ be the temperature at time t and position x. With proper choice of scales heat propagation is described by the partial differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial^2 x}, \tag{6.12}$$

and the point of interest can be assumed to have $x = 1$. The boundary condition for the partial differential equation is

$$\theta(0, t) = u(t)$$

To determine the transfer function we assume that the input is $u(t) = e^{st}$. Assume that there is a solution to the partial differential equation of the form $\theta(x, t) = \psi(x)e^{st}$, and insert this into (6.12) gives

$$s\psi(x) = \frac{d^2\psi}{dx^2},$$

with boundary condition $\psi(0) = e^{st}$. This ordinary differential equation has the solution

$$\psi(x) = Ae^{x\sqrt{s}} + Be^{-x\sqrt{s}}.$$

Matching the boundary conditions gives $A = 0$ and that $B = e^{st}$ and the solution is

$$y(t) = x(1, t) = \theta(1, t) = \psi(1)e^{st} = e^{-\sqrt{s}}e^{st} = e^{-\sqrt{s}}u(t)$$

The system thus has the transfer function $G(s) = e^{-\sqrt{s}}$.

6.3 Frequency Response

Frequency response is a method where the behavior of a system is characterized by its response to sine and cosine signals. The idea goes back to Fourier, who introduced the method to investigated heat propagation in metals. He observed that a periodic signal can be approximated by a Fourier series. Since

$$e^{i\omega t} = \sin \omega t + i \cos \omega t$$

it follows that sine and cosine functions are special cases of exponential functions. The response to sinusoids is thus a special case of the response to exponential functions.

Consider the linear time-invariant system (6.1). Assume that all eigenvalues of the matrix A have negative real parts. Let the input be $u(t) = e^{i\omega t}$

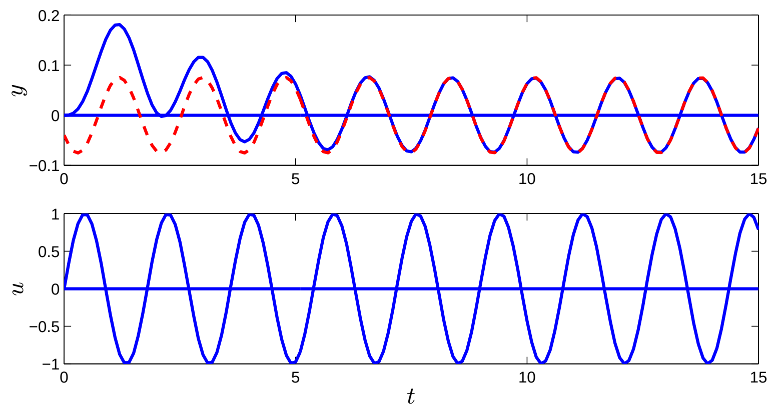

Figure 6.1: Response of a linear time-invariant system with transfer function $G(s) = (s+1)^{-2}$ to a sinusoidal input (full lines). The dashed line shows the steady state output calculated from (6.13).

and let $G(s)$ be the transfer function of the system. It follows from (6.3) that the output is

$$y(t) = Cx(t) + Du(t) = Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + G(i\omega)e^{i\omega t}.$$

Since the matrix all eigenvalues of A have negative real parts the first term decays exponentially and the solution converges to the steady state response

$$y(t) = \operatorname{Im}\big(G(i\omega)e^{i\omega t}\big).$$

Since $u(t) = \sin\omega t = \operatorname{Im}\big(e^{i\omega t}\big)$ we can obtain the response to a sinusoid by taking the imaginary parts, hence

$$\begin{aligned} y(t) &= \operatorname{Im}\big(G(i\omega)e^{i\omega t}\big) = \operatorname{Im}\big(|G(i\omega)|e^{i\arg G(i\omega)}e^{i\omega t}\big) \\ &= |G(i\omega)|\operatorname{Im}\big(e^{i(\arg G(i\omega)+\omega t)}\big) = |G(i\omega)|\sin\big(\omega t + \arg G(i\omega)\big). \end{aligned}$$

The steady state output generated by the input $u(t) = \sin(\omega t)$ is thus

$$y(t) = |G(i\omega)|\sin(\omega_t + \arg G(i\omega)), \tag{6.13}$$

where $|G(i\omega)|$ is called the gain of the system and $\arg G(i\omega)$ is called the phase of the system. This is illustrated in Figure 6.1 which shows the response of a linear time-invariant system to a sinusoidal input. The figure shows the output of the system when it is initially at rest and the steady state output given by (6.13). The figure shows that after a transient the output is indeed a sinusoid with the same frequency as the input.

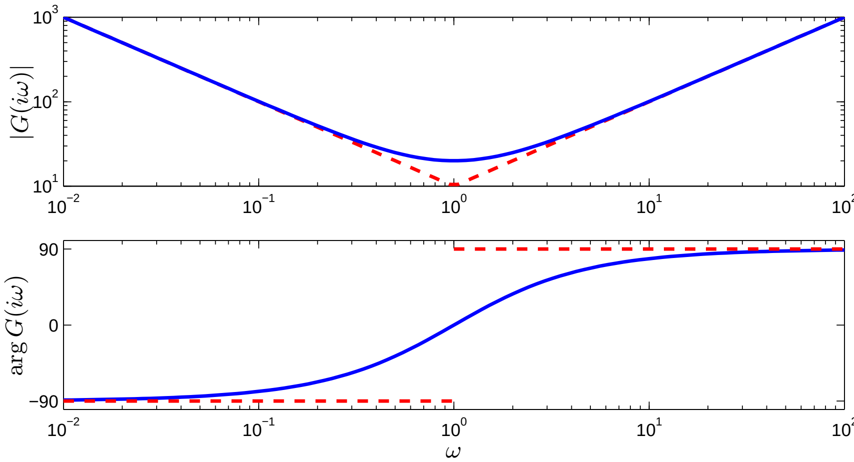

Figure 6.2: Bode plot of the transfer function $C(s) = 20 + 10/s + 10s$ of an ideal PID controller. The top plot is the gain curve and bottom plot is the phase curve. The dashed lines show straight line approximations of the gain curve and the corresponding phase curve.

