CDS 101: Lecture 5.1
Controllability and State Space Feedback

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28 October 2002

Goals:
• Define controllability of a control system
• Give tests for controllability of linear systems and apply to examples
• Describe the design of state feedback controllers for linear systems

Reading:
• Packard, Poola and Horowitz, Dynamic Systems and Feedback, Section 25
• Optional: Friedland, Sections 5.1-5.4 (controllability only)
• Optional: A. D. Lewis, A Mathematical Introduction to Feedback Control, Chapter 2 (available on web page)

Review from Last Week

\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]
\[ x(0) = 0 \]

\[ y(t) = Ce^\lambda t x(0) + \int_{t_0}^{t} Ce^{\lambda (t-\tau)} Bu(\tau) d\tau + Du(t) \]

Properties of linear systems
• Linearity with respect to initial condition and inputs
• Stability characterized by eigenvalues
• Many applications and tools available
• Provide local description for nonlinear systems
Control Design Concepts

System description: single input, single output nonlinear system (MIMO also OK)
\[
\begin{align*}
\dot{x} &= f(x,u) & x \in \mathbb{R}^n, x(0) \text{ given} \\
y &= h(x,u) & u \in \mathbb{R}, y \in \mathbb{R}
\end{align*}
\]

Stability: stabilize the system around an equilibrium point
- Given equilibrium point \( x_e \in \mathbb{R}^n \), find control “law” \( u = \alpha(x) \) such that
  \[
  \lim_{t \to \infty} x(t) = x_e \text{ for all } x(0) \in \mathbb{R}^n
  \]

Controllability: steer the system between two points
- Given \( x_0, x_f \in \mathbb{R}^n \), find an input \( u(t) \) such that
  \[
  x(t) = f(x,u(t)) \text{ takes } x(t) = x_0 \rightarrow x(T) = x_f
  \]

Tracking: track a given output trajectory
- Given \( y_d(t) \in \mathbb{R} \), find \( u = \alpha(x,t) \) such that
  \[
  \lim_{t \to \infty} (y(t) - y_d(t)) = 0 \text{ for all } x(0) \in \mathbb{R}^n
  \]

Controllability of Linear Systems

\[
\begin{align*}
\dot{x} &= Ax + Bu & x \in \mathbb{R}^n, x(0) \text{ given} \\
y &= Cx + Du & u \in \mathbb{R}, y \in \mathbb{R}
\end{align*}
\]

Defn A linear system is \textit{controllable} if for any \( x_0, x_f \in \mathbb{R}^n \) and any time \( T > 0 \) there exists an input \( u : [0,T] \to \mathbb{R} \) such that the solution of the dynamics starting from \( x(0) = x_0 \) and applying input \( u(t) \) gives \( x(T) = x_f \).

Remarks
- In the definition, \( x_0 \) and \( x_f \) do not have to be equilibrium points \( \Rightarrow \) we don’t necessarily stay at \( x_f \) after time \( T \).
- Controllability is defined in terms of states \( \Rightarrow \) doesn’t depend on output
- Can characterize controllability by looking at the general solution of a linear system:
  \[
  x(T) = e^{AT}x_0 + \int_{T=0}^T e^{A(T-\tau)}Bu(\tau) \, d\tau
  \]
  If integral is “full rank”, then we can find an input to achieve any desired final state.
Tests for Controllability

\[ \dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, x(0) \text{ given} \]
\[ y = Cx + Du \quad u \in \mathbb{R}, y \in \mathbb{R} \]

\[ x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau \]

**Thm** A linear system is controllable if and only if the \( n \times n \) controllability matrix

\[
\begin{bmatrix}
B & AB & A^2B & \ldots & A^{n-1}B
\end{bmatrix}
\]

is full rank.

**Remarks**

- Very simple test to apply. In MATLAB, use `ctrb(A,B)`
- If this test is satisfied, we say “the pair (A,B) is controllable”
- Some insight into the proof can be seen by expanding the matrix exponential

\[
e^{A(T-\tau)}B = \left( I + AT - \tau + \frac{1}{2} A^2(T-\tau)^2 + \ldots + \frac{1}{2} A^{n-1}(T-\tau)^{n-1} + \ldots \right) B
\]

\[
\begin{aligned}
e^{A(T-\tau)}B &= \left( B + AB(T-\tau) + \frac{1}{2} A^2B(T-\tau)^2 + \ldots + \frac{1}{2} A^{n-1}B(T-\tau)^{n-1} + \ldots \right)
\end{aligned}
\]

