CDS 101: Lecture 5.1
Reachability and State Space Feedback

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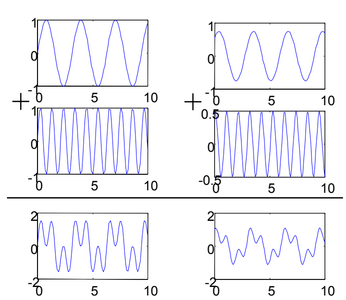
27 October 2003

Goals:
- Define reachability of a control system
- Give tests for reachability of linear systems and apply to examples
- Describe the design of state feedback controllers for linear systems

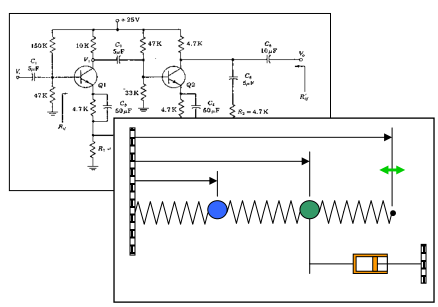
Reading:
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 5
- Packard, Poola and Horowitz, Dynamic Systems and Feedback, Section 25
- *Advanced*: A. D. Lewis, *A Mathematical Introduction to Feedback Control*, Chapter 2 (available on web page)

Review from Last Week

$u \rightarrow$
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$
$\rightarrow y$



$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$



Properties of linear systems
- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems

Control Design Concepts

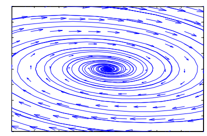
System description: single input, single output system (MIMO also OK)

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n, x(0) \text{ given}$$
$$y = h(x,u) \qquad u \in \mathbb{R}, y \in \mathbb{R}$$

Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" $u=\alpha(x)$ such that

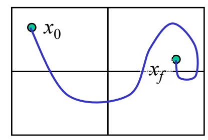
$$\lim_{t\to\infty} x(t) = x_e \text{ for all } x(0) \in \mathbb{R}^n$$



Reachability: steer the system between two points

- Given x_0, $x_f \in \mathbb{R}^n$, find an input $u(t)$ such that

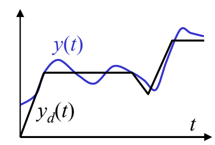
$$\dot{x} = f(x,u(t)) \text{ takes } x(t_0) = x_0 \to x(T) = x_f$$



Tracking: track a given output trajectory

- Given $y_d(t) \in \mathbb{R}$, find $u=\alpha(x,t)$ such that

$$\lim_{t\to\infty} \left(y(t) - y_d(t) \right) = 0 \text{ for all } x(0) \in \mathbb{R}^n$$



Reachability of Input/Output Systems

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n, x(0) \text{ given}$$
$$y = h(x,u) \qquad u \in \mathbb{R}, y \in \mathbb{R}$$

Defn An input/output system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ and any time $T > 0$ there exists an input $u:[0,T] \to \mathbb{R}$ such that the solution of the dynamics starting from $x(0)=x_0$ and applying input $u(t)$ gives $x(T)=x_f$.

Remarks

- In the definition, x_0 and x_f do not have to be equilibrium points $\Rightarrow$ we don't necessarily stay at x_f after time T.
- Reachability is defined in terms of states $\Rightarrow$ doesn't depend on output
- For *linear systems,* can characterize reachability by looking at the general solution:

$$\dot{x} = Ax + Bu \qquad\qquad x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$
$$y = Cx + Du$$

- If integral is "surjective" (as a linear operator), then we can find an input to achieve any desired final state.

Tests for Reachability

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, x(0) \text{ given}$$
$$y = Cx + Du \qquad u \in \mathbb{R}, y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Thm A linear system is reachable if and only if the $n \times n$ *reachability matrix*

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

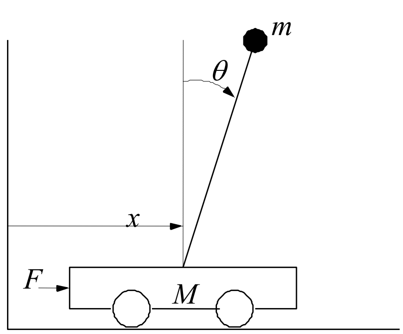
is full rank.

