CDS 101: Lecture 3.1
Stability and Performance

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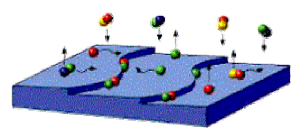
Goals:
- Describe different types of local stability of an equilibrium point
- Explain the difference between local stability, global stability, and related concepts
- Describe performance measures for (controlled) systems, including transients and steady state response

Reading:
- *Optional*: K. J. Astrom, Control Systems Design, Sections 3.5 (pp 90-92 and 102-109) and 7.4

Review from Last Week

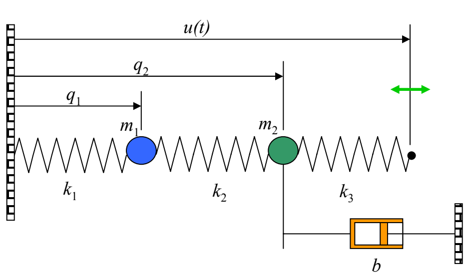
Model = state, inputs, outputs, dynamics

$$\frac{dx}{dt} = f(x,u)$$
$$y = h(x)$$



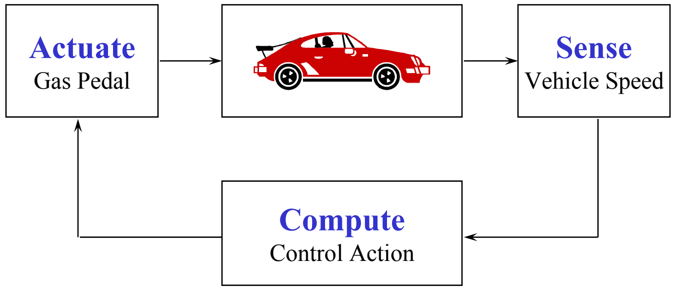
$$\frac{d}{dt}P_H = \sum_{H'} W_u(H',H)P_{H'} - W_u(H,H')P_H$$

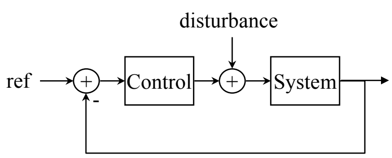
***Principle*: Choice of model depends on the questions you want to answer**




```
function dydt = f(t,y, k1, k2,
k3, m1, m2, b, omega)
u = 0.00315*cos(omega*t);
dydt = [
  y(3);
  y(4);
  -(k1+k2)/m1*y(1) +
      k2/m1*y(2);
  k2/m2*y(1) - (k2+k3)/m2*y(2)
      - b/m2*y(4) + k3/m2*u ];
```

Today: Stability and Performance





$$\dot{x} = f(x, k(y, r), d)$$
$$y = h(x)$$

Goal #1: Stability

- Check if *closed loop* response is stable

Goal #2: Performance

- Look at ability to track changes in reference and reject disturbances

Goal #3: Robustness (later)

Systems of Ordinary Differential Equations (ODEs)

Systems of coupled nonlinear equations (closed loop system ⇒ no inputs)

Component form

$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n)$$
$$\dot{x}_2 = f_1(x_1, x_2, \ldots, x_n)$$
$$\ldots$$
$$\dot{x}_n = f_1(x_1, x_2, \ldots, x_n)$$

Vector form

$$\dot{x} = f(x)$$
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

MATLAB form

```
function dxdt = f(t,x)
dxdt= [
  f1(x);
  f2(x);
  …
  fn(x);
];
```

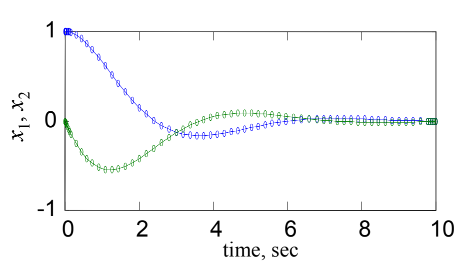
Example: damped oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

$$f(x) = \begin{bmatrix} x_1 \\ -x_1 - x_2 \end{bmatrix} \in \mathbb{R}^2$$

```
ode45('dosc', [0 10], [1 0]);
```

function — time interval — initial condition

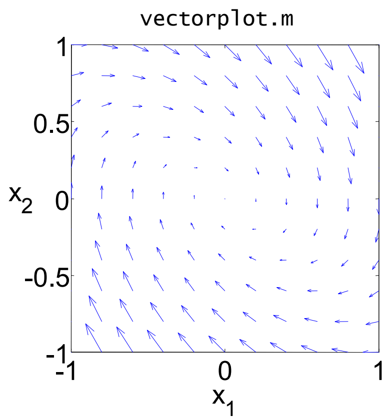


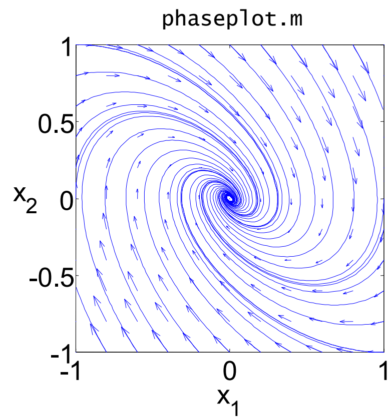
Phase Portraits (2D systems only)

