29 Root Locus

Given two polynomials N and D, the goal of the Root Locus method (developed by W.R. Evans, in 1948-1954) is to accurately sketch the roots of the polynomial

$$kN(s) + D(s) = 0$$

as k varies from $0 \to \infty$. This is to be done under the following assumptions:

- N and D are polynomials with real-valued coefficients with leading coefficients equal to 1
- The roots of N and D are individually known. Moreover, since N and D have real coefficients, we know that the roots of N and D are real and/or come in complex-conjugate pairs.
- D and N have no common roots
- $n := \mathrm{ord}(D) \geq \mathrm{ord}(N) =: m$

Note, if N and D do have a common root (or common roots), at say α, then there exist polynomials $\tilde{N}$ and $\tilde{D}$ such that

$$N(s) = (s - \alpha)\,\tilde{N}(s), \qquad D(s) = (s - \alpha)\,\tilde{D}(s)$$

Hence, for every k, we have

$$kN(s) + D(s) = (s - \alpha)\left[k\tilde{N}(s) + \tilde{D}(s)\right]$$

Therefore, the roots of $kN(s) + D(s) = 0$ are simply α (regardless of k) along with the roots of $k\tilde{N}(s) + \tilde{D}(s) = 0$. In this manner, you can pre-factor out any common roots, until you obtain an $\hat{N}$ and $\hat{D}$ that have no common roots, and proceed using the method described below.

29.1 Motivation

Suppose that L is a linear system, with transfer function

$$L(s) = \frac{N(s)}{D(s)}$$

Consider the feedback system, with proportional feedback

The closed-loop characteristic equation is $kN(s) + D(s) = 0$. Hence, in order to choose an appropriate value for k, it would be useful to know how the roots of $kN(s) + D(s) = 0$ are functions of the parameter k.

This is the canonical example. For this reason, when explaining the Root Locus method, it is common to refer to the roots of N as the "zeros," while the roots of D are referred to as the "poles."

The canonical example seems to be of very limited usefulness. It only concerns the stability of a system with proportional gain feedback, and only considers the variations in the closed-loop poles as functions of the proportional gain value. But, in fact, many problems which appear to be more complicated can ultimately be cast in this manner.

Recall for a linear system, *stability* is a property of the system, and that we had two essentially equivalent definitions – either that all bounded inputs produce bounded outputs, or that under no input, all initial conditions decay exponentially to zero. Hence in a block diagram, the stability is determined by the feedback loops, not the exogenous inputs that enter through summing junctions.

Fact: Any block diagram connection of linear systems, summing junctions, and a single real parameter

can be massaged into the following diagram

where all of the external inputs are ignored (ie., set to zero). Here $-L$ is simply the overall transfer function that the parameter interacts with in feedback. The stability of the original diagram can be determined from studying the stability of L is negative feedback with the parameter ρ.

29.1.1 Examples

- K_P or K_I or K_D in a PID controller
- A parameter in the plant state equations
- The stiffness, k_L of the flexible cord in the 2-mass experimental setup.

29.2 Rules for Constructing Root Locus

Given N and D, the main theorem is as follows:

Theorem: Suppose k is real, and $s_0 \in \mathbf{C}$. Then s_0 is a root of the equation

$$kN(s) + D(s) = 0$$

if and only if $N(s_0) \neq 0$ and either

- $k = 0$ and $D(s_0) = 0$, or

- $k \neq 0$, and $D(s_0) \neq 0$, the quotient $\dfrac{D(s_0)}{N(s_0)}$ is real, and

$$-\frac{D(s_0)}{N(s_0)} = k$$

Proof:

$\Rightarrow$ Suppose $N(s_0) = 0$. Then, since N and D have no common roots, it mus be that $D(s_0) \neq 0$, and hence $kN(s_0) + D(s_0) = D(s_0) \neq 0$. If $k = 0$, then since $D(s_0) = -kN(s_0) = 0$. If $k \neq 0$, then since $N(s_0) \neq 0$, it follows that $D(s_0) \neq 0$ either. Dividing out gives that $k = -\frac{D(s_0)}{N(s_0)}$ as desired.