---

**Example #1: Linearized pendulum on a cart**

**Question:** can we locally control the position of the cart by proper choice of input?

**Approach:** look at the linearization around the upright position (good approximation to the full dynamics if \( \theta \) remains small)

\[
\begin{bmatrix}
dx/dt \\
dx/dt
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & (J + m l^2) & 0 \\
0 & J(M + m) + Mml^2 & J(M + m) + Mml^2 & 0 \\
0 & mgl(M + m) & mgl(M + m) & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}
\]

\[
y =
\begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
\]

\[
\begin{bmatrix}
J + ml^2 \\
J(M + m) + Mml^2 \\
ml \ J(M + m) + Mml^2
\end{bmatrix}
\]
Example #1, con’t: Linearized pendulum on a cart

\[
\frac{d}{dt} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{m^2g^2}{q} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} \frac{mgl(M + m)}{q} \\ \frac{-mb}{q} \end{bmatrix} u
\]

\[q = J(M + m) + Mm^2\]

Controllability matrix

\[
M_c = \begin{bmatrix}
0 & \frac{J + ml^2}{q} & 0 & \frac{ml(J + ml^2)}{q^2} \\
0 & \frac{ml}{q} & 0 & \frac{m^2gl^2(M + m)}{q^2} \\
\frac{J + ml^2}{q} & 0 & \frac{ml(J + ml^2)}{q^2} & 0 \\
\frac{ml}{q} & 0 & \frac{m^2gl^2(M + m)}{q^2} & 0 \\
B & AB & A^2B & A^3B
\end{bmatrix}
\]

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- \(\Rightarrow\) controllable as long as \(g(M + m) \neq 1\)
- \(\Rightarrow\) can "steer" linearization between points by proper choice of input

Constructive Controllability

Given that system is controllable, how do we find input to transfer between states?

Simple case: chain of integrators

\[
\frac{d}{dt} x = Ax + Bu
\]

- Find curve \(x_i(t)\) such that \(x_i(0) = x_0, x_i(T) = x_1\)
- Choose input as \(u(t) = x_i^{(n)}(t)\)

Controllable canonical form

- If controllable, can show there exists a linear change of coordinates such that

\[
z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \ddots & \ddots \\ -a_1 & -a_2 & \ddots & -a_n \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} u
\]

\[
z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}
\]

\[
u(t) = z_1^{(n)}(t) + a_n z_1^{(n-1)} + \cdots + a_1 z_1^{(1)}
\]
Control Design Concepts

System description: single input, single output nonlinear system (MIMO also OK)
\[
\dot{x} = f(x,u) \quad x \in \mathbb{R}^n, \quad x(0) \text{ given} \\
y = h(x,u) \quad u \in \mathbb{R}, \quad y \in \mathbb{R}
\]

Stability: stabilize the system around an equilibrium point
- Given equilibrium point \( x_e \in \mathbb{R}^n \), find control “law” \( u = \alpha(x) \)
  such that \( \lim_{t \to \infty} x(t) = x_e \) for all \( x(0) \in \mathbb{R}^n \)

Controllability: steer the system between two points
- Given \( x_0, x_f \in \mathbb{R}^n \), find an input \( u(t) \) such that \( \dot{x} = f(x,u(t)) \) takes \( x(t_0) = x_0 \to x(t_f) = x_f \)

Tracking: track a given output trajectory
- Given \( y_r(t) \in \mathbb{R} \), find \( u = \alpha(x,t) \) such that \( \lim_{t \to \infty} (y(t) - y_r(t)) = 0 \) for all \( x(0) \in \mathbb{R}^n \)

State space controller design for linear systems
\[
\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, \quad x(0) \text{ given} \\
y = Cx + Du \quad u \in \mathbb{R}, \quad y \in \mathbb{R}
\]

Goal: find a linear control law \( u = Kx \) such that the closed loop system
\[
\dot{x} = Ax + BKx = (A + BK)x
\]
is stable at \( x = 0 \).

Remarks
- Stability determined by eigenvalues \( \Rightarrow \) use \( K \) to make eigenvalues of \((A+BK)\) stable
- Can also link eigenvalues to performance (eg, initial condition response)
- Question: when can we place the eigenvalues anywhere that we want?

Thm The eigenvalues of \((A+BK)\) can be set to arbitrary values if and only if the pair \((A,B)\) is controllable.

MATLAB: \( K = \text{place}(A, B, \text{eigs}) \)
Example #2: Predator prey

Natural dynamics
\[
\begin{align*}
\dot{x}_1 &= b_r x_1 - a x_1 x_2 - b x_1^2 \\
\dot{x}_2 &= a x_1 x_2 - d_f x_2 - b x_2^2
\end{align*}
\]

Controlled dynamics: modulate food supply
\[
\begin{align*}
\dot{x}_1 &= b_r (1 + u) x_1 - a x_1 x_2 - b x_1^2 \\
\dot{x}_2 &= a x_1 x_2 - d_f x_2 - b x_2^2
\end{align*}
\]

Q1: can we move from some initial population of foxes and rabbits to a specified one in time \( T \) by modulation if the food supply?