6.4 The Bode Plot

The frequency response $G(i\omega)$ can be represented by two curves, the gain curve and the phase curve. The gain curve gives gain $|G(i\omega)|$ as a function of frequency ω and the phase curve gives phase $\arg G(i\omega)$ as a function of frequency ω. The curves obtained when logarithmic scales for frequency and gain and linear scale for phase are used is called the Bode plot, see Figure 6.2 An useful feature of the Bode plot of a rational transfer function is that the gain curve can be approximated by piecewise by straight lines with integer slopes. The lines change slope at the poles and zeros of the transfer function. Tracing the curve for increasing frequencies the slope increases with one unit at a zero and it decreases with one unit at a pole. These straight lines correspond to phase curves that are horizontal at values that are integer multiples of 90°. This is illustrated in Figure 6.2 which gives the Bode plot of an ideal PID controller with the transfer function

$$C(s) = 20 + \frac{10}{s} + 10s = \frac{10(s+1)^2}{s}.$$

The Bode plot is shown in full lines and the straight line approximation in dashed lines. For $\omega < 0.1$ we have $G(s) \approx 10/s$, the approximation of the gain curve is a line with slope -1, and the phase curve is horizontal $\arg G(i\omega) = -90°$. For $\omega > 10$ we have $G(s) \approx 10/s$, the approximation of the gain curve is a straight line, and the phase curve is horizontal, $\arg G(i\omega) = 90°$.

It is easy to sketch Bode plots because they have linear asymptotes. This is useful in order to get a quick estimate of the behavior of a system. It is also a good way to check numerical calculations. Consider a transfer function which is a polynomial $G(s) = b(s)/a(s)$. We have

$$\log G(s) = \log b(s) - \log a(s)$$

Since a polynomial is a product of terms of the type :

$$k, s, \quad s + a, \quad s^2 + 2\zeta as + a^2$$

it suffices to be able to sketch Bode diagrams for these terms. The Bode plot of a complex system is then obtained by adding the gains and phases of the terms.

Example 6.4 *(Bode Plot of an Integrator).* Consider the transfer function

$$G(s) = \frac{k}{s}$$

We have $G(i\omega) = k/i\omega$ which implies

$$\log |G(i\omega)| = \log k - \log \omega, \quad \arg G(i\omega) = -\pi/2$$

The gain curve is thus a straight line with slope -1 and the phase curve is a constant at $-90°$. Bode plots of a differentiator and an integrator are shown in Figure 6.3

Example 6.5 *(Bode Plot of a Differentiator).* Consider the transfer function

$$G(s) = ks$$

We have $G(i\omega) = ik\omega$ which implies

$$\log |G(i\omega)| = \log k + \log \omega, \quad \arg G(i\omega) = \pi/2$$

The gain curve is thus a straight line with slope 1 and the phase curve is a constant at $90°$. The Bode plot is shown in Figure 6.3.

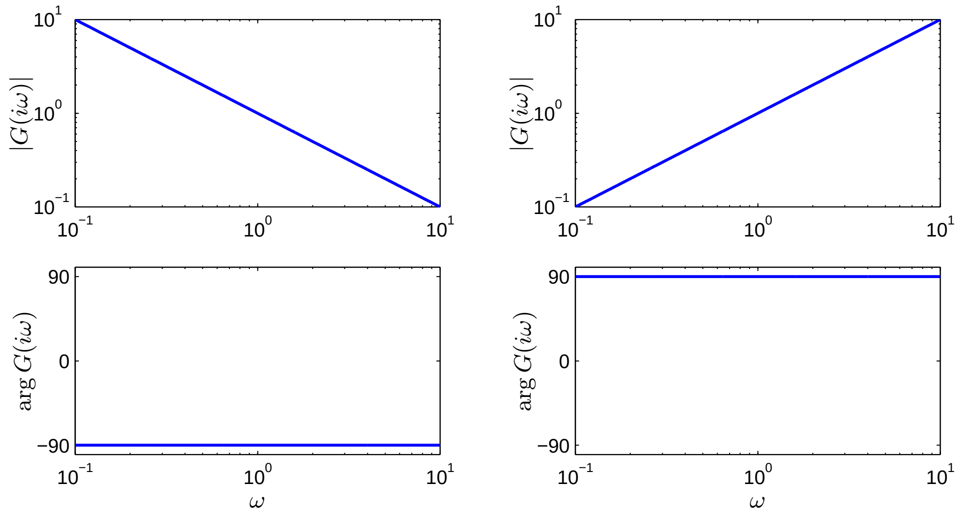

Figure 6.3: Bode plot of the transfer functions $G(s) = 1/s$ (left) and $G(s) = s$ (right).

