CDS 101: Lecture 7.1
Loop Analysis of Feedback Systems
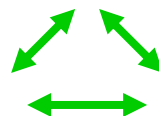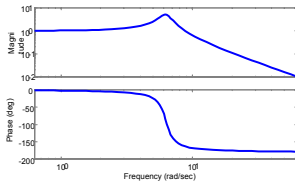
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10 November 2003

Goals:
- Show how to compute closed loop stability from open loop properties
- Describe the Nyquist stability criterion for stability of feedback systems
- Define gain and phase margin and determine it from Nyquist and Bode plots

Reading:
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 7
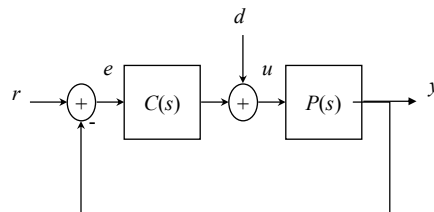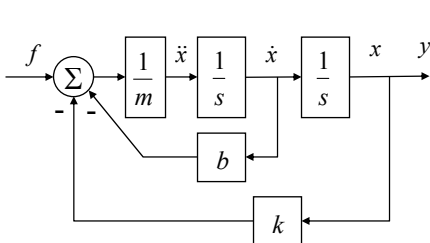- *Advanced*: Lewis, Chapter 7

Review from Last Week

$$u = A\sin(\omega t)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

$$y_{ss} = |H(j\omega)|A \cdot$$
$$\sin(\omega t + \angle H(j\omega))$$

$$H(s) = C(sI - A)^{-1}B + D$$

$$H_{y_2 u_1} = H_{y_2 u_2} H_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$

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Closed Loop Stability

Q: how do open loop dynamics affect the closed loop stability?
- Given open loop transfer function $C(s)P(s)$ determine when system is stable

Brute force answer: compute poles closed loop transfer function

$$H_{yr} = \frac{PC}{1+PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of H_{yr} = zeros of $1 + PC$
- Easy to compute, but not so good for design

Alternative: look for conditions on *PC* that lead to instability
- Example: if $PC(s) = -1$ for some $s = j\omega$, then system is *not* asymptotically stable
- Condition on *PC* is much nicer because we can *design PC(s)* by choice of *C(s)*
- However, checking $PC(s) = -1$ is not enough; need more sophisticated check

Game Plan: Frequency Domain Design

Goal: figure out how to *design C(s)* so that 1+C(s)P(s) is stable *and* we get good performance
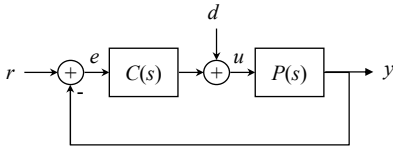
$$H_{yr} = \frac{PC}{1+PC}$$

- Poles of H_{yr} = zeros of $1 + PC$
- Would also like to "shape" H_{yr} to specify performance at different frequencies

- Low frequency range:

$$PC \gg 1 \;\Rightarrow\; \frac{PC}{1+PC} \approx 1$$

(good tracking)
- Bandwidth: frequency at which closed loop gain = ½ ⇒ open loop gain ≈ 1
- Idea: use *C(s)* to *shape PC* (under certain constraints)
- Need tools to analyze stability and performance for closed loop given *PC*

Nyquist Criterion

Determine stability from (open) loop transfer function, $L(s) = P(s)C(s)$.
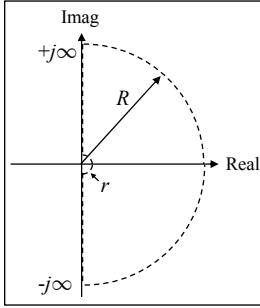
• Use "principle of the argument" from complex variable theory (see reading)

