



## CDS 101: Lecture 9.1 Limits of Performance



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24 November 2003

### Goals:

- Describe limits of performance on feedback systems
- Introduce Bode's integral formula and the "waterbed" effect
- Show some of the limitations of feedback due to RHP poles and zeros

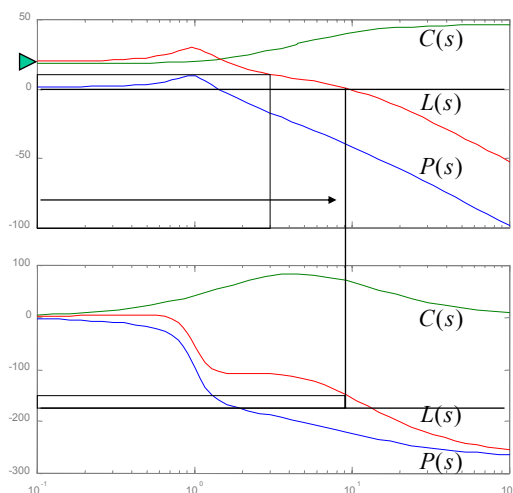
### Reading:

- Åström and Murray, *Analysis and Design of Feedback Systems*, Ch 9
- *Advanced*: Lewis, Chapter 9

## Review from Last Week

### Loop Shaping for Stability & Performance

- Steady state error, bandwidth, tracking



### Main ideas

- Performance specifications give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, lead, PI



### “Gang of Four”

$$\begin{bmatrix} e \\ u \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC} \\ \frac{C}{1+PC} & PC & \frac{C}{1+PC} \\ \frac{PC}{1+PC} & P & \frac{PC}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

**Noise and disturbances**

- d = process disturbances
- n = sensor noise
- Keep track of transfer functions between all possible inputs and outputs

**Design represents a tradeoff between the quantities**

- Keep L=PC large for good performance ( $H_{er} \ll 1$ )
- Keep L=PC small for good noise rejection ( $H_{yn} \ll 1$ )

**Four unique transfer functions define performance**

- Stability is always determined by  $1/(1+PC)$
- Numerator determined by forward path between input and output

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### Algebraic Constraints on Performance

**Goal: keep S & T small**

- S small  $\Rightarrow$  low tracking error
- T small  $\Rightarrow$  good noise rejection (and robustness [CDS 110b])

**Problem: S + T = 1**

- Can't make *both* S & T small at the same frequency
- Solution: keep S small at low frequency and T small at high frequency
- Loop again interpretation: keep L large at low frequency, and small at high frequency

$$H_{er} = \frac{1}{1+PC} =: S \quad \text{Sensitivity function}$$

$$H_{yn} = \frac{PC}{1+PC} =: T \quad \text{Complementary sensitivity function}$$

- Transition between large gain and small gain complicated by stability (phase margin)

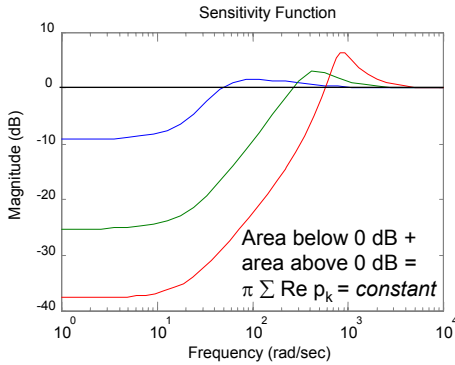
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### Bode's Integral Formula and the Waterbed Effect

**Bode's integral formula for  $S = 1/(1+PC) = 1/(1+L)$ :**

- Let  $p_k$  be the *unstable* poles of  $L(s)$  and assume relative degree of  $L(s) \geq 2$
- **Theorem:** the area under the sensitivity function is a conserved quantity:

$$\int_0^\infty \log |S(j\omega)| d\omega = \int_0^\infty \log \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re } p_k$$



**Waterbed effect:**

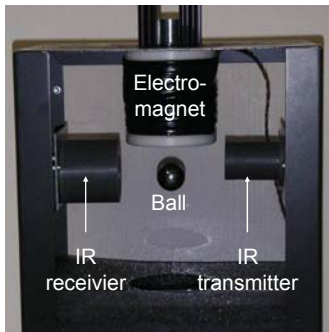
- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in  $\omega$ , Bode plots are logarithmic