Compare the Bode plots for the differentiator in and the integrator in Figure 6.3. The plot for the differentiator is obtained by mirror imaging the gain and phase curves for the integrator in the horizontal axis. This follows from the following property of the logarithm.

$$\log \frac{1}{G} = -\log G = -\log |G| - i \arg G$$

Example 6.6 *(Bode Plot of a First Order System).* Consider the transfer function

$$G(s) = \frac{a}{s+a}$$

We have

$$\log G(s) = \log a - \log s + a$$

Hence

$$\log |G(i\omega)| = \log a - \frac{1}{2} \log (\omega^2 + a^2)), \quad \arg G(i\omega) = -\arctan \omega / a$$

The Bode Plot is shown in Figure 6.4. Both the gain curve and the phase

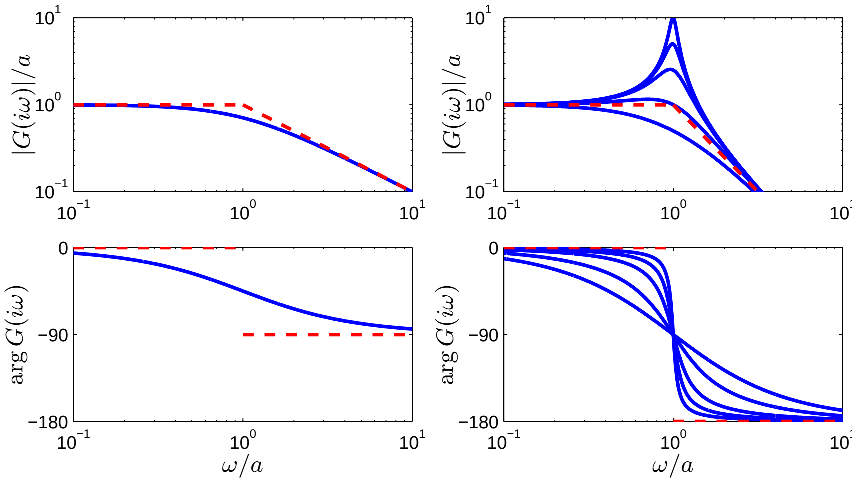

Figure 6.4: Bode plots of the systems $G(s) = a/(s + a)$ (left) and $G(s) = a^2/(s^2 + 2\zeta as + a^2)$ (right). The full lines show the Bode plot and the dashed lines show the straight line approximations to the gain curves and the corresponding phase curves. The plot for second order system has $\zeta = 0.1$, 0.2, 0.5 and 1.0.

curve can be approximated by the following straight lines

$$\log |G(i\omega)| \approx \begin{cases} \log a & \text{if } \omega << a, \\ -\log \omega & \text{if } \omega >> a \end{cases},$$

$$\arg G(i\omega) \approx \begin{cases} 0 & \text{if } \omega << a, \\ -\frac{\pi}{2} & \text{if } \omega >> a \end{cases}.$$

Notice that a first order system behaves like a constant for low frequencies and like an integrator for high frequencies. Compare with the Bode plot in Figure 6.3.

Example 6.7 *(Bode Plot of a Second Order System).* Consider the transfer function

$$G(s) = \frac{a^2}{s^2 + 2a\zeta s + a^2}$$

We have

$$\log G(i\omega) = 2 \log a - \log(-\omega^2 + 2ia\zeta\omega + a^2).$$

Hence

$$\log |G(i\omega)| = 2\log a - \frac{1}{2}\log\left(\omega^4 + 2a^2\omega^2(2\zeta^2 - 1) + a^4\right)$$
$$\arg G(i\omega) = -\arctan\frac{2\zeta a\omega}{a^2 - \omega^2}$$

The gain curve has an asymptote with zero slope for $\omega << a$. For large values of ω the gain curve has an asymptote with slope -2. The largest gain $Q = \max_\omega |G(i\omega)| = 1/(2\zeta)$, called the Q *value*, is obtained for $\omega = a$. The phase is zero for low frequencies and approaches $180°$ for large frequencies. The curves can be approximated with the following piece-wise linear expressions

$$\log |G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega << a, \\ -2\log\omega & \text{if } \omega >> a \end{cases},$$
$$\arg G(i\omega) \approx \begin{cases} 0 & \text{if } \omega << a, \\ -\pi & \text{if } \omega >> a \end{cases},$$

The Bode plot is shown in Figure 6.4.

Interpretations of Bode Plots

The Bode plot gives a quick overview of a system. Many properties can be read from the plot. Because logarithmic scales are used the plot gives the properties over a wide range of frequencies. Since any signal can be decomposed into a sum of sinusoids it is possible to visualize the behavior of a system for different frequency ranges. Furthermore when the gain curves are close to the asymptotes the system can be approximated by integrators of differentiators. Consider for example the Bode plot in Figure 6.2. For low frequencies the gain curve of the Bode plot has the slope -1 which means that the system acts like an integrator. For high frequencies the gain curve has slope +1 which means that the system acts like a differentiator.

6.5 Frequency Responses from Experiments

Modeling can be difficult and time consuming. One reason why control has been so successful is that the frequency response of a system can be determined experimentally by perturbing the input sinusoidally. When steady

Figure 6.5: The Hewlett Packard signal analyzer can be used to determine frequency response experimentally.

state is reached the amplitude ratio and the phase lag gives the frequency response for the excitation frequency. The complete frequency response is obtained by sweeping over frequency. By using correlation techniques it is possible to determine the frequency response very accurately. An analytic transfer function can be obtained from the frequency response by curve fitting. Nice instruments are commercially available, see Figure 6.5.

Example 6.8 *(A Piezoelectric Drive).* Experimental determination of frequency responses is particularly attractive for systems with fast dynamics. A typical example is given if Figure 6.6. In this case the frequency was obtained in less than a second. The full line shows the measured frequency response. The transfer function

$$G(s) = \frac{k\omega_2^2\omega_3^2\omega_5^2(s^2 + 2\zeta_1\omega_1 + \omega_1^2)(s^2 + 2\zeta_4\omega_4 + \omega_4^2)}{\omega_1^2\omega_4^2(s^2 + 2\zeta_2\omega_2 + \omega_2^2)(s^2 + 2\zeta_3\omega_3 + \omega_3^2)(s^2 + 2\zeta_5\omega_5 + \omega_5^2)}e^{-sT}$$

with $\omega_1 = 2420$, $\zeta_1 = 0.03$, $\omega_2 = 2550$, $\zeta_2 = 0.03$, $\omega_3 = 6450$, $\zeta_3 = 0.042$, $\omega_4 = 8250$, $\zeta_4 = 0.025$, $\omega_5 = 9300$, $\zeta_5 = 0.032$, $T = 10^{-4}$, and $k = 5$. was fitted to the data. The frequencies associated with the zeros are located where the gain curve has minima and the frequencies associated with the poles are located where the gain curve has local maxima. The relative damping are adjusted to give a good fit to maxima and minima. When a good fit to the gain curve is obtained the time delay is adjusted to give a good fit to the phase curve. The fitted curve is shown i dashed lines. Experimental determination of frequency response is less attractive for systems with slow dynamics because the experiment takes a long time.