Phase plane plots show 2D dynamics as *vector fields* and *stream functions*

- Plot $f(x)$ as a vector on the plane; stream lines follow the flow of the arrows

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$



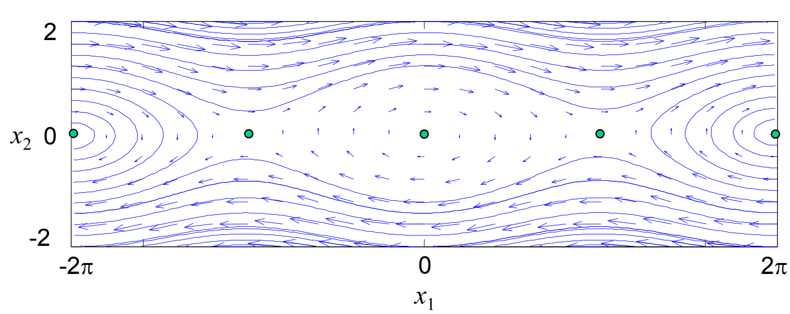


Equilibrium Points

Equilibrium points represent stationary conditions for the dynamics

The *equilibria* of the system $\dot{x} = f(x)$ are the points x_e such that $f(x_e) = 0$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) \end{bmatrix} \quad \Rightarrow \quad x_e = \begin{bmatrix} 0 \\ \pm n\pi \end{bmatrix}$$

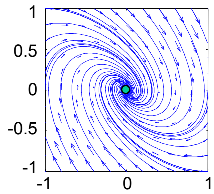


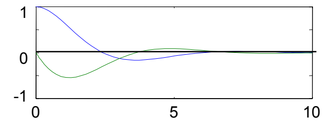
Stability of Equilibrium Points

An equilibrium point is:

Asymptotically stable if all initial conditions converge to equil. point

- Eq point is an *attractor* or *sink*

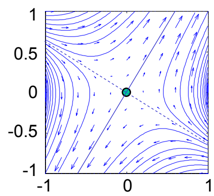


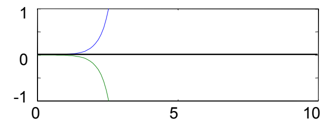


$\lim_{t\to\infty} x(t) = x_e$ for all $x(0) \in \mathbb{R}^n$

Unstable if some initial conditions diverge from the equilibrium point

- Eq point is a *source* (or *saddle*)

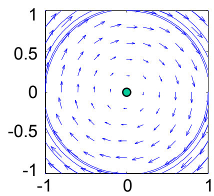


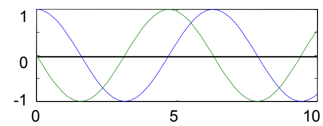


$\lim_{t\to\infty} \|x(t)\| = \infty$ for some $x(0) \in \mathbb{R}^n$

Stable if initial conditions that start near the equilibrium point, stay near

- Eq point is a *center*



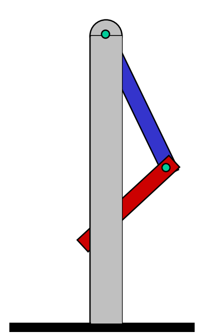


$\lim_{t\to\infty} \|x(t)\| < \varepsilon$ for all $\|x(0)\| < \delta_\varepsilon$

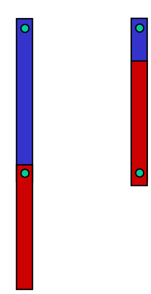
Example: Double Inverted Pendulum

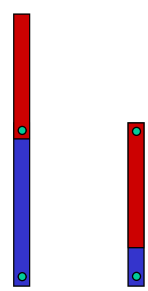
Two series coupled pendula

- States: pendulum angles (2), velocities (2)
- Dynamics: $F = ma$ (balance of forces)
- Dynamics are very nonlinear



Eq #1 Eq #2





Eq #3 Eq #4

Stability of equilibria

- Eq #1 is stable
- Eq #3 is unstable
- Eq #2 and #4 are unstable, but with some stable "modes"