Q2: can we stabilize the population around the current equilibrium point

Approach: try to answer this question locally, around the equilibrium point

Example #2: Problem setup

Equilibrium point calculation
\[
\begin{align*}
\dot{x}_1 &= b_r (1 + u) x_1 - a x_1 x_2 - b x_1^2 \\
\dot{x}_2 &= a x_1 x_2 - d_f x_2 - b x_2^2
\end{align*}
\]
\[
\begin{align*}
x_e &= (78.1726, 94.4162)
\end{align*}
\]

Linearization

\[
\begin{align*}
&\text{Compute linearization around equil. point, } x_e: \\
&\begin{align*}
A &= \begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial u}
\end{bmatrix}_{(x_e,u_e)} \\
B &= \begin{bmatrix}
\frac{\partial f}{\partial u}
\end{bmatrix}_{(x_e,u_e)}
\end{align*}
\end{align*}
\]

\[
\begin{align*}
&\text{Redefine local variables: } z=x-x_e, v=u-u_e
\end{align*}
\]

\[
\begin{align*}
&\frac{dz_1}{dt} = -b_r - ax_2 e - 2 bx_1 e \\
&\frac{dz_2}{dt} = ax_2 e - d_f x_2 e - 2 bx_2 e
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} = \begin{bmatrix}
-b_r - ax_2 e - 2 bx_1 e \\
ax_2 e - d_f x_2 e - 2 bx_2 e
\end{bmatrix} + \begin{bmatrix}
b_r x_v e \\
0
\end{bmatrix} v
\end{align*}
\]

\[
\begin{align*}
&\text{Controllable? YES, if } b_r, a \neq 0 \text{ (check } [B AB] \Rightarrow \text{ can locally steer to any point)}
\end{align*}
\]
Example #2: Stabilization via eigenvalue assignment

\[
\frac{dx}{dt} = \begin{bmatrix}
-h_1 - ax_{1,\omega} - 2bx_{1,\omega} \\
ax_{1,\omega} \\
-d_1 + ax_{1,\omega} - 2bx_{1,\omega} \\
-bx_{1,\omega}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} u
\]

Control design:

\[
v = Kz = K(x - x_c)
\]
\[
u = u_c + v
\]

Place poles at stable values
- Choose \(\lambda = -1, -2\)
- \(K = \text{place}(A, B, [-1; -2]);\)

Modify dynamics to include control

\[
\dot{x}_1 = b_1(1 - K(x - x_c))x_1 - ax_1x_2 - bx_1^2
\]
\[
\dot{x}_2 = ax_1x_2 - d_1x_2 - bx_2^2
\]

How to Assign Eigenvalues

Eigenvalue location determines time response
- For each eigenvalue \(\lambda = \sigma + j\omega\), get contribution of the form

\[
y(t) = e^{-\sigma t} (a \sin(\omega t) + b \cos(\omega t))
\]

- Repeated eigenvalues can give additional terms of the form \(t^p e^{\sigma t} + j\omega
\)

Eigenvalue assignment changes the dynamics of the system
- Illustrates one of the main principles of feedback
- Can be used to make unstable points stable, increase the speed of the response, etc

Caution: eigenvalue assignment affects magnitude of input required
- The amount of actuator effort required to change the dynamics from the natural (open loop) dynamics to the desired (closed loop) dynamics can be large
- We will learn more sophisticated ways to deal with this tradeoff later in the course
State Space Feedback Design Tools

Eigenvalue assignment (also called “pole placement”)

- Choose eigenvalue locations to correspond to desired dynamics

Optimal control (CDS 110b)

- Choose the feedback that minimizes a cost function:

\[ J = \int_0^\infty (x^T Q x + u^T R u) dt \]

\[ u = K x \]

- Key advantage is the explicit tradeoff between state error and input magnitude
- Often very hard to relate the weights \((Q, R)\) to the desired system behavior

Nonlinear systems (CDS 221)

- If linearization is controllable, can show that linear feedback provides local stability
- More generally, can look at nonlinear state feedback, \(u = d(x)\)
- Optimal control problem can’t be solved in closed form
- Alternative: Lyapunov based design (“integrator backstepping”, “Lyapunov redesign”, etc)

Summary: Controllability and State Space Feedback

\[ \dot{x} = Ax + Bu \]

\[ y = Cx + Du \]

Key concepts

- Controllability: find \(u\) s.t. \(x_0 \rightarrow x_f\)
- Controllability rank test for linear systems
- State feedback to assign eigenvalues

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