$\Leftarrow$ If $k = 0$ and $D(s_0) = 0$, then since $N(s_0)$ is just some finite number, we have $kN(s_0) + D(s_0) = 0$. If $k \neq 0$ and $D(s_0) \neq 0$ and $k = -\frac{D(s_0)}{N(s_0)}$, then multiplying out gives $kN(s_0) + D(s_0) = 0$ as desired. $\sharp$

So, in order to draw the root locus, without directly calculating any roots of $kN + D$, you "simply" do the following:

1. For <u>every</u> complex number λ, check to see if either $D(\lambda) = 0$, or if $\frac{N(\lambda)}{D(\lambda)}$ is real.

2. If $D(\lambda) = 0$, then λ is a root of $kN(s) + D(s)$ for $k = 0$. In this case, λ is said to "be on the root locus of the pair (N, D)."

3. If $D(\lambda) \neq 0$, but $\frac{N(\lambda)}{D(\lambda)}$ is real, then λ is a root of $kN(s)+D(s)$ for a real-valued k, namely $k := -\frac{D(\lambda)}{N(\lambda)}$. In this case:

   - If $k > 0$, then λ is said to *be on the Positive Root Locus of the pair* (N, D).
   - If $k < 0$, then λ *is on the Negative Root Locus of the pair* (N, D).

4. If $\frac{N(\lambda)}{D(\lambda)}$ is <u>not</u> real, then λ is <u>not</u> a root of $kN(s) + D(s) = 0$ for <u>any</u> real value of k. In this case, λ is not on the root locus of the pair (N, D).

Of course, you can't actually do this at every complex number - "so many complex numbers, so little time." But, there are a few basic rules that can be used to accurately get a good idea of what the roots, as functions of k, look like. The first rule follows from the reasoning above:

Basic Rule 1: For any complex number $s_0 \in \mathbf{C}$, it is either not on the root locus of (N, D), or it is on the root locus for one, and only one value of k. Put another way, if k_1 and k_2 are different real numbers, then it is impossible for a complex number s_0 to be a root of both $k_1N(s) + D(s) = 0$ and $k_2N(s) + D(s) = 0$. This fact *greatly* limits the complexity of the root locus diagram. This simple fact is not explicitly stated in most textbooks, and as such, causes students much grief the first few times sketching root locus plots.

Next, we recall a few basic facts from arithmetic:

1. if q and p are integers, then $q-p$ is odd if and only if $q+p$ is odd. Obviously then, $q-p$ is even if and only if $q+p$ is even.
2. If A and B are complex, then $\angle(AB) = \angle A + \angle B$
3. If A is complex, $A \neq 0$, then $\angle \frac{1}{A} = -\angle A$
4. If A is complex, $A \neq 0$, then A is real if and only if $\angle A$ is an integer multiple of π. Moreover,
   - $A > 0$ iff $\frac{1}{A} > 0$ iff $\angle A$ is an even multiple of π
   - $A < 0$ iff $\frac{1}{A} < 0$ iff $\angle A$ is an odd multiple of π
5. The n roots of -1 are $e^{j\pi(2q+1)/n}$, for $q = 0, 1, \ldots n-1$.
6. The n roots of 1 are $e^{j\pi 2q/n}$, for $q = 0, 1, \ldots n-1$.

Now, let $\{p_1, p_2, \cdots, p_n\}$ be the roots of $D(s) = 0$. Since we assume that the leading coefficient of D is one, it follows tha D can be factored as

$$D(s) = (s - p_1)(s - p_2)\cdots(s - p_n)$$

Similarly, let $\{z_1, z_2, \cdots, z_m\}$ be the roots of $N(s) = 0$. Hence, N can be factored as

$$N(s) = (s - z_1)(s - z_2)\cdots(s - z_m)$$

This yields the 2nd basic rule:

Basic Rule 2: A complex number s_0 is on the root locus of (N, D) if and only if for some integer q

$$\angle \frac{N(s_0)}{D(s_0)} = q\pi$$

Depending on whether q is even or odd determines whether the point s_0 is on the Positive or Negative Root Locus. Note that

$$\begin{aligned}\angle \tfrac{N(s_0)}{D(s_0)} = {} & [\angle(s_0 - z_1) + \angle(s_0 - z_2) + \cdots + \angle(s_0 - z_m)] \\ & - [\angle(s_0 - p_1) + \angle(s_0 - p_2) + \cdots + \angle(s_0 - p_n)]\end{aligned}$$