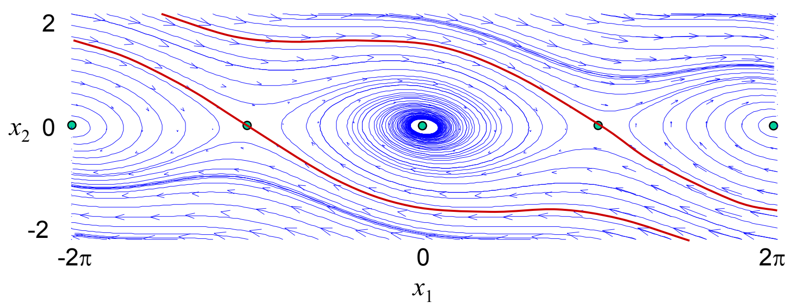
Local versus Global Behavior

Stability is a *local* concept

- Equilibrium points define the local behavior of the dynamical system
- Single dynamical system can have stable *and* unstable equilibrium points

Region of attraction

- Set of initial conditions that converge to a given equilibrium point



Example #2: Predator Prey (ODE version)

Alternative version of predator prey dynamics:

$$\dot{x}_1 = b_r x_1 - a x_1 x_2 - b x_1^2$$
$$\dot{x}_2 = a x_1 x_2 - d_f x_2 - b x_2^2$$

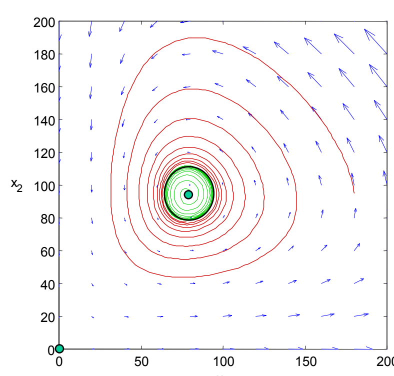
- written as ODE instead of difference equation
- includes second order effects that limit growth of rabbits (x_1) and foxes (x_2)

Equilibrium points (2)

- (0,0): unstable ⇒ species don't die out
- ~(77,92): unstable ⇒ no steady state population

Global features

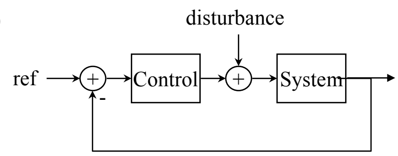
- Limit cycle ⇒ popultion of each species oscillates over time
- This is a *global* feature of the dynamics (not local to an equilibrium point)



Performance of Feedback Systems

Return to control system with inputs (ref, dist)

- How well does controller track reference?
- How well does controller reject disturbances?



Look at input/output measures:

- Step response: How well does output track a step change in reference value?
- Frequency response: How well does output track/reject oscillations at different frequencies

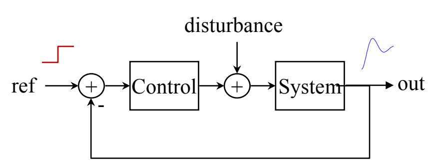
Note: be careful with nonlinear behavior!

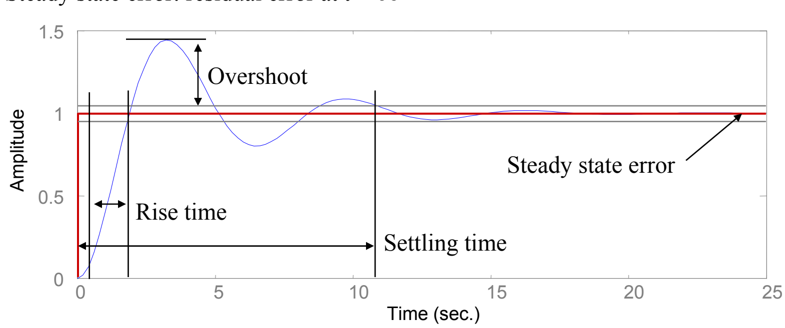
- For *linear* systems (next week), these questions are independent of input amplitude and initial condition (after transients die out)
- For *nonlinear* systems, answers can change depending on amplitude and starting condition (eg, which equilibrium point we are near)

Step Response Specifications

Standard specification for regulation performance

- Rise time: time required to move from 10% to 90% of final value
- Overshoot: ratio between amplitude of first peak and desired steady state value
- Settling time: time required to remain w/in *p*% (usually 2%) of final value
- Steady state error: residual error at t = ∞

