CDS 101: Lecture 7.1
Loop Analysis of Feedback Systems

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Goals:

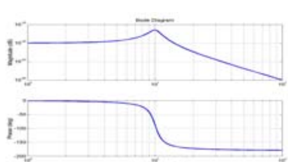
- Show how to compute closed loop stability from open loop properties
- Describe the Nyquist stability criterion for stability of feedback systems
- Define gain and phase margin and determine it from Nyquist and Bode plots

Reading:

- Astrom, Section 3.5
- *Optional*: Packard, Poola and Horowitz, Chapter 30-31
- *Advanced*: Lewis, Chapter 7

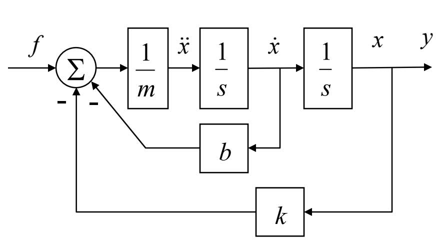
Review from Last Week

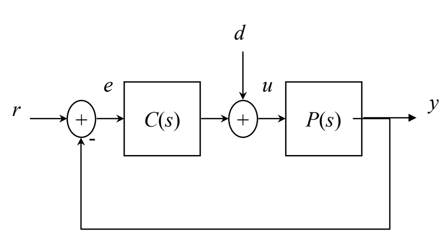
$u = A\sin(\omega t) \rightarrow$ $\dot{x} = Ax + Bu$, $y = Cx + Du$, $x(0) = 0$ $\rightarrow y_{ss} = |H(j\omega)| A \cdot \sin(\omega t + \angle H(j\omega))$



$$H(s) = C(sI - A)^{-1}B + D$$

$$H_{y_2 u_1} = H_{y_2 u_2} H_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$

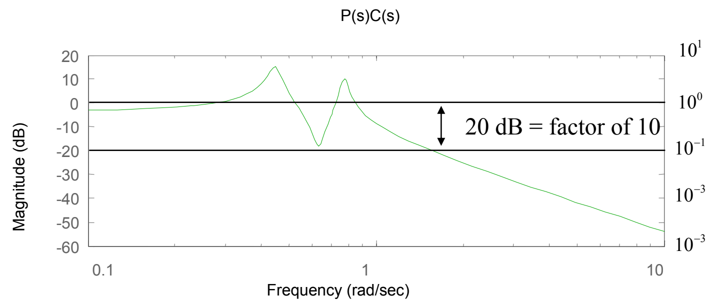




Additional info from last week’s lecture

Decibel (dB)

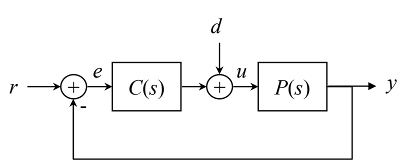
- Logarithmic scale used to define ratio between amplitudes (gain)
- 20 decibel (dB) = factor of 10 in gain: plot $20*\log_{10}(\text{gain})$



History and background

- 1 bel = logarithm of the ratio of *power*. Decibel = 1/10 bel.
- Since power goes as the square of the amplitude, we use 20*log(gain) for ratio of *amplitudes* (power goes as the square of the amplitude in electrical circuits)

Closed Loop Stability



Q: how do open loop dynamics affect the closed loop stability?

- Given open loop transfer function $C(s)P(s)$ determine when system is stable
- Useful for design since we specify C(s)

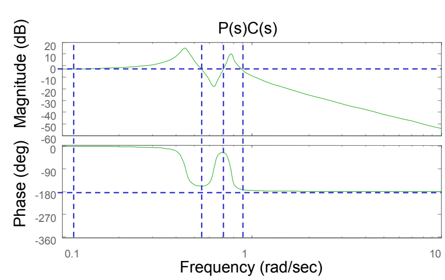
Brute force answer: compute poles closed loop transfer function

$$H_{yr} = \frac{PC}{1+PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of H_{yr} = zeros of $1 + PC$
- Easy to compute, but not so good for design

Alternative: look for conditions on *PC* that lead to instability

- Example: if $PC(s) = -1$ for some $s = j\omega$, then system is *not* asymptotically stable
- Condition on *PC* is much nicer because we can *design* $PC(s)$ by choice of $C(s)$
- However, checking $PC(s) = -1$ is not enough; need more sophisticated check