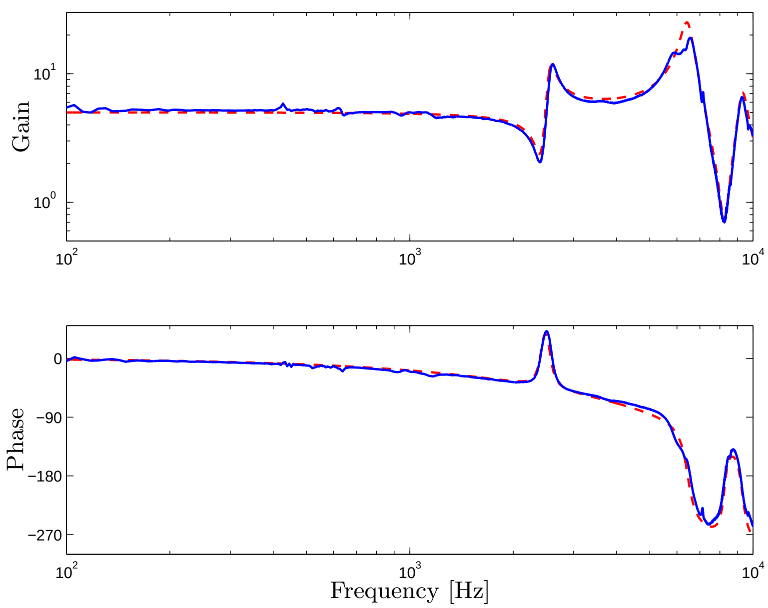

Figure 6.6: Frequency response of a piezoelectric drive for an atomic force microscope. The input is the voltage to the drive amplifier and the output is the output of the amplifier that measures beam deflection.

The human eye is an organ that is easily accessible for experiments. It has a control system which adjust the pupil opening to regulate the light intensity at the retina. This control system was explored extensively by Stark in the late 1960s. To determine the dynamics light intensity on the eye was varied sinusoidally and the pupil opening was measured. A fundamental difficulty is that the closed loop system is insensitive to internal system parameters. Analysis of a closed loop system thus gives little information about the internal properties of the system. Stark used a clever experimental technique which allowed him to investigate both open and closed loop dynamics. He excited the system by varying the intensity of a light beam focused on the eye and he measured pupil area, see Figure 6.7. By using a wide light beam that covers the whole pupil the measurement gives the closed loop dynamics. The open loop dynamics was obtained by using a narrow beam. By focusing a narrow beam on the edge of the pupil opening the gain of the system can be increased so that the pupil oscillates. The result of one experiment for determining open loop dynamics is given in

Example 6.9 *(Pupillary Light Reflex Dynamics).*

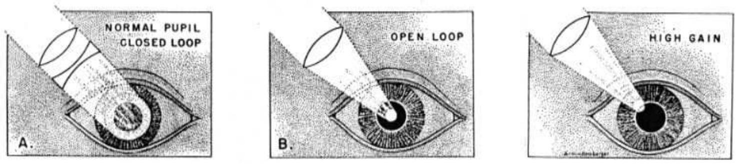

Figure 6.7: Light stimulation of the eye. In A the light beam is so large that it always covers the whole pupil. This experiment gives the closed loop dynamics. In B the light is focused into a beam which is so narrow that it is not influenced by the pupil opening. This experiment gives the open loop dynamics. In C the light beam is focused on the edge of the pupil opening. The pupil will oscillate if the beam is sufficiently small. From [**?**].

Figure 6.8. Fitting a transfer function to the gain curves gives a good fit for $G(s) = 0.17/(1 + 0.08s)^3$. This curve gives a poor fit to the phase curve as as shown by the dashed curve in Figure 6.8. The fit to the phase curve is improved by adding a time delay. Notice that a time delay does not change the gain curve. The final fit gives the model

$$G(s) = \frac{0.17}{(1 + 0.08)^3} e^{-0.2s}.$$

The Bode plot of this is shown with dashed curves in Figure 6.8.

Example 6.10 *(Determination of Thermal Diffusivity).* The Swedish Physicist Ångström used frequency response to determine thermal conductivity in 1861. A long metal rod with small cross section was used. A heat-wave is generate by periodically varying the temperature at one end of the sample. Thermal diffusivity is then determined by analyzing the attenuation and phase shift of the heat wave. A schematic diagram of Ångström's apparatus is shown in Figure 6.9. The input was generated by switching from steam to cold water periodically. Switching was done manually because of the low frequencies used. Heat propagation is described by the one-dimensional heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \mu T, \qquad (6.14)$$

where $\kappa = \frac{\lambda}{\rho C}$, and λ the thermal conductivity, ρ density, and C specific heat. The term μT is an approximation of heat losses due to radiation, and convection. The transfer function relating temperatures at points with the distance a is

