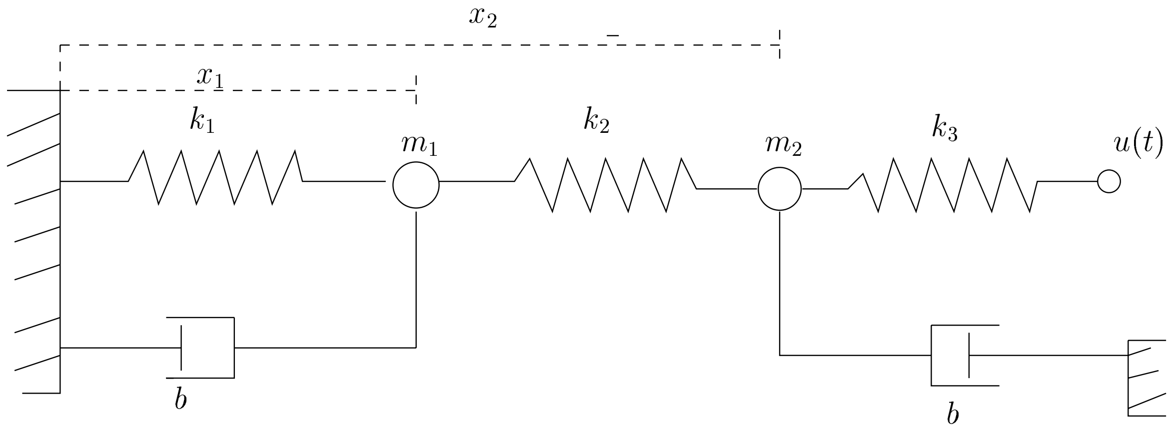
Matrix Diagonalization and Systems of ODEs

Outline

- Why do we care of matrix **diagonalization**?
- What are **eigenvalues** and **eigenvectors**?
- How do we compute them?
- How do we use eigenvalues and eigenvectors to diagonalize a matrix?
- How do we solve **systems of ODEs**?
- How to infer **stability** information from the eigenvalues

Motivation to Diagonalization



$$m_1 = m_2 = m$$
$$k_1 = k_2 = k_3 = k$$
$$u(t) = 0$$

Figure 1: Mass spring system

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2 - b\dot{x}_1 \\ m\ddot{x}_2 &= kx_1 - 2kx_2 - b\dot{x}_2 \end{aligned}$$

WHAT IS THE SOLUTION $(x_1(t), x_2(t))$?

<u>Hint</u>: yesterday you saw the solution of odes of the kind

$$m\ddot{y} + b\dot{y} + ky = 0$$

Try to change the coordinates:

$$\begin{aligned} z_1 &= \frac{1}{2}(x_1 + x_2) \\ z_2 &= \frac{1}{2}(x_2 - x_1) \end{aligned} \tag{1}$$

then

$$\begin{aligned} m\ddot{z}_1 &= -kz_1 - b\dot{z}_1 \\ m\ddot{z}_2 &= -kz_2 - b\dot{z}_2 \end{aligned}$$

which is now decoupled.

Then you solve the first one to find $z_1(t)$ and the second one to find $z_2(t)$, and inverting (1) you find

$$\begin{aligned} x_1(t) = z_1(t) - z_2(t) \\ x_2(t) = z_2(t) + z_1(t) \end{aligned} \tag{2}$$

and you have solved the problem.

In matrix notation we can write

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and change of coordinates (2) becomes

$$x = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} z$$

and the inverse change of coordinates (1) becomes

$$z = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} x.$$

Let

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$p^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

We can picture this in the following schematic:

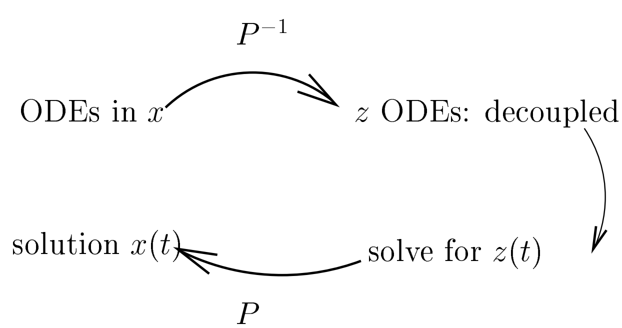


Figure 2: Schematic for spring mass ODEs solution

In general for $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ and the system of ODEs