Moreover, since for any point on the root locus, the corresponding value of k is $-\frac{D}{N}$, it is easy to see that λ is on the Positive Root Locus if and only if

$$\begin{aligned}(2q+1)\pi &= [\angle(s_0 - z_1) + \angle(s_0 - z_2) + \cdots + \angle(s_0 - z_m)] \\ & \quad - [\angle(s_0 - p_1) + \angle(s_0 - p_2) + \cdots + \angle(s_0 - p_n)]\end{aligned}$$

for some integer q. Similarly, λ is on the Negative Root Locus if and only if

$$\begin{aligned} 2q\pi \;=\; & [\angle(s_0 - z_1) + \angle(s_0 - z_2) + \cdots + \angle(s_0 - z_m)] \\ & - [\angle(s_0 - p_1) + \angle(s_0 - p_2) + \cdots + \angle(s_0 - p_n)] \end{aligned}$$

for some integer q.

29.2.1 Number of Roots

Since $n \geq m$, for $K \geq 0$, the polynomial is always n'th order, hence there must always be n roots.

Special Case: When $n = m$, and $K = -1$, the polynomial is no longer n'th order. What happens is that as $k \to -1$ (from either side), at least one of the roots goes to ∞. For example, try the case $N(s) = s^2$ and $D(s) = s^2 + s + 1$.

29.2.2 Roots when $K = 0$

With $K = 0$, the polynomial becomes simply $D(s) = 0$. Hence, the roots are located at the roots of D.

29.2.3 Roots on real axis

Take any point $\lambda \in \mathbf{R}$, with λ not a root of N or D. Let $\#_{zR}$ be the number of real zeros (from the list of $z_1, z_2, \ldots, z_m$) that are to the right of λ. Also, let $\#_{pR}$ be the number of real poles (from the list of $p_1, p_2, \ldots, p_n$) that are to the right of

λ. From the picture below,

it follows that

$$\begin{aligned}&[\angle(s_0 - z_1) + \angle(s_0 - z_2) + \cdots + \angle(s_0 - z_m)] \\ &\qquad - [\angle(s_0 - p_1) + \angle(s_0 - p_2) + \cdots + \angle(s_0 - p_n)] \;=\; \pi\,(\#_{zR} - \#_{pR})\end{aligned}$$

This is *always an integer multiple of* π, hence every point on the real axis is part of the root locus of the pair (N, D). We can determine whether it is on the Positive or Negative root locus as follows:

- λ is on the Positive Root Locus iff $\#_{zR} - \#_{pR}$ is odd iff $\#_{zR} + \#_{pR}$ is odd
- λ is on the Negative Root Locus iff $\#_{zR} - \#_{pR}$ is even iff $\#_{zR} + \#_{pR}$ is even

29.2.4 Behavior as $K \to \infty$

Here, it is best to consider first a simple example: $N(s) = s+2$, $D(s) = s^2+5s+1$. Now, by quadratic formula, we have that the roots of $kN(s) + D(s) = 0$ are

$$\frac{-(5+k) \pm \sqrt{(5+k)^2 - 4(1+2k)}}{2}$$

Completing the square inside the square root gives that the roots are at

$$\frac{-(5+k) \pm \sqrt{(1+k)^2 + 20}}{2}$$

For very large values of k the roots are near

$$\frac{-(5+k) + (1+k)}{2}, \quad \frac{-(5+k) - (1+k)}{2}$$

which are at -2 and $-3-k$. Hence, as $k \to \infty$, one root goes to 2, while the other goes to $-\infty$.