$$G(s) = e^{-a\sqrt{(s+\mu)/\kappa}}, \qquad (6.15)$$

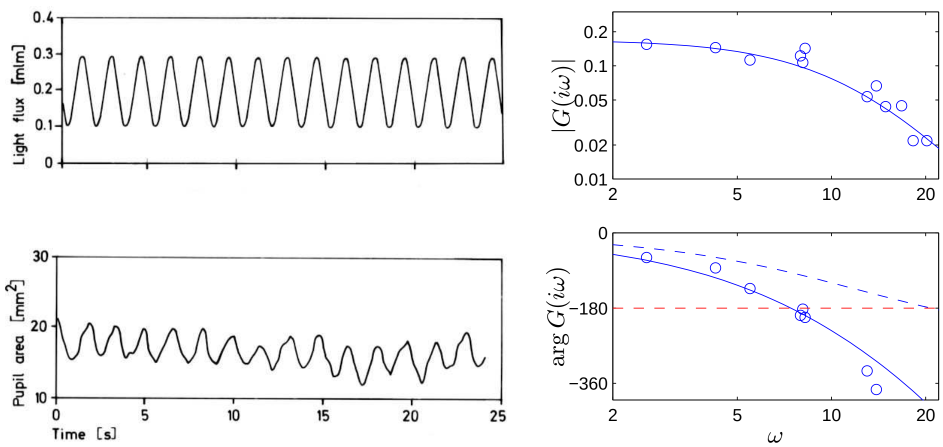

Figure 6.8: Sample curves from open loop frequency response of the eye (left) and Bode plot for the open loop dynamics (right). Redrawn from the data of [**?**]. The dashed curve in the Bode plot is the minimum phase curve corresponding to the gain curve.*Perhaps both Bode and Nyquist plots?*

and the frequency response is given by

$$\log|G(i\omega)| = -a\sqrt{\frac{\mu + \sqrt{\omega^2+\mu^2}}{2\kappa}}$$

$$\arg G(i\omega) = -a\sqrt{\frac{-\mu + \sqrt{\omega^2+\mu^2}}{2\kappa}}.$$

Multiplication of the equations give

$$\log|G(i\omega)| \ \arg G(i\omega) = \frac{a^2\omega}{2\kappa}. \tag{6.16}$$

Notice that the parameter μ which represents the thermal losses disappears in this formula. Ångström remarked that (6.16) is indeed a remarkable simple formula. Earlier values of thermal conductivity for copper obtained by Peclet was 79 W/mK. Ångström obtained 382 W/mK which is very close to modern data. Since the curves shown in Figure 6.9 are far from sinusoidal Fourier analysis was used to find the sinusoidal components. The procedure developed by Ångström quickly became the standard method for determining thermal conductivity.

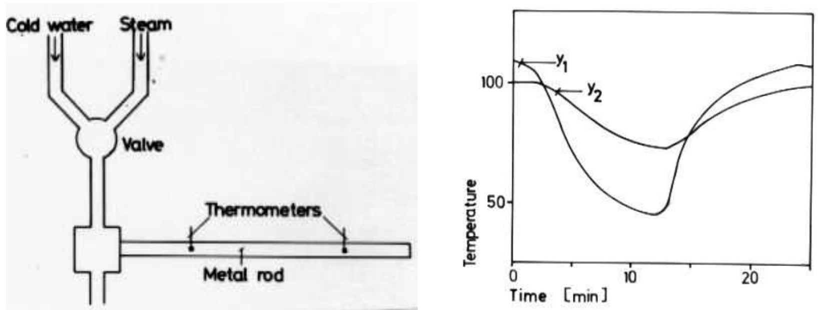

Figure 6.9: Schematic diagram of AAngström's equipment (left) and sample of his data(right). The copper rod had length 0.57 m, and diameter of 23.75 mm. Holes were drilled at distances of 0.05 m.

Notice that both for the piezo drive and for the pupillary dynamics it is not easy to derive appropriate models from first principle. It is often fruitful to use a combination.

6.6 Block Diagrams and Transfer Functions

The combination of block diagrams and transfer functions is a powerful way to represent control systems. Transfer functions relating different signals in the system can be derived by pure algebraic manipulations of the the transfer functions of the blocks using *block diagram algebra*. To show how this can be done we will begin with simple combination of systems.

Since the transfer function is characterized by transmission of exponential signals we let the input be $u(t) = e^{st}$. Recall from Section 6.2 that the pure exponential output a linear system with transfer function $G(s)$ is

$$y(t) = G(s)e^{st} = G(s)u(t)$$

which we simply write as $y = Gu$.

Now consider a system which is a cascade combination of systems with the transfer functions $G_1(s)$ and $G_2(s)$, see Figure 6.10A. Let the input of the system be $u = e^{st}$. The pure exponential output of the first block is the exponential signal G_1u, which is also the input to the second system. The pure exponential output of the second system is

$$y = G_2(G_1u) = G_1G_2u$$

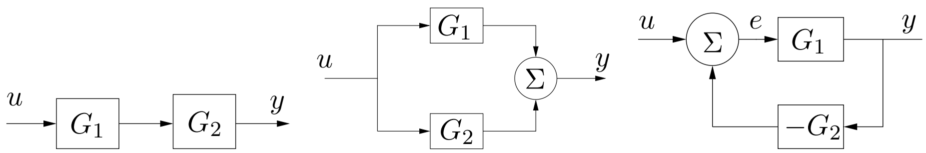

Figure 6.10: Series (left), parallel (middle) and feedback (right) connections of two linear systems.

The transfer function of the system is thus $G = G_1 G_2$, i.e. the product of the transfer functions.

Consider a parallel connection of systems with the transfer functions G_1 and G_2, see Figure 6.10b. Let $u = e^{st}$ be the input to the system. The pure exponential output of the first system is then $y_1 = G_1 u$ and the output of the second system is $y_2 = G_2 u$. The pure exponential output of the parallel connection is thus

$$y = G_1 u + G_2 u = (G_1 + G_2)u.$$

The transfer function of the parallel connection is thus $G = G_1 + G_2$.