Remarks

- *Very* simple test to apply. In MATLAB, use ctrb(A,B) and check rank w/ det()
- If this test is satisfied, we say "the pair (A,B) is reachable"
- Some insight into the proof can be seen by expanding the matrix exponential

$$e^{A(T-\tau)}B = \left(I + A(T-\tau) + \frac{1}{2}A^2(T-\tau)^2 + \cdots + \frac{1}{2}A^{n-1}(T-\tau)^{n-1} + \cdots \right)B$$
$$= \left(B + AB(T-\tau) + \frac{1}{2}A^2B(T-\tau)^2 + \cdots + \frac{1}{2}A^{n-1}B(T-\tau)^{n-1} + \cdots \right)$$

Example #1: Linearized pendulum on a cart



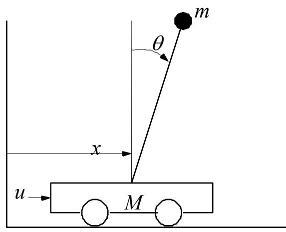
Question: can we locally control the position of the cart by proper choice of input?

Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2gl^2}{J(M+m)+Mml^2} & \frac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \frac{mgl(M+m)}{J(M+m)+Mml^2} & \frac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J+ml^2}{J(M+m)+Mml^2} \\ \frac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2gl^2}{q} & \frac{-(J+ml^2)b}{q} & 0 \\ 0 & \frac{mgl(M+m)}{q} & \frac{-mlb}{q} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J+ml^2}{q} \\ \frac{ml}{q} \end{bmatrix} u$$

$$q = J(M+m) + Mml^2$$

- Simplify by setting b = 0

Reachability matrix

$$M_c = \begin{bmatrix} 0 & \frac{J+ml^2}{q} & 0 & \frac{ml(J+ml^2)}{q^2} \\ 0 & \frac{ml}{q} & 0 & \frac{m^2gl^2(M+m)}{q^2} \\ \frac{J+ml^2}{q} & 0 & \frac{ml(J+ml^2)}{q^2} & 0 \\ \frac{ml}{q} & 0 & \frac{m^2gl^2(M+m)}{q^2} & 0 \end{bmatrix}$$

$B \quad AB \quad A^2B \quad A^3B$

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- $\Rightarrow$ reachable as long as $g(M+m) \neq 1$
- $\Rightarrow$ can "steer" linearization between points by proper choice of input

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Control Design Concepts

System description: single input, single output system (MIMO also OK)

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n, x(0) \text{ given}$$
$$y = h(x,u) \qquad u \in \mathbb{R}, y \in \mathbb{R}$$

Stability: stabilize the system around an equilibrium point
- Given equilibrium point xe 2 ℝn, find control "law" u=α(x) such that

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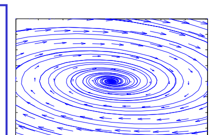
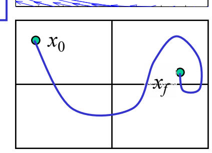


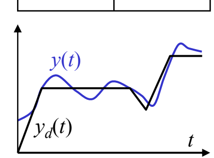
✓ **Reachability: steer the system between two points**
- Given x_0, $x_f \in \mathbb{R}^n$, find an input u(t) such that

$$\dot{x} = f(x,u(t)) \text{ takes } x(t_0) = x_0 \rightarrow x(t_f) = x_f$$



Tracking: track a given output trajectory
- Given $y_d(t) \in \mathbb{R}$, find $u=\alpha(x,t)$ such that

$$\lim_{t\to\infty}\left(y(t) - y_d(t)\right) = 0 \text{ for all } x(0) \in \mathbb{R}^n$$



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State space controller design for linear systems

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, x(0) \text{ given}$$
$$y = Cx + Du \qquad u \in \mathbb{R}, y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Goal: find a linear control law $u=Kx$ such that the closed loop system

$$\dot{x} = Ax + BKx = (A + BK)x$$

is stable at $x_e=0$.

Remarks

- Stability based on eigenvalues ⇒ use K to make eigenvalues of $(A+BK)$ stable
- Can also link eigenvalues to *performance* (eg, initial condition response)
- Question: when can we place the eigenvalues anyplace that we want?

Theorem The eigenvalues of $(A+BK)$ can be set to arbitrary values if and only if the pair (A,B) is reachable.

MATLAB: K = place(A, B, eigs)

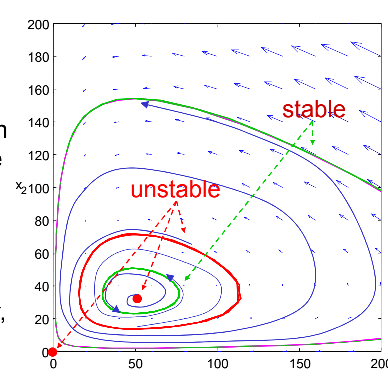
Example #2: Predator prey

Natural dynamics

$$\dot{x}_1 = b_r x_1 - a x_1 x_2$$
$$\dot{x}_2 = a x_1 x_2 - d_f x_2$$

Controlled dynamics: modulate food supply

$$\dot{x}_1 = b_r(1+u)x_1 - a x_1 x_2$$
$$\dot{x}_2 = a x_1 x_2 - d_f x_2$$



Q1: can we move from some initial population of foxes and rabbits to a specified one in time *T* by modulation if the food supply?

