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CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101/110
Homework Set #4

R. M. Murray
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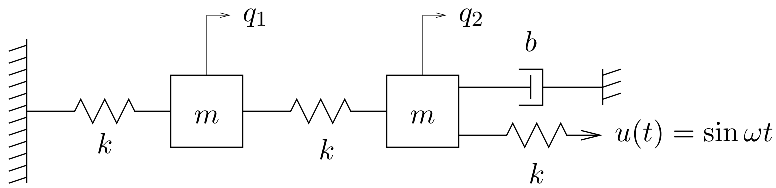
Due: 28 Oct 02

Note: In the upper left hand corner of the first page of your homework set, please put the class you are taking (CDS 101, CDS 110) and the number of hours that you spent on this homework set (including reading).

All students should complete the following problems:

1. For each of the following linear systems, determine whether the equilibrium point is asymptotically stable and, if it is, plot the step response and Bode plot for the system. If there are multiple inputs or outputs, plot the response for each pair of inputs and outputs.

   (a) *Coupled mass spring system.* Consider the coupled mass spring system we saw in class, which has a damper on only one of the masses:



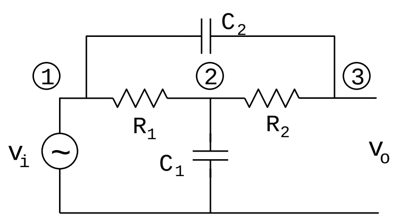
   The equations of motion are given by

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot q_1 \\ \dot q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k/m & k/m & 0 & 0 \\ k/m & -2k/m & 0 & -b/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k/m \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

   Use $m = 250$, $k = 50$, $b = 10$ for the parameter values. Note that this system does not diagonalize in the same way as the version given in homework set #2 (due to the asymmetric damper).

   (b) *Bridged Tee Circuit.* Consider the following electrical circuit, with input v_i and output v_o:

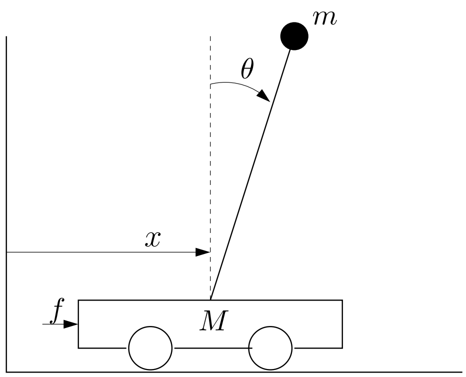


$$\frac{d}{dt}\begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\ \frac{1}{C_2 R_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + v_i$$

   where v_{c1} and v_{c2} are the voltages across the two capacitors. Assume that $R_1 = 100\Omega$, $R_2 = 100\Omega$ and $C_1 = C_2 = 1\mu\text{F}$.

2. (MATLAB/SIMULINK) Consider the inverted pendulum on a cart, as show in the figure below:



$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + F$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

$M = 0.5$ kg $\quad m = 0.2$ kg

$b = 0.1$ N/m/sec $\quad l = 0.3$ m

J $= 0.006$ kg m^2

This system has been modeled in SIMULINK in the file `hw4cartpend.mdl`, available from the course web page.

The linearization of the system was given in class:

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2gl^2}{J(M+m)+Mml^2} & \dfrac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \dfrac{mgl(M+m)}{J(M+m)+Mml^2} & \dfrac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J+ml^2}{J(M+m)+Mml^2} \\ \dfrac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Note: in the SIMULINK model, the output has been set to include all of the states ($y = x$). You will need this for part (c) below.

(a) Use the MATLAB `linmod` command to numerically compute the linearization of the original nonlinear system at the equilibrium point $(x, \theta, \dot{x}, \dot{\theta}) = (0, \pi, 0, 0)$. Compare the eigenvalues of the analytical linearization to those of the one you obtained with `linmod` and verify they agree.

(b) We can design a stabilizing control law for this system using "state feedback", which is a control law of the form $u = -Kx$ (we will learn about this more next week). The closed loop system under state feedback has the form

$$\dot{x} = (A - BK)x.$$

Show that the following state feedback stabilizes the linearization of the inverted pendulum on a cart: $K = (-1, 18.7, -1.7, 3.5)$.

(c) Now build a simulation for the closed loop, *nonlinear* system in SIMULINK. Use the file `hw4cartpend.mdl` for the nonlinear equations of motion in it (you should look in the file and try to understand how it works). Simulate several different initial conditions and show that the controller *locally* asymptotically stabilizes x_0. Include plots of a representative simulation for an initial condition that is in the region of attraction of the controller and one that is outside the region of attraction. (Hint: remember that the equilibrium point that we linearized about was not zero. You will need to account for this in your controller by implementing $u = -K(x - x_0)$ for the nonlinear system.)

(d) *Optional.* Use MATLAB to write an animation of your results and post them on the web.

Only CDS 110a/ChE 105 students need to complete the following additional problems:

3. Consider the following discrete time system

$$
\begin{aligned}
z[k+1] &= Az[k] + Bu[k] \\
y[k+1] &= Cz[k+1].
\end{aligned}
$$

In this problem we will derive the stability conditions, step response, and "convolution equation" for discrete time systems. (Don't worry, they are much easier than the ODE versions.)