$$\dot{x} = Ax$$

we want to find (if it exists) the change of coordinates represented by the matrix P such that

$$P^{-1}AP = \Lambda, \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda_n \end{pmatrix}$$

because if we change the coordinates in the z variables where

$$z = P^{-1}x \qquad \text{and} \qquad x = Pz$$

we have

$$\dot{z} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APz = \Lambda z$$

which means that we have n decoupled dynamics which we can treat indipendently. In fact $\dot{z} = \Lambda z$ can be rewritten in scalar form as

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 \\ \dot{z}_2 &= \lambda_2 z_2 \\ &\vdots \\ \dot{z}_n &= \lambda_n z_n \end{aligned}$$

which are n first order ODEs that we can solve idependently as

$$z_i(t) = z_i(0)e^{\lambda_i t}$$

for all i, and then we can go back to the x coordinates so to get the $x(t)$ solution as

$$x(t) = Pz(t).$$

This is one of the reasons why it is useful to find a change of coordinates P that transforms matrix A to its diagonal form Λ (when it exists).

The process of finding P and the diagonal matrix Λ is called **diagonalization**. This process needs the computation of eigenvalues and eigenvectors of the matrix A.

Eigenvectors and Eigenvalues

Let $A \in \mathbb{R}^{n \times n}$. A vector $v \in \mathbb{R}^n$ is said to be an **eigenvector** of A with **eigenvalue** λ if

$$Av = \lambda v$$

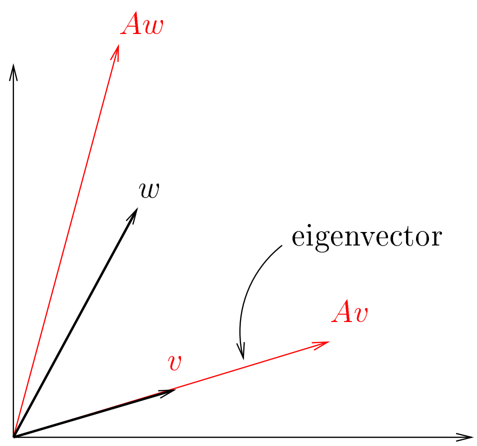


Figure 3: A matrix acts on its eigenvectors by scaling them.

How do we find the eigenvalues of A?
Theorem: λ is an eigenvalue of A if and only if

$$det(A - \lambda I) = 0$$

(I is the $n \times n$ identity matrix)
The expression $det(A - \lambda I)$ is a function of λ of the form

$$\lambda^n + c1\lambda^{n-1} + ... + c_{n-1}\lambda + c_n$$

which is called the **characteristic polynomial** of A. Then to find the eigenvalues of the matrix A we need to find the roots of the characteristic polynomial.

EXAMPLE (step 1)

Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Compute the eigenvalues of A.

How do we find the eigenvalues of A?

Once you have found the eigenvalues λ_i of the matrix A, you can find the corresponding eigenvector v_i by solving the system of equations

$$(A - \lambda_i I)v_i = 0$$

for v_i.

EXAMPLE (step 2)

Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

On the basis of the computed eigenvalues of A (step 1) compute the corresponding eigenvectors of A.

Diagonalization

Diagonalization theorem :If the eigenvalues of an $n \times n$ matrix are real and distinct, then any set of corresponding eigenvectors $\{v_1, ..., v_n\}$ form a matrix $P = (v_1, ..., v_n)$ that is invertible and

$$P^{-1}AP = \Lambda$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda_n \end{pmatrix}$$

and λ_i is the eigenvalue with eigenvector v_i.

idea: assume for example that $A \in R^{2\times 2}$ and v_1 and v_2 are eigenvectors of A. Then

$$P = (v_1, v_2)$$

and

$$P^{-1} = \begin{pmatrix} w_1^T \\ w_2^T \end{pmatrix}$$

where w_1 and w_2 are vectors such that $w_1^T v_1 = 1$, $w_1^T v_2 = 0$, $w_2^T v_1 = 0$ and $w_2^T v_2 = 1$. Then

$$P^{-1}AP = P^{-1}(Av_1, Av_2) = P^{-1}(\lambda_1 v_1, \lambda_2 v_2) = \begin{pmatrix} \lambda_1 w_1^T v_1 & \lambda_2 w_1^T v_2 \\ \lambda_1 w_2^T v_1 & \lambda_2 w_2^T v_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

EXAMPLE (step3)
given the matrix A of steps 1 and 2, compute P and verify that $P^{-1}AP = \Lambda$.