Thm (Nyquist). Consider the Nyquist plot for loop transfer function $L(s)$. Let

P # RHP poles of $L(s)$

N # clockwise encirclements of -1

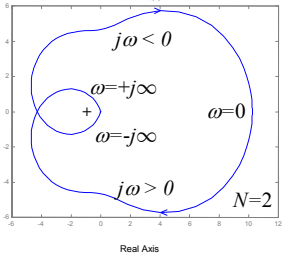Z # RHP zeros of $1 + L(s)$

Then

$$Z = N + P$$

• Nyquist "D" contour
• Take limit as $r \to 0$, $R \to \infty$
• Trace from $-\infty$ to $+\infty$ along imaginary axis
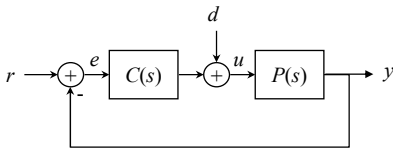• Trace frequency response for $L(s)$ along the Nyquist "D" contour
• Count net # of clockwise encirclements of the -1 point

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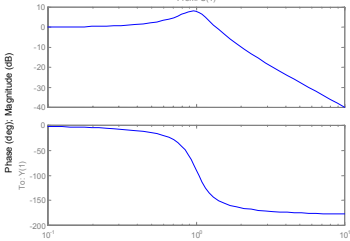# Simple Interpretation of Nyquist

Basic idea: avoid positive feedback

• If $L(s)$ has 180° phase (or greater) and gain greater than 1, then signals are amplified around loop
• Use when phase is monotonic
• General case requires Nyquist

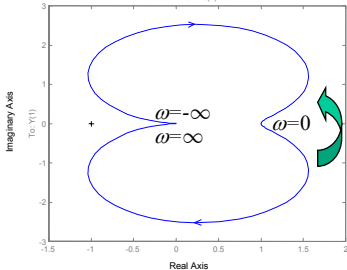**Can generate Nyquist plot from Bode plot + reflection around real axis**
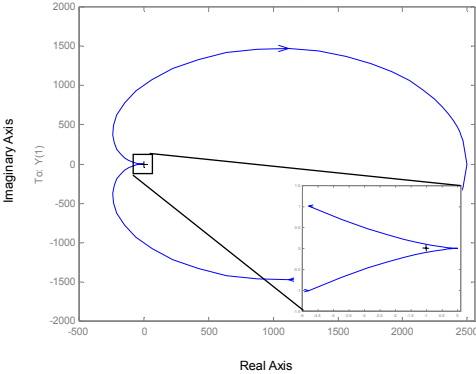
bode(sys)   nyquist(sys)
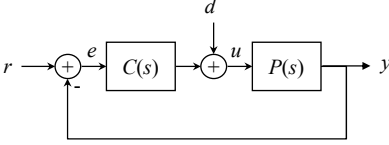
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Example: Proportional + Integral* speed controller

$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

Remarks

- $N = 0$, $P = 0 \Rightarrow Z = 0$ (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don't have to compute closed loop response

* slightly modified; more on the design of this compensator in next week's lecture
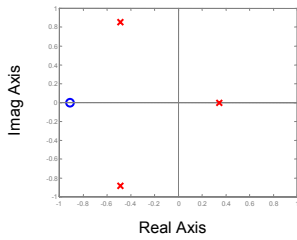
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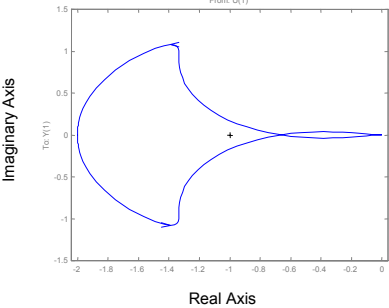

More complicated systems

What happens when open loop plant has RHP poles?