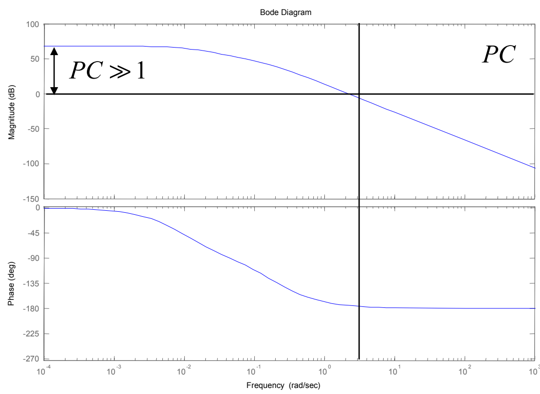


Game Plan: Frequency Domain Design

Goal: figure out how to *design* $C(s)$ so that $1+C(s)P(s)$ is stable + good performance

$$H_{yr} = \frac{PC}{1+PC}$$

- Poles of H_{yr} = zeros of $1 + PC$
- Would also like to "shape" H_{yr} to specify performance at differenct frequencies



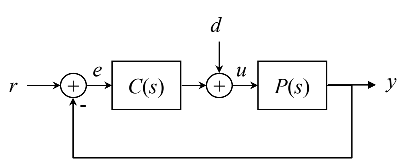
- Low frequency range:

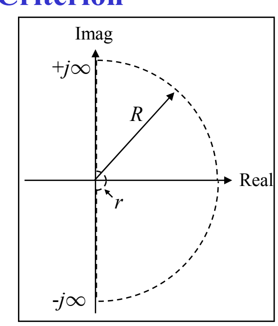
$$PC \gg 1 \quad \Rightarrow \quad \frac{PC}{1+PC} \approx 1$$

(good tracking)

- Bandwidth: as high as possible, but stay stable
- Idea: use $C(s)$ to *shape* PC (under certain constraints)
- Need tools to analyze stability and performance for closed loop given PC

Nyquist Criterion

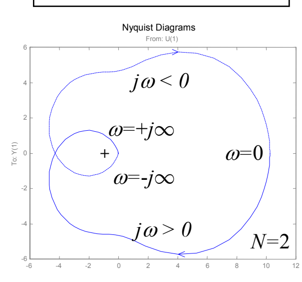




- Nyquist "D" contour
- Take limit as $r \to 0$, $R \to \infty$
- Trace from $-\infty$ to $+\infty$ along imaginary axis

Can determine stability from (open) loop transfer function, $L(s) = P(s)C(s)$.

- Use "principle of the argument" from complex variable theory (see reading)



- Trace frequency response along the Nyquist "D" contour
- Count net # of clockwise encirclements of the -1 point

Thm (Nyquist). Consider the Nyquist plot for loop transfer function $L(s)$. Let

P # RHP poles of $L(s)$

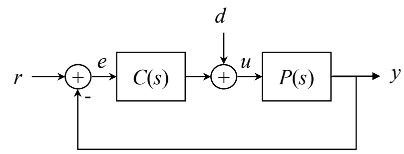
N # clockwise encirclements of -1

Z # RHP zeros of $1 + L(s)$

Then

$$Z = N + P$$

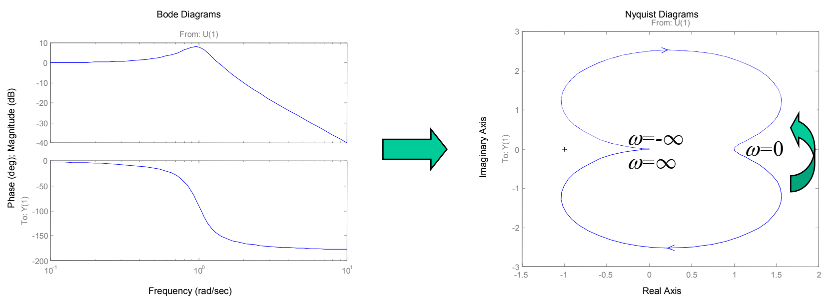
Simple Interpretation of Nyquist



Basic idea: avoid positive feedback

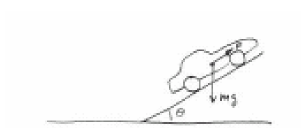
- If $L(s)$ has 180° phase (or greater) and gain greater than 1, then signals are amplified around the loop
- Use when phase is monotonic
- General case requires Nyquist

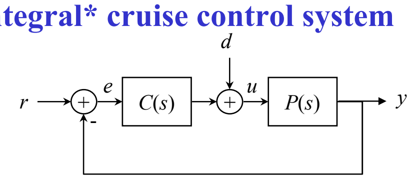
Can generate Nyquist plot from the Bode plot + reflection around real axis