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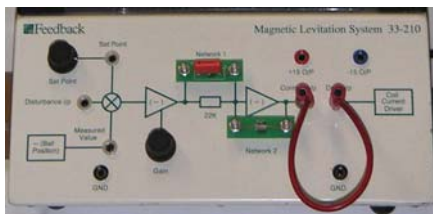
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### Example: Magnetic Levitation



**System description**

- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States:  $z, \dot{z}$
- Dynamics:  $F = ma, F = \text{magnetic force generated by wire coil}$
- See MATLAB handout for details



**Controller circuit**

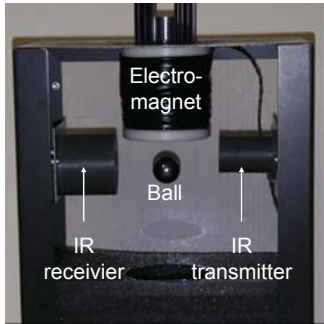
- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current

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### Equations of Motion



**Process: actuation, sensing, dynamics**

$$m\ddot{z} = k_m(k_A u)^2 / z^2 - mg$$

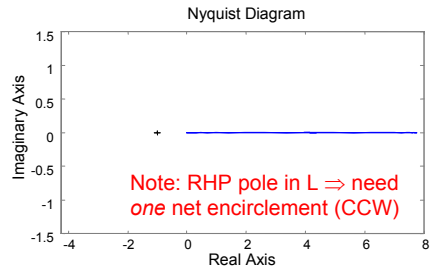
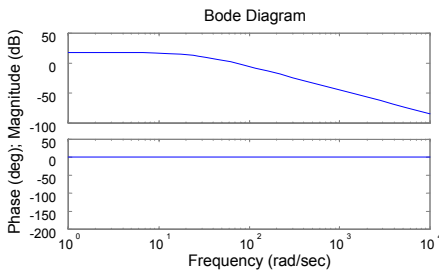
$$v_{ir} = k_T z + v_0$$

- $u$  = current to electromagnet
- $v_{ir}$  = voltage from IR sensor

**Linearization:**

$$P(s) = \frac{k}{s^2 - r^2} \quad k, r > 0$$

- Poles at  $s = \pm r \Rightarrow$  open loop unstable



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### Control Design

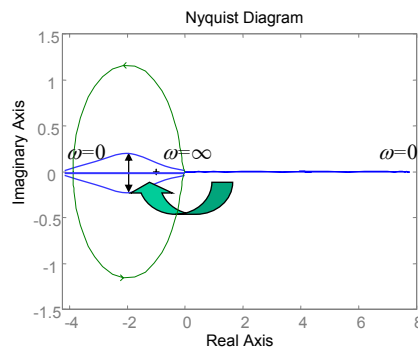
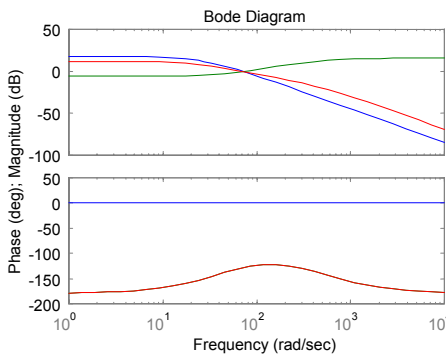
**Need to create encirclement**

- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

**Can accomplish using a lead compensator**

- Produce phase lead at crossover
- Generates loop in Nyquist plot

$$C(s) = -k \frac{s + a}{s + b}$$



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### Performance Limits

**Nominal design gives low perf**

- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

**Bode integral limits improvement**

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi r$$

- Must increase sensitivity at some point

Sensitivity Function

Step Response

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### Right Half Plane Zeros

**Right half plane zeros produce “non-minimum phase” behavior**

- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move *opposite* from input for a short period of time

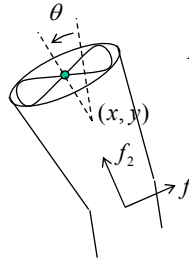
**Example:**  $H_1(s) = \frac{s+a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$  vs  $H_2(s) = \frac{s-a}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Bode Diagrams

Step Response

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### Example: Lateral Control of the Ducted Fan

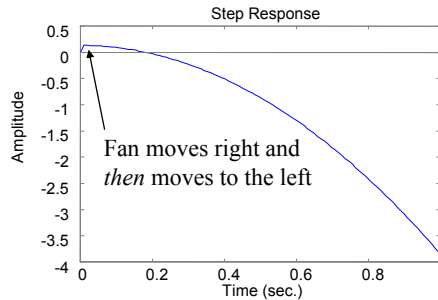


$$H_{xf_1}(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: 0, 0,  $-\sigma \pm j \omega_d$
- Zeros:  $\pm \sqrt{mgl}$