This can be generalized to higher order N and D. First note that for large values of k, but finite values of s_0,

$$kN(s) + D(s) \approx kN(s)$$

Hence, as $k \to \pm\infty$, m of the roots of $kN(s) + D(s) = 0$ approach the m roots of $N(s) = 0$. The remaining $n - m$ roots get large as k gets large, in a predictable fashion. First, multiply out N and D and write them as

$$D(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

and

$$N(s) = s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m$$

Note that in doing so, it is easy to see that

$$a_1 = \sum_{i=1}^{n} -p_i = -\sum_{i=1}^{n} p_i, \qquad b_1 = \sum_{j=1}^{m} -z_j = -\sum_{j=1}^{m} z_j$$

(you can prove this by induction...). Also, since the collections $\{p_i\}_{i=1}^{n}$ and $\{z_j\}_{j=1}^{m}$ appear in complex-conjugate pairs, only the real parts matter in the sums,

$$a_1 = -\sum_{i=1}^{n} \mathrm{Re}(p_i), \qquad b_1 = -\sum_{j=1}^{m} \mathrm{Re}(z_j) \tag{95}$$

Now, suppose $N(\lambda) \neq 0$, dividing gives

$$\begin{aligned} -k &= \frac{D(\lambda)}{N(\lambda)} \\ &\approx \lambda^{n-m}\left[1 + \frac{a_1 - b_1}{\lambda} + \cdots\right] \end{aligned}$$

Therefore,

$$\begin{aligned} (-k)^{\frac{1}{n-m}} &\approx \lambda\left[1 + \frac{a_1 - b_1}{\lambda}\right]^{\frac{1}{n-m}} \\ &\approx \lambda\left[1 + \frac{a_1 - b_1}{\lambda(n-m)}\right] \\ &= \lambda + \frac{a_1 - b_1}{n-m} \end{aligned}$$

(to see these, note that a first-order Taylor's series of $(1+x)^r$ around $x = 0$ is $1 + rx$). Take the Positive Root Locus (so $k > 0$). Using the expression for the roots of -1 gives that

$$\lambda + \frac{a_1 - b_1}{n-m} = k^{\frac{1}{n-m}} e^{\frac{j\pi}{n-m}(2q+1)}$$

for $q = 0, 1, \ldots, n-m-1$. Solving for λ gives

$$\lambda \;=\; k^{\frac{1}{n-m}} e^{\frac{j\pi}{n-m}(2q+1)} \;-\; \frac{a_1 - b_1}{n-m}$$

Note that for a given θ and real number α, the quantity $re^{j\theta} + \alpha = (\alpha + r\cos\theta) + jr\sin\theta$, and as r goes from $0 \to \infty$ looks like

Hence, as $k \to \infty$, $n-m$ of the roots of $kN(s) + D(s) = 0$ go to ∞ in the complex plane, along lines radiating from a common point, $-\frac{a_1-b_1}{n-m}$, called the centroid, at angles $\frac{\pi(2q+1)}{n-m}$ for $q = 0, 1, \ldots, n-m-1$.

A similar argument shows that as $k \to -\infty$, $n-m$ of the roots of $kN(s)+D(s) = 0$ go to ∞ in the complex plane, along lines radiating from the same centroid, at angles $\frac{\pi(2q)}{n-m}$ for $q = 0, 1, \ldots, n-m-1$.

The last thing to notice is that the centroid can also be written in terms of the zeros of N and D. Using equation 95, we see that the centroid is at

$$\begin{aligned} \text{centroid} &= -\frac{a_1 - b_1}{n-m} \\ &= \frac{\sum_{i=1}^{n} \operatorname{Re}(p_i) \;-\; \sum_{j=1}^{m} \operatorname{Re}(z_j)}{n-m} \end{aligned}$$

Summarizing: As $k \to \pm\infty$, m of the root loci tend toward the zeros of N, namely the $\{z_1, z_2, \ldots, z_m\}$. The other $n-m$ roots go to ∞ along determinable asymptotes. There are $n-m$ asymptotes for $k \to \infty$. There are $n-m$ asymptotes for $k \to -\infty$. The centroid of all of the asymptotes is at

$$\frac{\sum_{i=1}^{n} \operatorname{Re}(p_i) \;-\; \sum_{j=1}^{m} \operatorname{Re}(z_j)}{n-m}$$

The angles of the asymptotes are

$$\textbf{PositiveRootLocus} \qquad \theta = \frac{(2q+1)\pi}{n-m}, q = 0, 1, \ldots, n-m-1$$

and

$$\textbf{NegativeRootLocus} \qquad \theta = \frac{(2q)\pi}{n-m}, q = 0, 1, \ldots, n-m-1$$