Consider a feedback connection of systems with the transfer functions G_1 and G_2, as shown in Figure 6.10c. Let $r = e^{st}$ be the input to the system, y the pure exponential output, and e be the pure exponential part of the error. Writing the relations for the different blocks and the summation unit we find

$$\begin{aligned} y &= G_1 e \\ e &= r - G_2 y. \end{aligned}$$

Elimination of e gives

$$y = G_1(r - G_2 y),$$

hence

$$(1 + G_1 G_2)y = G_1 r,$$

which implies

$$y = \frac{G_1}{1 + G_1 G_2} r.$$

The transfer function of the feedback connection is thus

$$G(s) = \frac{G_1(s)}{G_1(s) + G_2(s)}.$$

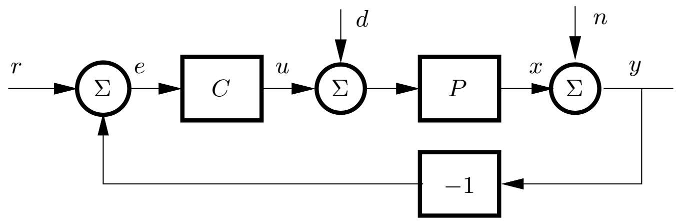

Figure 6.11: Block diagram of a feedback system.

With a little practice the equation relating the signals of interest can be written by inspection. We illustrate this by an example

Example 6.11 *(Relations Between Signals in a Block Diagram).* Consider the system in Figure 6.11. The system has two blocks representing a process P and a controller C. There are three external signals, the reference r, the load disturbance d and the measurement noise n. A typical problem is to find out how the error e related to the signals r d and n? To derive the transfer function we simply assume that all signals are exponential functions and we drop the arguments of signals and transfer functions.

To obtain the desired relation we simply trace the signals around the loop. We begin with the signal we are interested in, i.e. e. The error e is given by

$$e = r - y.$$

The signal y is the sum of n and x

$$y = n + x,$$

where x is the output of the process, i.e.

$$x = Pv = P(d + u),$$

where u is the output of the controller, i.e.

$$u = Ce.$$

Combining the equations gives

$$\begin{aligned} e &= r - y = r - (n + x) = r - \big(n + P(d + u)\big) \\ &= r - \big(n + P(d + Ce)\big). \end{aligned}$$

Hence

$$e = r - \big(n + P(d + Ce)\big) = r - n - Pd - PCe.$$

Solving this equation for e gives

$$e = \frac{1}{1+PC}r - \frac{1}{1+PC}n - \frac{P}{1+PC}d = G_{er}r + G_{en}n + G_{ed}d \qquad (6.17)$$

The error is thus the sum of three terms, depending on the reference r, the measurement noise n and the load disturbance d. The function

$$G_{er} = \frac{1}{1+PC}$$

is the transfer function from reference r to error e, G_{en} is the transfer functions from measurement noise n to error e and G_{ed} is the transfer functions from load disturbance d to error e.

The example illustrates an effective way to manipulate the equations to obtain the relations between inputs and outputs in a feedback system. The general idea is to start with the signal of interest and to trace signals around the feedback loop until coming back to the signal we started with. With a some practice the equation (6.17) can be written directly by inspection of the block diagram. Notice that all terms in Equation (6.17) formally have the same denominators, there may, however, be factors that cancel.

The combination of block diagrams and transfer functions is a powerful tool because it is possible both to obtain an overview of a system and find details of the behavior of the system. When manipulating block diagrams and transfer functions it may happen that poles and zeros are canceled. Some care must be exercised as is illustrated by the following example.

Example 6.12 *(Cancellation of Poles and Zeros).* Consider a system described by the equation.

$$\frac{dy}{dt} - y = \frac{du}{dt} - u \qquad (6.18)$$

Integrating the equation gives

$$y(t) = u(t) + ce^t$$

where c is a constant. The transfer function of the system is

$$\frac{Y(s)}{U(s)} = \frac{s-1}{s-1} = 1$$

Since s is a complex variable the cancellation is clearly permissible and we find that the transfer function is $G(s) = 1$ and we have seemingly obtained a contradiction because the system is not equivalent to the system

$$y(t) = u(t)$$

In the next section we will give a fundamental results that gives insight into the cancellation problem.

6.7 The Kalman Decomposition

Equation (6.2) gives a formula for the transfer function of a linear system on state space form. It appears at first sight that the denominator of the transfer function has the characteristic polynomial $\det(sI_A)$ as a numerator. This is not necessarily true because there may be cancellations as illustrated in Example 6.12. A very good insight into what happens can be obtained by using the concepts of reachability and observability discussed in Chapter 5. The key result is Kalman's decomposition theorem, which says that a linear system can be divided into four subsystems. A linear system can thus be decomposed into four subsystems: S_{ro} which is reachable and observable, $S_{r\bar{o}}$ which is reachable not observable, $S_{\bar{r}o}$ which is not reachable observable, and $S_{\bar{r}\bar{o}}$ which is not reachable not observable.

Diagonal Systems

We will show this in the special case of systems where the matrix A has distinct eigenvalues. Such a system can be represented as

$$\begin{aligned} \frac{dz}{dt} &= \begin{bmatrix} \Lambda_{ro} & 0 & 0 & 0 \\ 0 & \Lambda_{r\bar{o}} & 0 & 0 \\ 0 & 0 & \Lambda_{\bar{r}o} & 0 \\ 0 & 0 & 0 & \Lambda_{\bar{r}\bar{o}} \end{bmatrix} z + \begin{bmatrix} \beta_{ro} \\ \beta_{ro} \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} \gamma_{ro} & 0 & \gamma_{\bar{r}o} & 0 \end{bmatrix} z + Du. \end{aligned}$$

All states z_k such that $\beta_k \neq 0$ are controllable and all states such that $\gamma_k \neq 0$ are observable. The system can be represented as The transfer function of the system is

$$G(s) = \gamma_{ro}(sI - A_{ro})^{-1}\beta_{ro} + D.$$

It is uniquely given by the subsystem which is reachable and observable. A block diagram of the system is given in Figure 6.12.