- 1 + PC has singularities inside D countour $\Rightarrow$ these must be taken into account

$$L(s) = \frac{s + 1}{s - 0.5} \cdot \frac{1}{s^2 + s + 1}$$

unstable pole

$$N = -1, P = 1 \Rightarrow Z = N+P = 0 \text{ (stable)}$$

$$\frac{1}{1 + L} = \frac{s + 1}{(s + 0.35)(s + 0.07 + 1.2j)(s + 0.07 - 1.2j)} \checkmark$$
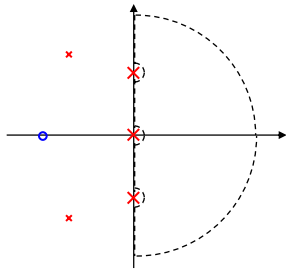
11 Nov 02   R. M. Murray, Caltech CDS   8
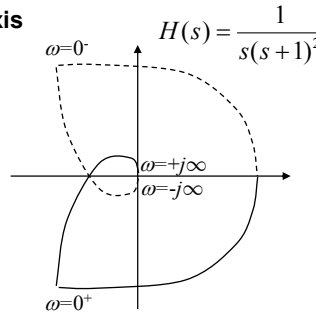
Comments and cautions

Why is the Nyquist plot *useful*?

- Old answer: easy way to compute stability (before computers and MATLAB)
- Real answer: gives *insight* into stability and robustness; very useful for reasoning about stability

Nyquist plots for systems with poles on the *jω* axis

$$H(s) = \frac{1}{s(s+1)^2}$$

- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\epsilon+0j)$ to determine direction

$\omega=0^-$

$\omega=+j\infty$
$\omega=-j\infty$

$\omega=0^+$

Cautions with using MATLAB

- MATLAB doesn't generate portion of plot for poles on imaginary axis
- These must be drawn in by hand (make sure to get the orientation right!)

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Relative stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not *how* stable
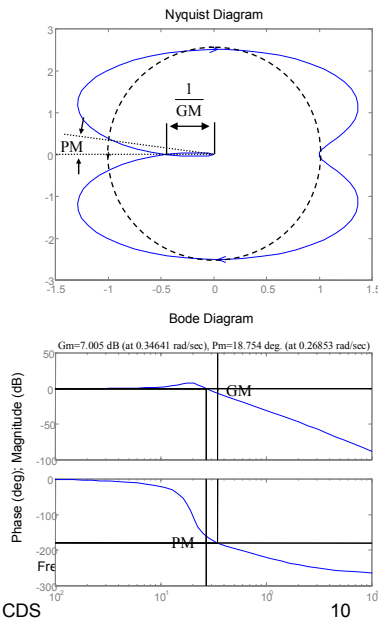
Gain margin

- How much we can modify the *loop gain* and still have the system be stable
- Determined by the location where the loop transfer function crosses 180° phase

Phase margin

- How much we can add "phase delay" and still have the system be stable
- Determined by the phase at which the loop transfer function has unity gain

Bode plot interpretation

- Look for gain = 1, 180° phase crossings
- MATLAB: margin(sys)

Nyquist Diagram

$\frac{1}{\text{GM}}$

PM

Bode Diagram

Gm=7.005 dB (at 0.34641 rad/sec), Pm=18.754 deg. (at 0.26853 rad/sec)

GM

PM

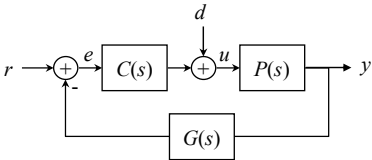11 Nov 02   R. M. Murray, Caltech CDS   10
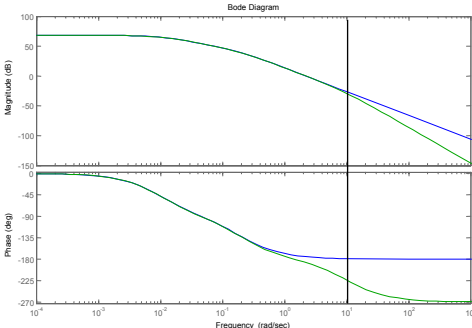
Example: cruise control

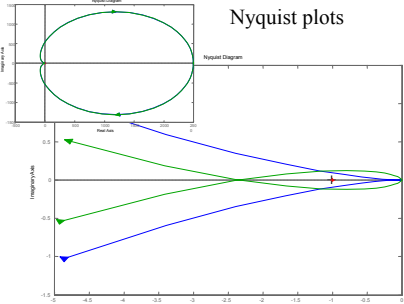$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

$$G(s) = \frac{10}{s + 10}$$

Effect of additional sensor dynamics
- New speedometer has pole at $s = 10$ (very fast); problems develop in the field
- What's the problem? A: insufficient phase margin in original design (not robust)
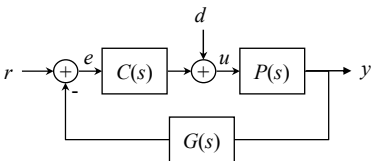