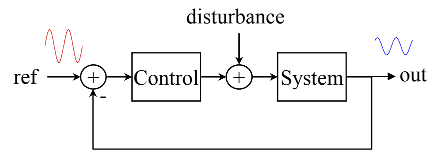




Frequency Response Specifications

In many applications, specify ability to track sinusoidal reference

- Example: stereo amplifier – would like flat response between 20 Hz & 20,000 Hz
- Can also be used for non-sinusoidal inputs by applying Fourier transform and superposition (linear systems only)

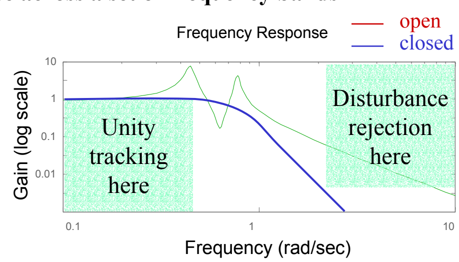


Typically specify the desired performance across a set of frequency bands

- Tracking: desire unity gain over range of frequencies that will be tracked

$$\text{Gain} = \frac{\text{output amplitude}}{\text{input amplitude}}$$

- Can also limit gain in high frequency range to keep from amplifying noise

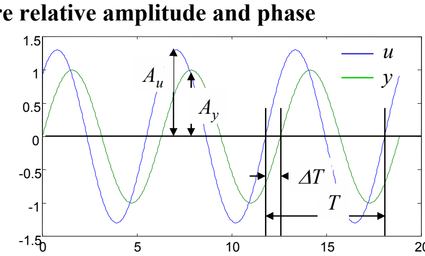


Caution: this specification implicitly assumes linear dynamics!

Computing Frequency Responses

Technique #1: plot input and output, measure relative amplitude and phase

- Use MATLAB or SIMULINK to generate response of system to sinusoidal output
- Gain = $A_y/A_{\mathbf{u}}$
- Phase = $2\pi \cdot \Delta T/T$
- Note: In general, gain and phase will depend on the input amplitude

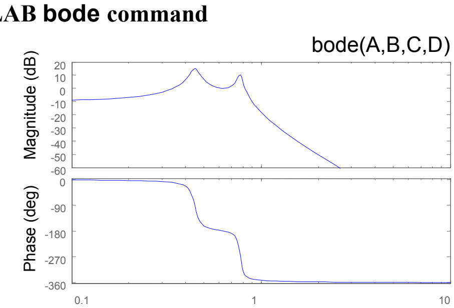


Technique #2 (linear systems): use MATLAB bode command

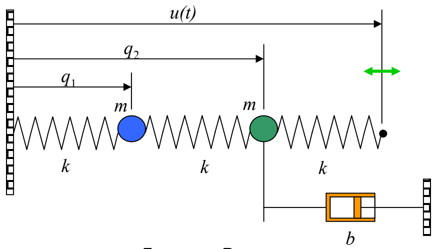
- Assumes linear dynamics in state space form:

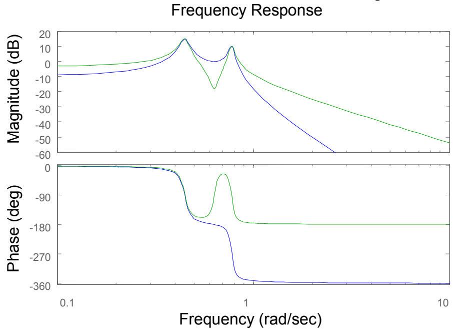
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Gain plotted on log-log scale
  - dB = 20 $\log_{10}$ (gain)
- Phase plotted on linear-log scale



Example: Coupled Spring System





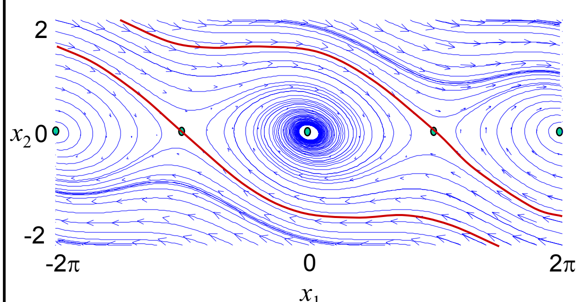

```
# springmass.m
m = 250; k = 50; b = 10;

A = [0       0     1   0;
     0       0     0   1;
   -2*k/m   k/m  -b/m  0;
   k/m    -2*k/m   0 -b/m];
B = [0; 0; 0; k/m];

C1 = [1 0 0 0]; D1 = [0];
C2 = [0 1 0 0]; D2 = [0];

bode(A,B,C1,D1);
hold;
bode(A,B,C2,D2);
```


Summary: Stability and Performance



Key topics for this lecture

- Stability of equilibrium points
- Local versus global behavior
- Performance specification via step and frequency response