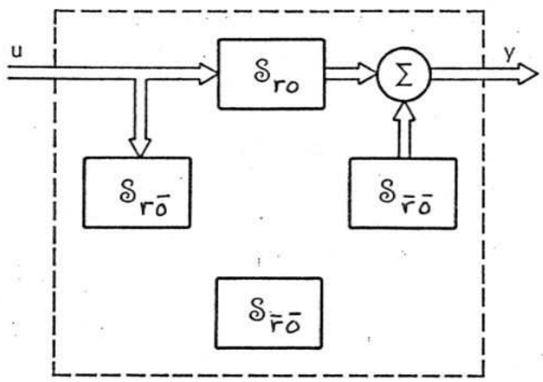

Figure 6.12: Kalman's decomposition of a linear system with distinct eigenvalues.

The Cancellation Problem

The Kalman decomposition gives a good insight into what happens when poles and zeros are canceled in a transfer function as illustrated in Example 6.12. In this particular case the system has two subsystems S_{ro} and $S_{\bar{r}o}$. The system S_{ro} is a static system with transfer function $G(s) = 1$ and the subsystem $S_{\bar{r}o}$ which is observable but non reachable has the dynamics.

$$\frac{dx}{dt} = x$$

This system is unstable and the unstable state will appear in the output.

The General Case

Some linear algebra is required to solve the general case. Introduce the reachable subspace $\mathcal{X}_r$ which is the linear subspace spanned by the columns of the reachability matrix W_r. The state space is the direct sum of $\mathcal{X}_r$ and another linear subspace $\mathcal{X}_{\bar{r}}$. Notice that $\mathcal{X}_r$ is unique but that $\mathcal{X}_{\bar{r}}$ can be chosen in many different ways. Choosing a coordinates with $x_r \in \mathcal{X}_r$ and $x_{barr} \in \mathcal{X}_{\bar{r}}$ the system equations can be written as

$$\frac{d}{dt}\begin{pmatrix} x_c \\ x_{\bar{c}} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} x_c \\ x_{\bar{c}} \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u \tag{6.19}$$

where the states x_c are reachable and $x_{\bar{c}}$ are non-reachable.

Introduce the unique subspace $\mathcal{X}_{\bar{o}}$ of non-observable states. This is right the null space of the observability matrix W_o. The state space is the direct sum of $\mathcal{X}_{\bar{o}}$ and the non-unique subspace $\mathcal{X}_o$. Choosing a coordinates with $x_o \in \mathcal{X}_o$ and $x_{baro} \in \mathcal{X}_{\bar{o}}$ the system equations can be written as

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x_o \\ x_{\bar{o}} \end{pmatrix} &= \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar{0}} \end{pmatrix} \\ y &= \begin{pmatrix} C_1 & 0 \end{pmatrix}\begin{pmatrix} x_o \\ x_{\bar{o}} \end{pmatrix} \end{aligned} \tag{6.20}$$

where the states x_o are observable and $x_{\bar{o}}$ not observable (quiet)

The intersection of two linear subspaces is also a linear subspace. Introduce $\mathcal{X}_{r\bar{o}}$ as the intersection of $\mathcal{X}_r$ and $\mathcal{X}_{\bar{o}}$ and the linear subspace $\mathcal{X}_{ro}$ which together with $\mathcal{X}_{r\bar{o}}$ spans $\mathcal{X}_r$. Finally we introduce the linear subspace $\mathcal{X}_{\bar{r}o}$ which together with $\mathcal{X}_{r\bar{o}}$, $\mathcal{X}_{r\bar{o}}$ and $\mathcal{X}_{r\bar{o}}$ spans the full state space. Notice that the decomposition is not unique because only the subspace $\mathcal{X}_{r\bar{o}}$ is unique.

Combining the representation (6.19) and (6.20) we find that a linear system can be transformed to the form

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & 0 & C_2 & 0 \end{pmatrix} x \end{aligned} \tag{6.21}$$

where the state vector has been partitioned as

$$x = \begin{pmatrix} x_{ro} \\ x_{r\bar{o}} \\ x_{\bar{r}o} \\ x_{\bar{r}\bar{o}} \end{pmatrix}^T$$

A block diagram of the system is shown in Figure 6.13. By tracing the arrows in the diagram we find that the input influences the systems S_{oc} and $S_{\bar{o}c}$ and that the output is influenced by S_{oc} and $S_{o\bar{c}}$. The system $S_{\bar{o}\bar{c}}$ is neither connected to the input nor the output. The transfer function of the system is

$$G(s) = C_1(sI - A_{11})^{-1}B_1 \tag{6.22}$$

It thus follows that the transfer function is uniquely given by the reachable and observable subsystem S_{oc}. When cancellations occur it means that there ares subsystems that are not reachable and observable.

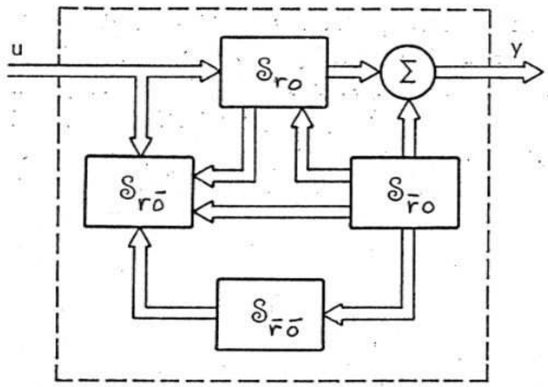

Figure 6.13: Kalman's decomposition of a linear system with distinct eigenvalues.