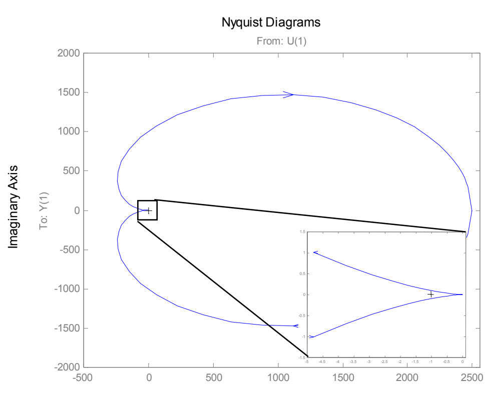


bode(sys) nyquist(sys)

Example: Proportional + Integral* cruise control system







$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s + a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

Remarks

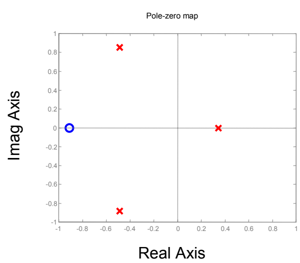
- $N = 0$, $P = 0 \Rightarrow Z = 0$ (stable)
- Need to zoom in to make sure there are no net encirclements
- Note that we don't have to compute closed loop response

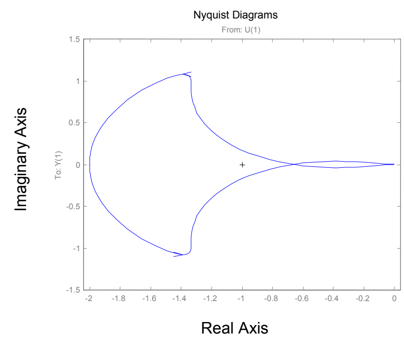
* slightly modified; more on the design of this compensator in next week's lecture

More complicated systems

What happens when open loop plant has RHP poles?

- 1 + PC has singularities inside D countour ⇒ these must be taken into account





$$L(s) = \frac{s+0.9}{s-0.5}\cdot\frac{1}{s^2+s+1}$$

unstable pole

N = -1, P = 1 ⇒ $Z = N+P = 0$ (stable)

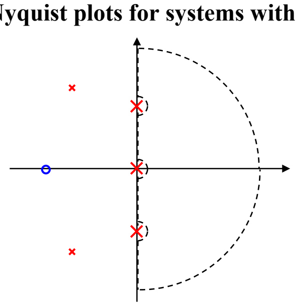
$$\frac{1}{1+L} = \frac{s+1}{(s+0.35)(s+0.07+1.2j)(s+0.07-1.2j)} \checkmark$$

Comments and cautions

Why is the Nyquist plot *useful*?

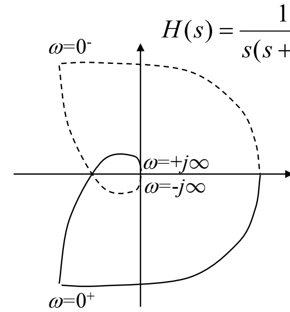
- Old answer: easy way to compute stability (before computers and MATLAB)
- Real answer: gives *insight* into stability and robustness; very useful for proofs

Nyquist plots for systems with poles on the $j\omega$ axis



- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\epsilon+0j)$ to determine direction

$$H(s) = \frac{1}{s(s+1)^2}$$



Cautions with using MATLAB

- MATLAB doesn't generate portion of plot corresponding to poles on imaginary axis
- These must be drawn in by hand (make sure to get the orientation right!)

Robust stability: gain and phase margins

Nyquist plot tells us if closed loop is stable, but not *how* stable

Gain margin

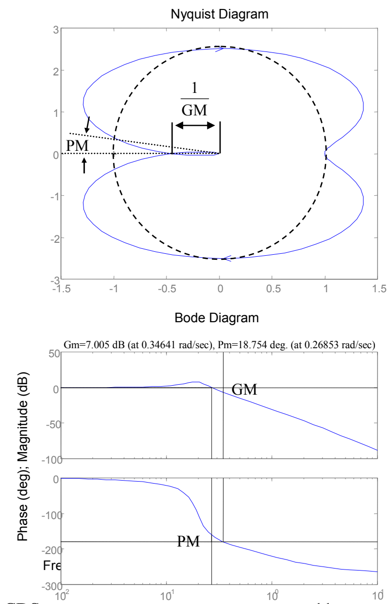
- How much we can modify the *loop gain* and still have the system be stable
- Determined by the location where the loop transfer function crosses 180° phase

Phase margin

- How much we can add *phase delay* and still have the system be stable
- Determined by the phase at which the loop transfer function has unity gain

Bode plot interpretation

- Look for unity gain & 180° phase crossings
- MATLAB: margin(sys)