#### Source of non-minimum phase behavior

- To move left, need to make  $\theta > 0$
- To generate positive  $\theta$ , need  $f_1 > 0$
- Positive  $f_1$  causes fan to move *right* initially
- Fan starts to move left after short time (as fan rotates)



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### Stability in the Presence of Zeros

#### Loop gain limitations

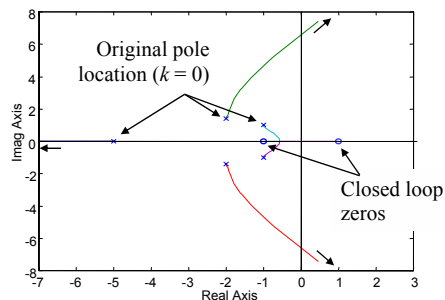
- Poles of closed loop = poles of  $1 + L$ . Suppose  $C = k n_c/d_c$ , where  $k$  is the gain of the controller

$$1 + L = 1 + k \frac{n_c n_p}{d_c d_p} = \frac{d_c d_p + k n_c n_p}{d_c d_p}$$

- For large  $k$ , closed loop poles approach open loop zeros
- RHP zeros limit maximum gain  $\Rightarrow$  serious design constraint!

#### Root locus interpretation

- Plot location of eigenvalues as a function of the loop gain  $k$
- Can show that closed loop poles go from open loop poles ( $k = 0$ ) to open loop zeros ( $k = \infty$ )
- More details in CDS 110 on Wed



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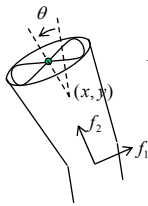
### Additional performance limits due to RHP zeros

**Another waterbed-like effect:** look at maximum of  $H_{er}$  over frequency range:

$$M_1 = \max_{\omega_1 \leq \omega \leq \omega_2} |H_{er}(j\omega)| \quad M_2 = \max_{0 \leq \omega \leq \infty} |H_{er}(j\omega)|$$

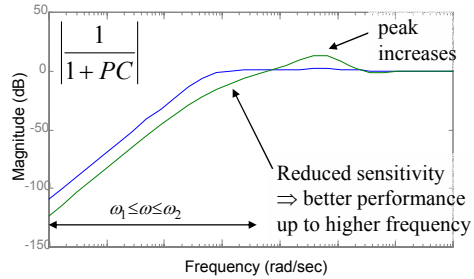
**Thm:** Suppose that  $P$  has a RHP zero at  $z$ . Then there exist constants  $c_1$  and  $c_2$  (depending on  $\omega_1, \omega_2, z$ ) such that  $c_1 \log M_1 + c_2 \log M_2 \geq 0$ .

- $M_1$  typically  $\ll 1 \Rightarrow M_2$  must be larger than 1 (since sum is positive)
- If we increase performance in active range (make  $M_1$  and  $H_{er}$  smaller), we must lose performance ( $H_{er}$  increases) some place else
- Note that this affects *peaks* not integrals (different from RHP poles)



$$H(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles:  $0, 0, -\sigma \pm j \omega_d$
- Zeros:  $\pm \sqrt{mgl}$



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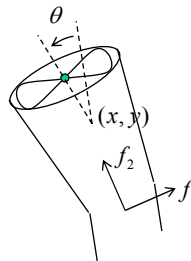
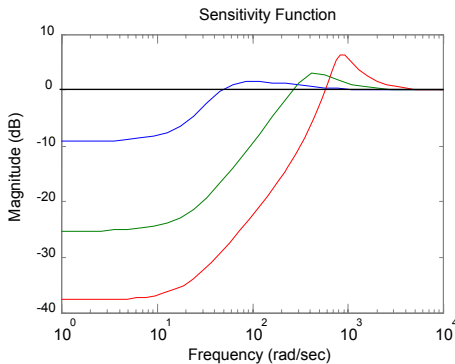
### Summary: Limits of Performance

**Many limits to performance**

- Algebraic:  $S + T = 1$
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of  $S$

**Main message: try to avoid RHP poles and zeros whenever possible (eg, re-design)**

$$\int_0^\infty \log |S(j\omega)| d\omega = \int_0^\infty \log \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re } p_k$$



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