



CDS 101: Lecture 4.1 Linear Systems



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21 October 2002

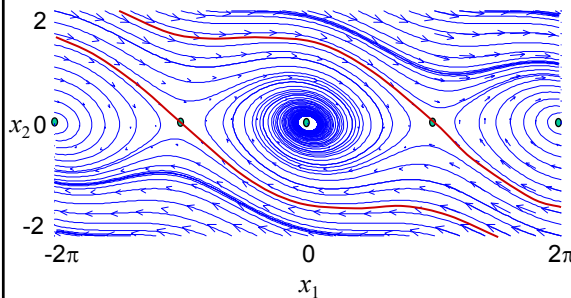
Goals:

- Describe linear system models: properties, examples, and tools
- Characterize the stability and performance of a linear system in terms of eigenvalues
- Compute linearization of a nonlinear systems around an equilibrium point

Reading:

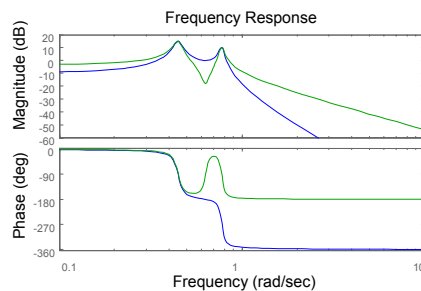
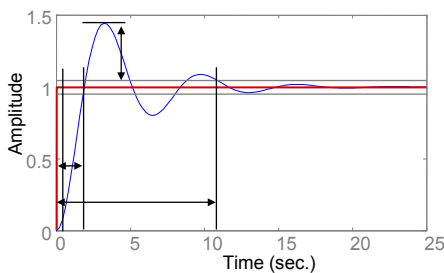
- Packard, Poola and Horowitz, *Dynamic Systems and Feedback*, Sections 19, 20, 22
- *Optional*: Astrom, Section 3.6, 3.7 (pp 125-134)

Review from Last Week



Key topics for this lecture

- Stability of equilibrium points
- Local versus global behavior
- Performance specification via step and frequency response



What is a linear system?

Linearity of functions: $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

- Zero at the origin: $f(0) = 0$
- Addition: $f(x + y) = f(x) + f(y)$
- Scaling: $f(\alpha x) = \alpha f(x)$

Canonical example: $f(x) = Ax$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Linearity of systems: sums of solutions

Dynamical system

$$\dot{x} = Ax$$

$$x(0) = x_{10} \quad \rightarrow x(t) = x_1(t)$$

$$x(0) = x_{20} \quad \rightarrow x(t) = x_2(t)$$

$$\Downarrow$$

$$x(0) = \alpha x_1 + \beta x_2$$

$$\rightarrow x(t) = \alpha x_1(t) + \beta x_2(t)$$

Control system

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(0) = 0, u(t) = u_1(t) \quad \rightarrow y(t) = y_1(t)$$

$$x(0) = 0, u(t) = u_2(t) \quad \rightarrow y(t) = y_2(t)$$

$$\Downarrow$$

$$x(0) = 0, u(t) = \alpha u_1(t) + \beta u_2(t)$$

$$\rightarrow y(t) = \alpha y_1(t) + \beta y_2(t)$$

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Linear input/output systems

Input/output linearity at $x(0) = 0$

- Linear systems are linear in initial condition *and* input \Rightarrow need to use $x(0) = 0$ to add the outputs together
- For different initial conditions, you need to be more careful (sounds like a good midterm question)

Linear system \Rightarrow step response and frequency response scale with input amplitude

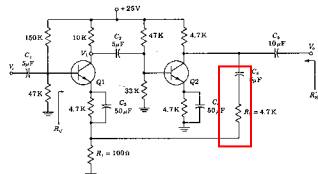
- 2X input \Rightarrow 2X output
- This is why we look at *ratios* and *percentages* in step/freq response. These are *independent* of input amplitude
- Limitation: input saturation \Rightarrow only holds up to certain input amplitude

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Why are linear systems important?

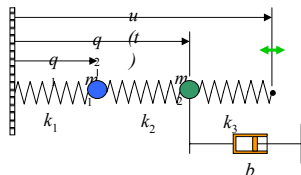
Many important examples

Electronic circuits



- Especially true after **feedback**
- Frequency response is key performance specification (think telephones)

Many mechanical systems



Quantum mechanics, Markov chains, ...