29.2.5 Angle of Departure from Pole

At $k = 0$ all of the roots of $kN(s)+D(s)$ are at the roots of D, namely $p_1, p_2, \ldots, p_n$. As k increases, the roots migrate away from each p_i. We can check the points very close to a p_i to see at what angle the root departs from p_i. Take $\lambda = p_1 + \epsilon e^{j\theta_d}$ where ϵ is small, and we want to determine θ_d so that λ is on the root loci. Note that

$$\begin{aligned}
\frac{N(\lambda)}{D(\lambda)} &= \frac{(\lambda - z_1)(\lambda - z_2)\cdot\cdot(\lambda - z_m)}{(\lambda - p_1)\ (\lambda - p_2)\ \cdots\ (\lambda - p_n)} \\
&= \frac{(p_1 + \epsilon e^{j\theta_d} - z_1)(p_1 + \epsilon e^{j\theta_d} - z_2)\cdots(p_1 + \epsilon e^{j\theta_d} - z_m)}{(p_1 + \epsilon e^{j\theta_d} - p_1)(p_1 + \epsilon e^{j\theta_d} - p_2)\cdots(p_1 + \epsilon e^{j\theta_d} - p_n)} \\
&\approx \frac{(p_1 - z_1)(p_1 - z_2)\cdots(p_1 - z_m)}{\epsilon e^{j\theta_d}(p_1 - p_2)\cdots(p_1 - p_n)}
\end{aligned}$$

In order for λ to be on the Positive Root Loci, we need this to be any odd multiple of π, giving

$$\begin{aligned}
(2q+1)\pi &= [\angle(p_1 - z_1) + \angle(p_1 - z_2) + \cdots + \angle(p_1 - z_m)] \\
&\quad - [\theta_d + \angle(p_1 - p_2) + \cdots + \angle(p_1 - p_n)]
\end{aligned}$$

Solve for θ_d as

$$\begin{aligned}
\theta_d &= [\angle(p_1 - z_1) + \angle(p_1 - z_2) + \cdots + \angle(p_1 - z_m)] \\
&\quad - [\angle(p_1 - p_2) + \cdots + \angle(p_1 - p_n)] - (2q+1)\pi
\end{aligned}$$

for any integer q. Note that regardless of what integer we chose, we just get a additive factor of 2π, which as an angle of departure, plays no role. Hence, for a simple way to remember the formula, chose $q = -1$, giving

$$\begin{aligned}
\theta_d &= \pi + [\angle(p_1 - z_1) + \angle(p_1 - z_2) + \cdots + \angle(p_1 - z_m)] \\
&\quad - [\angle(p_1 - p_2) + \cdots + \angle(p_1 - p_n)]
\end{aligned}$$

This holds for the **Positive Root Locus**.

For the Negative Root Locus, we use an even multiple of π which yields a similar formula, without the last term,

$$\begin{aligned}\theta_d &= [\angle(p_1 - z_1) + \angle(p_1 - z_2) + \cdots + \angle(p_1 - z_m)] \\ &\quad - [\angle(p_1 - p_2) + \cdots + \angle(p_1 - p_n)]\end{aligned}$$

This is for the **Negative Root Locus**.

29.2.6 Angle of Arrival to Zero

As $k \to \pm\infty$, m of the loci approach the m zeros of N, namely $\{z_1, z_2, \ldots, z_m\}$. The angle at which they arrive to the zero can be calculated in an identical manner as we used to determine the angle of departure from a pole. We can check the points very close to a z_j to see at what angle the root loci arrives at z_j. Take $\lambda = z_1 + \epsilon e^{j\theta_a}$ where ϵ is small, and we want to determine θ_a so that λ is on the root loci. Mimicing the technique for departure, we get

- For the Positive Root Locus, the angle of arrival at a zero (in this case, z_1) is

$$\begin{aligned}\theta_a &= \pi + [\angle(z_1 - p_1) + \angle(z_1 - p_2) + \cdots + \angle(z_1 - p_n)] \\ &\quad - [\angle(z_1 - z_2) + \cdots + \angle(z_1 - z_m)]\end{aligned}$$

- For the Negative Root Locus, the angle of arrival at the zero z_1 is

$$\begin{aligned}\theta_a &= [\angle(z_1 - p_1) + \angle(z_1 - p_2) + \cdots + \angle(z_1 - p_n)] \\ &\quad - [\angle(z_1 - z_2) + \cdots + \angle(z_1 - z_m)]\end{aligned}$$

29.2.7 Imaginary Axis Crossings

If we plot a Bode plot of $\frac{N(j\omega)}{D(j\omega)}$ versus ω, we can easily determine values of ω for which N/D is real. The values of ω such that N/D is negative correspond to points on the Positive root locus, achieved with $k = \frac{D(j\omega)}{N(j\omega)}$.