Nyquist plots

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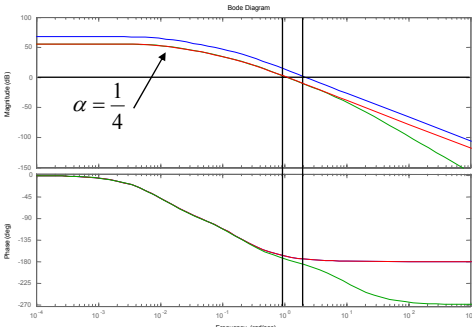
Preview: control *design*

$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s + a}$$
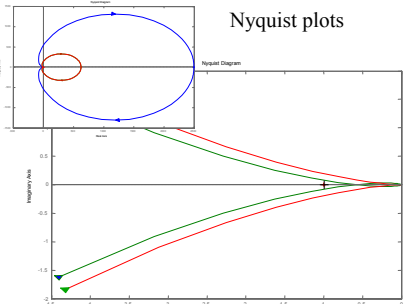
$$C(s) = \alpha \left(K_p + \frac{K_i}{s + 0.01} \right)$$

$$G(s) = \frac{10}{s + 10}$$

Approach: Increase phase margin
- Increase phase margin by reducing gain $\Rightarrow$ can accommodate new sensor dynamics
- Tradeoff: lower gain at low frequencies $\Rightarrow$ less bandwidth, larger steady state error
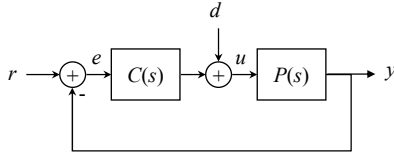

$\alpha = \frac{1}{4}$

Nyquist plots

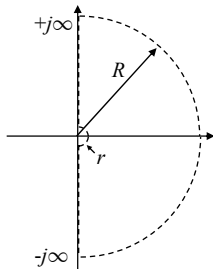11 Nov 02   R. M. Murray, Caltech CDS   12
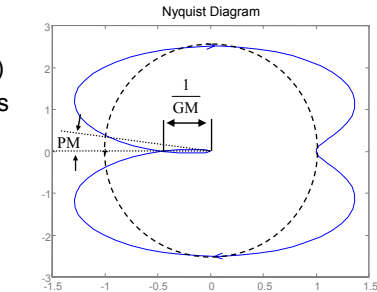
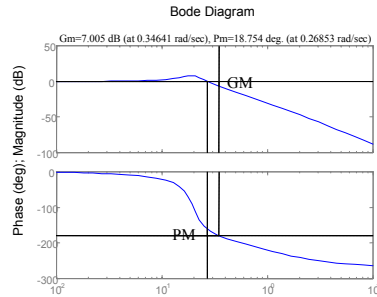## Summary: Loop Analysis of Feedback Systems

- Nyquist criteria for loop stability
- Gain, phase margin for robustness

Thm (Nyquist).
P # RHP poles of $L(s)$
N # CW encirclements
Z # RHP zeros

$$Z = N + P$$

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What's Next

Homework problems
- Nyquist plots; gain, phase margins
- Cruise control design
- CDS 110: PI control + time delay

Wednesday
- Nyquist analysis
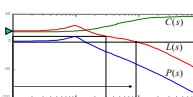- Control of second order systems
- Time delays

Friday
- Hideo Mabuchi, Closed-loop atomic magnetometry (quantum control)
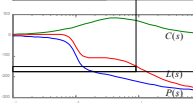- Friday, 2 pm, 74 JRG

Next week: PID Control
- Design of controllers using PID
- Relative stability and performance

Don't forget to fill out MUD CARDS

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