Many important tools

Frequency response, step response, etc

- Traditional tools of control theory
- Developed in the 1930's at Bell Labs; intercontinental telecommunications

Classical control design toolbox

- Nyquist plots, gain/phase margin
- Loop shaping

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Optimal control/estimator design

- Linear quadratic regulators
- Kalman estimators

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Robust control design

- H_∞ control design
- μ analysis for structured uncertainty

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Solutions of linear systems: the matrix exponential

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \longrightarrow y(t) = ???$$

Scalar linear system, with no input

$$\begin{aligned} \dot{x} &= ax \\ y &= cx \end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{at} x_0 \longrightarrow y(t) = ce^{at} x_0$$

Matrix version, with no input

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{At} x_0 \longrightarrow y(t) = Ce^{At} x_0$$

initial(A,B,C,D,x0);

Matrix exponential

- Analog to the scalar case; defined by series expansion:

$$e^M = I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \dots \quad P = \exp(M)$$

Stability of linear systems

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(t) = e^{At} x_0$$

Q: when is the system asymptotically stable?

$$\lim_{t \rightarrow \infty} x(t) = 0$$

Stability is determined by the eigenvalues of the matrix A

- Simple case: diagonal system

$$\dot{x} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} x \Rightarrow x(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} x_0$$

Stable if $\lambda_i \leq 0$
 Asy stable if $\lambda_i < 0$
 Unstable if $\lambda_i > 0$

- More generally: transform to "Jordan" form

$$\dot{x} = T^{-1} J T x \quad J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix} \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix}$$

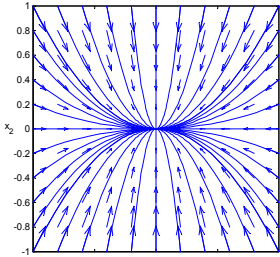
Stable if $\text{Re}(\lambda_i) \leq 0$
 Asy stable if $\text{Re}(\lambda_i) < 0$
 Unstable if $\text{Re}(\lambda_i) > 0$

Form of eigenvalues determines system behavior
Linear systems are automatically globally stable or unstable

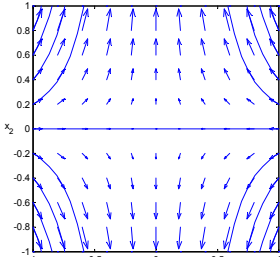
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Eigenstructure of linear systems

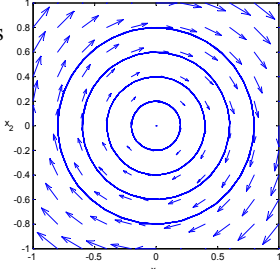
Real e-values
 $\text{Re}(\lambda_1) < 0$



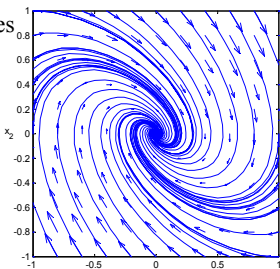
Real e-values
 $\text{Re}(\lambda_1) > 0$



Complex e-values
 $\text{Re}(\lambda_1) = 0$



Complex e-values
 $\text{Re}(\lambda_1) < 0$



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Impulse, step and frequency response

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$u(t) = 1(t)$$

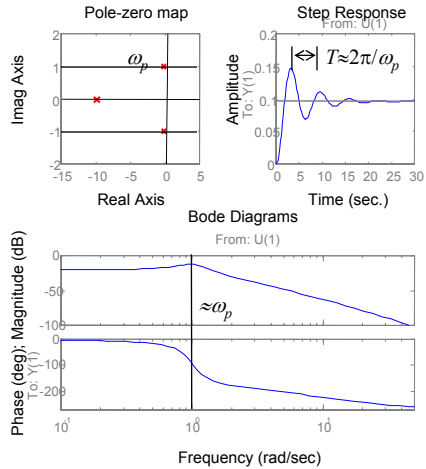
$$u(t) = A \sin(\omega t)$$

Effect of eigenstructure on impulse and step response

- Complex eigenvalues with small real part lead to oscillatory response
- Frequency of oscillations $\approx \omega_i$

Effects of eigenstructure on frequency response

- Eigenvalues determine “break points” for frequency response
- Complex eigenvalues lead to peaks in response function near ω_i

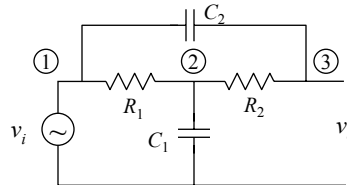
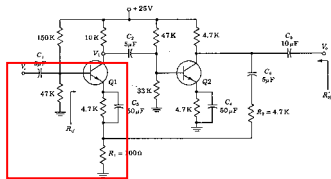


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Example: electrical circuit



“Bridged Tee Circuit”

Derivation based on Kirchoff's laws for electrical circuits (Ph 2)

- Sum of currents at nodes = 0:

$$C_1 \frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1} - \frac{v_2 - v_3}{R_2}$$

$$C_2 \frac{d(v_3 - v_1)}{dt} = -\frac{v_3 - v_2}{R_2}$$

- Rewrite in terms of new states: $v_{c1} = v_2$, $v_{c2} = v_3 - v_1$

$$\frac{d}{dt} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \\ \frac{1}{C_2} \end{bmatrix} v_i \quad \left| \quad v_o = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + v_i$$

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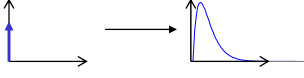
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Linear control systems and convolution

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

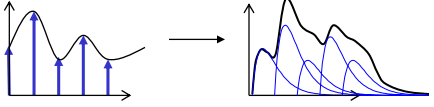
Impulse response, $h(t) = Ce^{At}B$

- Response to input “impulse”
- Equivalent to “Green’s function”