Note: This is really the brute-force methodology described in section 29.2, but applied only to purely imaginary values of s_0.

On problems with small n (say $n \leq 5$), a direct approach is sometimes successful. Separate the equation

$$kN(j\omega) + D(j\omega) = 0$$

into real and imaginary parts, giving two real equations in two real unknowns (k and ω). Since the coefficients of N and D are real, the real-part equation will involve even powers of ω, while the imaginary-part will involve odd power of ω. Exploit the fact that not all powers of ω appear in each equation, and use the quadratic formula to get two expressions of ω in terms of k. In some cases, these can be solved analytically by hand.

29.2.8 Symmetry of Root Loci

Since $kN(s)+D(s)$ is a polynomial with real coefficients, the roots are real and/or come in complex-conjugate pairs. This implies that the root locus plot is symmetrical about the real axis.

29.2.9 Breakaway Points/Multiple Roots

As k varies, it is possible that a real-valued root turns into a complex-valued root. For example, consider the roots of s^2+4s+k. For $k<1$, the roots are real, but for $k>1$ the roots are complex. The polynomial has real coefficients, so the roots, when complex, come in complex-conjugate pairs. Since complex-conjugate roots must have equal real parts, and since the roots change continuously with the parameter k, it must be that there are two identical roots at $k=1$, the transition point. Indeed, at $k=1$, the polynomial has two real roots, **both** at $s=-2$.

Also, from **Basic Rule 1**, we know that any fixed value $s_0 \in \mathbf{C}$ can be on the root locus for atmost one value of k. Hence, if two real roots approach each other as k varies, they must split into complex-conjugate pairs at the value of k which makes the roots equal. This occurance is called a *breakaway point*.

More generally, since the polynomial $kN(s)+D(s)$ has real coefficients, the roots are real and/or come in complex-conjugate pairs. This means that if a real-root becomes complex, it's complex conjugate must also appear at the same real value (but with an oppositely signed imaginary part). Hence at the value of k where the real root becomes complex, there must in fact be two identical real roots.

A polynomial $p(s)$ with root at $s=s_0$ actually has more than one root at $s=s_0$ if and only if

$$p(s_0)=0, \quad \text{and} \quad \left.\frac{dp}{ds}\right|_{s=s_0}=0$$

Hence, for s_0 to be on the root locus of (N,D), it must be that

$$\frac{N(s_0)}{D(s_0)} \in \mathbf{R} \tag{96}$$

is real. Denote $k_0 := -\frac{D(s_0)}{N(s_0)}$. For s_0 to be a multiple root, we also need that

$$\frac{d}{ds}\left[k_0 N(s) + D(s)\right]\Big|_{s=s_0} = 0$$

Carrying out the differentiation, and using what k_0 is, we have that at a multiple root, in addition to equation (96), it must be that

$$-D(s_0)\frac{dN}{ds}(s_0) + N(s_0)\frac{dD}{ds}(s_0) = 0 \tag{97}$$

So, to find multiple-roots (and root crossings, often called "breakaway points") we simply need to determine if any of the roots of

$$-D(s)\frac{dN}{ds}(s) + N(s)\frac{dD}{ds}(s) = 0$$

also satisfy $\frac{N(s)}{D(s)} \in \mathbf{R}$ (ie., are actually on the root locus).

29.3 Modern Root Locus Methods

Most good computer-aided control system design packages have a root locus command (in the MatLab control toolbox, the command is `rlocus`). These work purely by brute force, direct calculation of the roots of the $kN(s) + D(s) = 0$ as k varies. The computer uses speed, and well-developed, polynomial root finding algorithms to quickly make accurate sketches of the dependencies of the roots on k.