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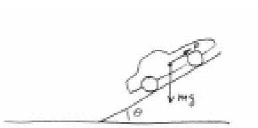
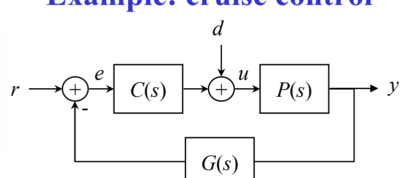
Example: cruise control





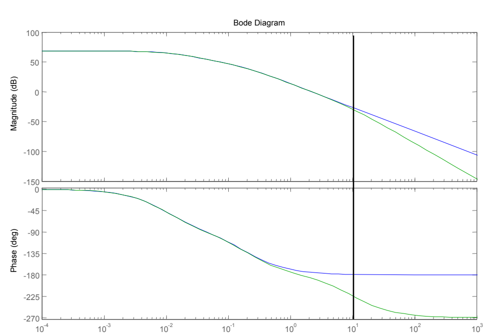
$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s + a}$$

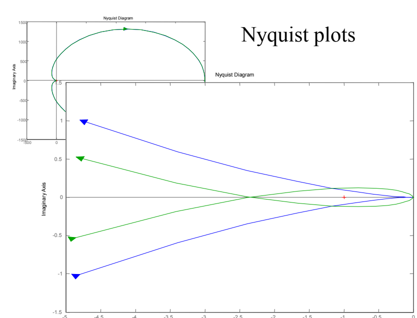
$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

$$G(s) = \frac{10}{s + 10}$$

Effect of additional sensor dynamics

- New speedometer has pole at $s = 10$ (very fast); problems develop in the field
- What's the problem? A: insufficient phase margin in original control design (not robust)





Nyquist plots

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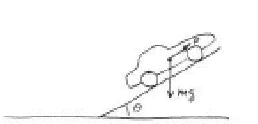
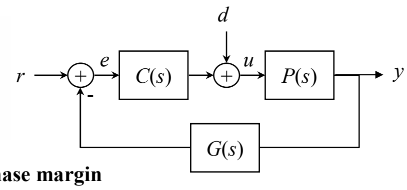
Preview: control *design*





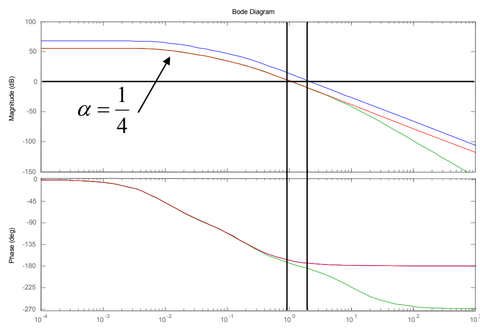
$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s+a}$$

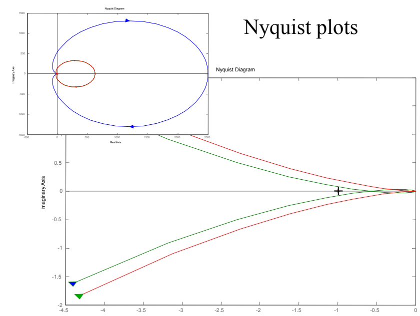
$$C(s) = \alpha \left(K_p + \frac{K_i}{s + 0.01} \right)$$

$$G(s) = \frac{10}{s+10}$$

Approach: Increase phase margin

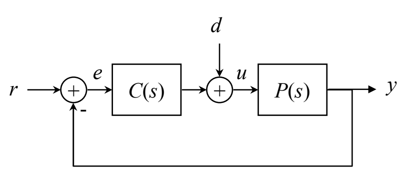
- Increase phase margin by reducing gain ⇒ can accommodate new sensor dynamics
- Tradeoff: lower gain at low frequencies ⇒ less bandwidth, larger steady state error

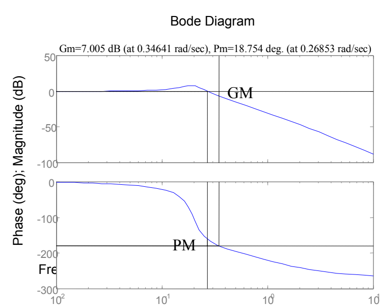




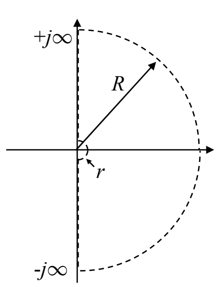
Nyquist plots

Summary: Loop Analysis of Feedback Systems





- Nyquist criteria for loop stability
- Gain, phase margin for robustness



Thm (Nyquist).

P # RHP poles of $L(s)$

N # CW encirclements

Z # RHP zeros

$$Z = N + P$$