Solution of Systems of ODEs

Given the linear dynamical system

$$\dot{x} = Ax$$

let $\{v_1, ..., v_n\}$ be a basis of eigenvectors with eigenvalues $\{\lambda_1, ..., \lambda_n\}$. Let $P = (v_1, ..., v_n)$, and consider the change of coordinates

$$z = P^{-1}x$$

then the dynamics in the new coordinates z becomes

$$\dot{z} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APz$$

then by virtue of the diagonalization theorem $P^{-1}AP = \Lambda$, so that

$$\dot{z} = \Lambda z$$

that is

$$\dot{z}_i = \lambda_i z_i \qquad \text{for all } i$$

which have solutions

$$z_i(t) = z_i(0)e^{\lambda_i t} \qquad \text{for all } i$$

so that

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix}$$

and

$$x(t) = Pz(t)$$

EXAMPLE

consider $\dot{x} = Ax$ with

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

find $x(t)$.

Stability of Systems of ODEs from Eigenvalues

Given the system $\dot{x} = Ax$, once we have the solution $x(t)$, we will say that the system is **unstable** if $\|x(t)\|$ becomes arbitrarily far from the equilibrium point, x=0, as time increases. Since P is not depending on time, we expect that if $\|z(t)\|$ is becoming arbitrarily big as time increases, then also $\|x(t)\|$ will, and if $\|z(t)\|$ is staing close to 0 for any time, also $\|x(t)\|$ is. Then to check if the system is unstable we can check the behavior of $\|z(t)\|$ in time.

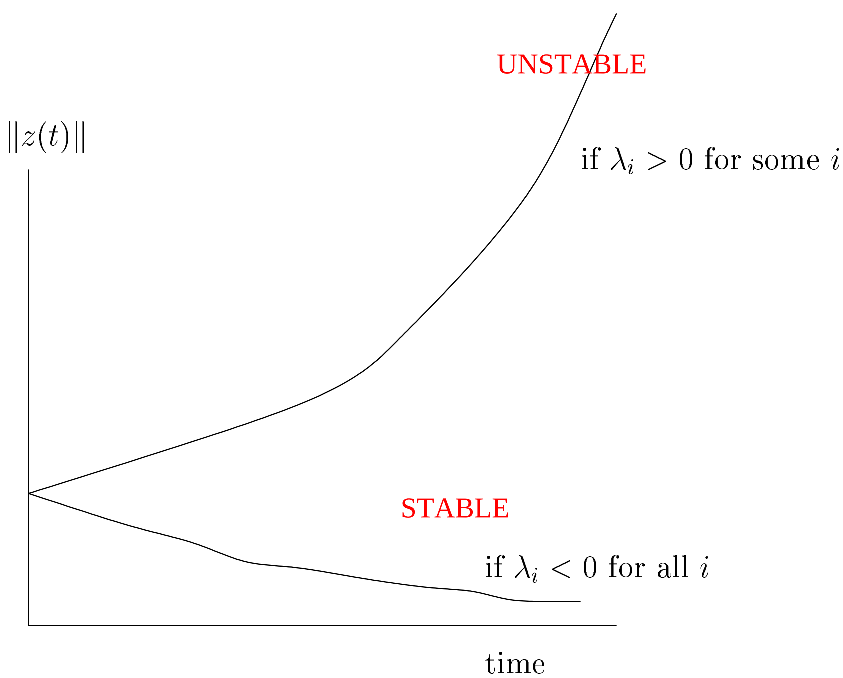


Figure 4: Stability property of the system of ODEs depending on the eigenvalues of A.

Useful Matlab Commands

1. `[V,D]=eig(A)` gives the eigenvalues of A in the matrix D and the eigenvectors of A in the matrix V.

Useful References

C. W. Curtis, *Linear Algebra, An Introductory Approach*, Springer Verlag, NY, 1984

K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, NJ, 1971