Linearity \Rightarrow compose response to arbitrary $u(t)$ using convolution

- Decompose input into “sum” of shifted impulse functions
- Compute impulse response for each
- “Sum” impulse response to find $y(t)$



Complete solution: use integral instead of “sum”

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input \Rightarrow 2X output when $x(0) = 0$

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Matlab tools for linear systems

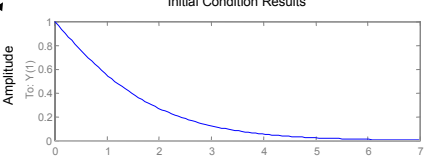
$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{Initial Condition Results}} + \underbrace{\int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{\text{Linear Simulation Results}}$$

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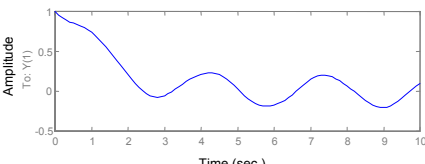
A = [-1 1; 0 -1]; B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impz(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
        
```



Initial Condition Results



Linear Simulation Results

Other MATLAB commands

- gensig, square, sawtooth – produce signals of different types
- step, impulse, initial, lsim – time domain analysis
- bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots

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Linearization around an equilibrium point

$$\begin{aligned} \dot{x} &= f(x, u) & \longrightarrow & \quad \dot{z} = Az + Bv \\ y &= h(x, u) & & \quad w = Cz + Dv \end{aligned}$$

“Linearize” around $x=x_e$

$f(x_e, u_e) = 0 \quad y_e = h(x_e, u_e)$

$z = x - x_e \quad v = u - u_e \quad w = y - y_e$

$$A = \frac{\partial f}{\partial x} \quad B = \frac{\partial f}{\partial u}$$

$$C = \frac{\partial h}{\partial x} \quad D = \frac{\partial h}{\partial u}$$

Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* to equilibrium point

Full nonlinear model Linear model (honest!)

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Local stability of nonlinear systems

Asymptotic stability of the linearization implies *local* asymptotic stability of eq pt

- Linearization around equilibrium point captures “tangent” dynamics

$$\dot{x} = f(x) = A \cdot (x - x_e) + o(x - x_e) \quad \leftarrow \text{higher order terms}$$

- If linearization is *unstable*, can conclude that nonlinear system is locally unstable
- If linearization is *stable* but not *asymptotically stable*, can’t conclude anything about nonlinear system:

$$\dot{x} = \pm x^3 \quad \xrightarrow{\text{linearize}} \quad \dot{x} = 0$$

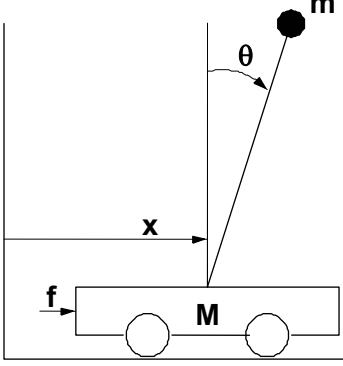
- linearization is stable (but not asy stable)
- nonlinear system can be asy stable or unstable

Local approximation particularly appropriate for control systems design

- Control often used to *ensure* system stays near desired equilibrium point
- If dynamics are well-approximated by linearization near equilibrium point, can use this to design the controller that keeps you there (!)

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Example: inverted pendulum on a cart



$$(M + m)\ddot{x} + ml \cos \theta \ddot{\theta} = -b\dot{x} + ml \sin \theta \dot{\theta}^2 + f$$

$$(J + ml^2)\ddot{\theta} + ml \cos \theta \ddot{x} = -mgl \sin \theta$$

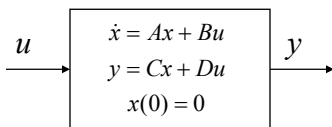
- State: $x, \theta, \dot{x}, \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around $\theta = \pi$

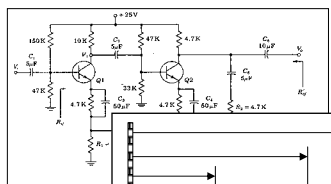
$$\frac{d}{dt} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 g l^2}{J(M+m) + M m l^2} & \frac{-(J + ml^2)b}{J(M+m) + M m l^2} & 0 \\ 0 & \frac{m g l (M+m)}{J(M+m) + M m l^2} & -m l b & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J + ml^2}{J(M+m) + M m l^2} \\ \frac{ml}{J(M+m) + M m l^2} \end{bmatrix} u$$

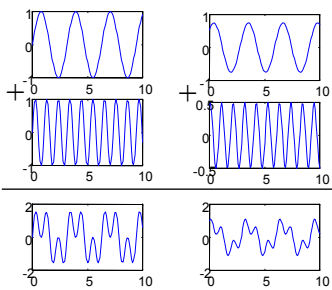
$$y = [1 \ 0 \ 0 \ 0]$$

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Summary: Linear Systems







$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Properties of linear systems

- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems

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