Q2: can we *stabilize* the population around the desired equilibrum point

Approach: try to answer this question *locally*, around the natural equilibrium point

Example #2: Problem setup

Equilibrium point calculation

$$\dot{x}_1 = b_r(1+u)x_1 - a x_1 x_2$$
$$\dot{x}_2 = a x_1 x_2 - d_f x_2$$

- x_e = (50, 35)

```
% Compute the equil point
% predprey.m contains dynamics
f = inline('predprey(0,x)');
xeq = fsolve(f, [50,50]);

% Compute linearization
A = [
  br - a*xeq(2) - a*xeq(1);
  a*xeq(2), -df + a*xeq(1)
];
B = [br*xeq(1); 0];
```

Linearization

- Compute linearization around equil. point, x_e:

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

- Redefine local variables: $z=x-x_e$, $v=u-u_e$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -b_r - a x_{2,e} & -a x_{1,e} \\ a x_{2,e} & -d_f + a x_{1,e} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} b_r x_{1,e} \\ 0 \end{bmatrix} v$$

- Reachable? YES, if $b_r,\ a \neq 0$ (check [B AB]) $\Rightarrow$ can locally steer to any point

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Example #2: Stabilization via eigenvalue assignment

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -b_r - a x_{2,e} & -a x_{1,e} \\ a x_{2,e} & -d_f + a x_{1,e} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} b_r x_{1,e} \\ 0 \end{bmatrix} v$$

Control design:

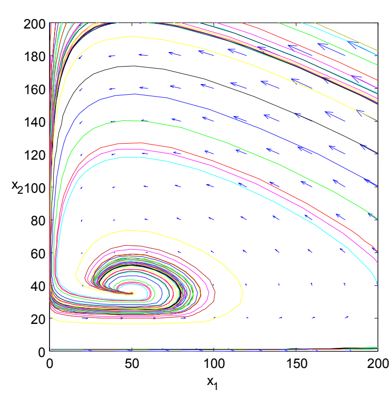
$$v = -Kz = -K(x - x_e)$$
$$u = u_e + v = u_e - K(x - x_e)$$

Place poles at stable values

- Choose λ=-1, -2
- K = place(A, B, [-1; -2]);

Modify dynamics to include control

$$\dot{x}_1 = b_r(1 - K(x - x_e))x_1 - a x_1 x_2$$
$$\dot{x}_2 = a x_1 x_2 - d_f x_2$$



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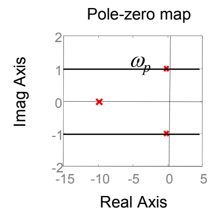
Implementation Details

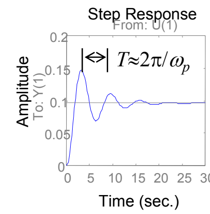
Eigenvalues determine performance

- For each eigenvalue $\lambda_i = \sigma_i + j\omega_i$, get contribution of the form

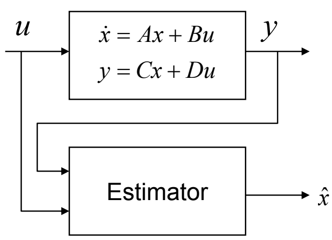
$$y_i(t) = e^{-\sigma t}\left(a\sin(\omega t) + b\cos(\omega t)\right)$$

- Repeated eigenvalues can give additional terms of the form $t^k e^{\sigma + j\omega}$





Use *estimator* to determine the current state if you can't measure it



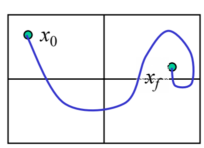
- Estimator looks at inputs and outputs of plant and estimates the current state
- Can show that if a system is *observable* then you can construct and estimator
- Use the *estimated* state as the feedback

$$u = K\hat{x}$$

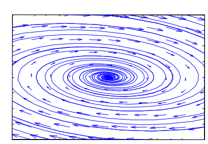
- *Kalman* filter is an example of an estimator

Summary: Reachability and State Space Feedback

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$



$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$



$$u = u_e - K(x - x_e)$$

Key concepts

- Reachability: find u s.t. $x_0 \rightarrow x_f$
- Reachability rank test for linear systems
- State feedback to assign eigenvalues