6.8 Laplace Transforms

Transfer functions are traditionally introduced using Laplace transforms. Students who are familiar with Laplace transform will get additional insight from this. Traditionally Laplace transforms were also used to compute responses of linear system to different stimuli. Today we can easily generate the responses using computers. Only a few elementary properties are needed for basic control applications. There is a however a beautiful theory for Laplace transforms which makes it possible to use many powerful tools of the theory of functions of a complex variable to get deep insights into the behavior of systems.

Consider a time function $f : R^+ \to R$ which is integrable and grows no faster than $e^{s_0 t}$ for large t. The Laplace transform maps f to a function $F = \mathcal{L}f : C \to C$ of a complex variable. It is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt, \mathrm{Re}s > s_0 \tag{6.23}$$

The transform has some properties that makes it very well suited to deal with linear systems. First we observe that the transform is linear because

$$\begin{aligned}\mathcal{L}(af + bg) &= \int_0^\infty e^{-st}(af(t) + bg(t))dt \\ &= a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt = a\mathcal{L}f + b\mathcal{L}g\end{aligned} \tag{6.24}$$

Next we will calculate the Laplace transform of the derivative of a function. We have

$$\mathcal{L}\frac{df}{dt} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt = -f(0) + s\mathcal{L}f$$

where the second equality is obtained by integration by parts. We thus obtain the following important formula for the transform of a derivative

$$\mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(0) = sF(s) - f(0) \tag{6.25}$$

This formula is particularly simple if the initial conditions are zero because *it follows that differentiation of a function corresponds to multiplication of the transform with s.*

Since differentiation corresponds to multiplication with s we can expect that integration corresponds to division by s. This is true as can be seen by calculating the Laplace transform of an integral. We have

$$\begin{aligned}\mathcal{L}\int_0^t f(\tau)d\tau &= \int_0^\infty \Big(e^{-st}\int_0^t f(\tau)d\tau\Big)dt\\ &= -\frac{e^{-st}}{s}\int_0^t e^{-s\tau} f(\tau)d\tau\Big|_0^\infty + \int_0^\infty \frac{e^{-s\tau}}{s} f(\tau)d\tau = \frac{1}{s}\int_0^\infty e^{-s\tau} f(\tau)d\tau\end{aligned}$$

hence

$$\mathcal{L}\int_0^t f(\tau)d\tau = \frac{1}{s}\mathcal{L}f = \frac{1}{s}F(s) \tag{6.26}$$

Integration of a time function thus corresponds to dividing the Laplace transform by s.

The Laplace Transform of a Convolution

Consider a linear time-invariant system with zero initial state. The relation between the input u and the output y is given by the convolution integral

$$y(t) = \int_0^\infty g(t-\tau)u(\tau)d\tau.$$

We will now consider the Laplace transform of such an expression. We have

$$\begin{aligned}Y(s) &= \int_0^\infty e^{-st} y(t)dt = \int_0^\infty e^{-st}\int_0^\infty g(t-\tau)u(\tau)d\tau dt\\ &= \int_0^\infty\int_0^t e^{-s(t-\tau)}e^{-s\tau} g(t-\tau)u(\tau)d\tau dt\\ &= \int_0^\infty e^{-s\tau}u(\tau)d\tau\int_0^\infty e^{-st}g(t)dt = G(s)U(s)\end{aligned}$$

The result can be written as $Y(s) = G(s)U(s)$ where G, U and Y are the Laplace transforms of g, u and y. The system theoretic interpretation is that the Laplace transform of the output of a linear system is a product of two terms, the Laplace transform of the input $U(s)$ and the Laplace transform of the impulse response of the system $G(s)$. A mathematical interpretation is that the Laplace transform of a convolution is the product of the transforms of the functions that are convoluted. The fact that the formula $Y(s) = G(s)U(s)$ is much simpler than a convolution is one reason why Laplace transforms have become popular in control.

The Transfer Function

The properties (6.24) and (6.25) makes the Laplace transform ideally suited for dealing with linear differential equations. The relations are particularly simple if *all initial conditions are zero.*

Consider for example a system described by (6.1). Taking Laplace transforms under the assumption that all initial values are zero gives

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

Elimination of $X(s)$ gives

$$Y(s) = \Big(C[sI - A]^{-1}B + D\Big)U(s) \tag{6.27}$$

The transfer function is thus $G(s) = C[sI - A]^{-1}B + D$. Compare with (6.2).

The formula (6.27) has a strong intuitive interpretation because it tells that the Laplace transform of the output is the product of the transfer function of the system and the transform of the input. In the transform domain the action of a linear system on the input is simply a multiplication with the transfer function. The transfer function is a natural generalization of the concept of gain of a system.

6.9 Further Reading

The concept of transfer function was an important part of classical control theory, see James Nichols Phillips. It was introduced via the Laplace transform Gardner Barnes which also was used to calculate response of linear systems. The Laplace transform is of less importance nowadays when responses to linear systems can easily be generated using computers.

6.10 Exercises

1. The linearized model of the pendulum in the upright position is characterized by the matrices
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$
Determine the transfer function of the system.

2. Consider the differential equation
$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \ldots + a_n y = 0$$
Let λ be a root of the polynomial
$$s^n + a_1 s^{n-1} + \cdots + a_n = 0.$$
Show that the differential equation has the solution $y(t) = e^{\lambda t}$.

3. Consider the system
$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + b_2 \frac{d^{n-2} u}{dt^{n-2}} + \ldots + b_n u,$$
Let λ be a zero of the polynomial
$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n$$
Show that if the input is $u(t) = e^{\lambda t}$ then there is a solution to the differential equation that is identically zero.

4. Consider the linear state space system
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned}$$
Show that the transfer function is
$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$
where
$$\begin{aligned} b_1 &= CB \\ b_2 &= CAB + a_1 CB \\ b_3 &= CA^2 B + a_1 CAB + a_2 CB \\ &\vdots \\ b_n &= CA^{n-1} B + a_1 CA^{n-1} B + \ldots + a_{n-1} CB \end{aligned}$$