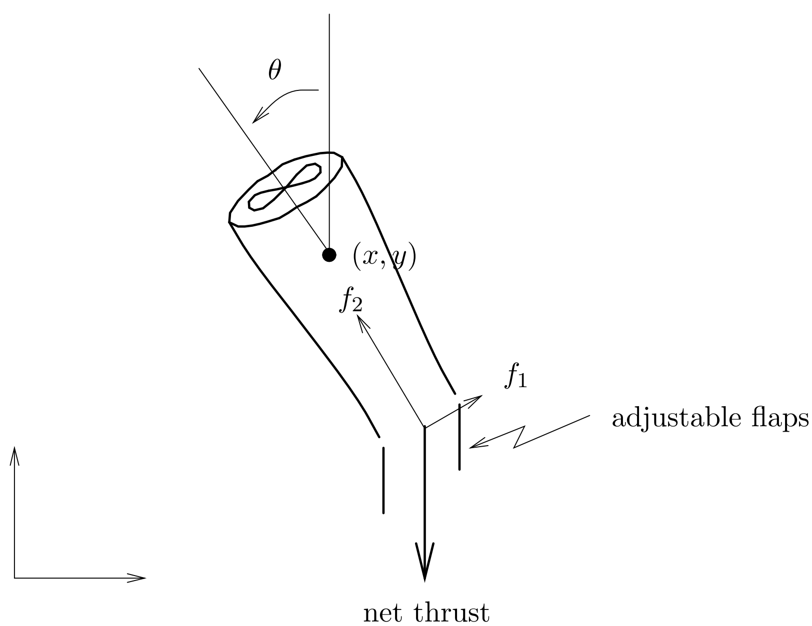
(a) Assume that the matrix A has a full basis of eigenvectors $\{v_i\}$, so that any initial condition can be written as a linear combination of these eigenvectors. Let λ_i be the associated eigenvectors and show the discrete time system is asymptotically stable if and only if $|\lambda_i| < 1$.

(b) Derive a formula for the transient response to an initial condition x_0 (the analog of $e^{AT}x_0$ for continuous time systems). Your answer should be in the form $y[k] = C\Phi[k]x_0$ where $\Phi[k]$ depends on the matrix A.

(c) Derive a formula for the output response of the system to a general input u_k. Your result should be expressed as a sum involving terms of the from $\Phi[k-j]$ (similar to the terms $e^{A(t-\tau)}$ for continuous time systems).

Note: you can find the answer to this in many books on linear control systems, including those on reserve in the library. You are encouraged to look at them, but make sure you *understand* the answer you write down and include enough detail for the TAs to follow your derivation.

4. Consider the motion of a small model aircraft powered by a vectored thrust engine, as shown below.



Let (x, y, θ) denote the position and orientation of the center of mass of the fan. We assume that the forces acting on the fan consist of a force f_1 perpendicular to the axis of the fan acting at a distance r and a force f_2 parallel to the axis of the fan. Let m be the mass of the fan, J the moment of inertia, γ the gravitational constant, and D the damping coefficient. Then the equations of motion for the fan are given by:

$$
\begin{aligned}
m\ddot{x} &= f_1 \cos\theta - f_2 \sin\theta - d\dot{x} \\
m\ddot{y} &= f_1 \sin\theta + f_2 \cos\theta - m\gamma - d\dot{y} \\
J\ddot{\theta} &= rf_1.
\end{aligned}
$$

It is convenient to redefine the inputs so that the origin is an equilibrium point of the system with zero input. If we let $u_1 = f_1$ and $u_2 = f_2 - mg$ then the equations become

$$\begin{aligned}
m\ddot{x} &= -mg\sin\theta - d\dot{x} + u_1\cos\theta - u_2\sin\theta \\
m\ddot{y} &= mg(\cos\theta - 1) - d\dot{y} + u_1\sin\theta + u_2\cos\theta \\
J\ddot{\theta} &= ru_1.
\end{aligned} \tag{1}$$

These equations are referred to as the *planar ducted fan equations* (this is an experiment that we use for CDS 111).

Use the following values for the parameters of the system:

$$\begin{array}{lll}
\gamma = 0.52\ \mathrm{m/sec}^2 & m = 4.25\ \mathrm{kg} & \\
 & & d = 0.1\ \mathrm{kg/sec} \\
r = 26\ \mathrm{cm} & J = 0.0475\ \mathrm{kg\ m}^2 &
\end{array}$$

The reason that gravity γ is not 9.8 $\mathrm{m/sec}^2$ is because of the presence of a counterweight to offset the weight of the fan.

(a) Rewrite the equations of motion in state space form (still nonlinear, still symbolic). Choose x and y as the outputs.

(b) Compute the linearization of the system around the "hover" state: $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = (0, 0, 0, 0, 0, 0)$. Your result should be in terms of the symbolic parameters (don't plug in the numbers, yet).

(c) Using the parameters above, determine if the linearization is stable, asymptotically stable, or unstable.

(d) Plot the step and frequency responses of the system from the two inputs to the two outputs (you should have eight plots total). (Hint: you might want to use the MATLAB `subplot` command to save some paper.)

5. *Optional.* Show that the Lyapunov equation

$$AP + P^T A = -Q$$

has a solution $P > 0$ for any $Q \geq 0$ if the matrix A is asymptotically stable (all eigenvalues in the left half plane). (This was an exam question in my linear systems class at Berkeley. You need to know some linear algebra in order to solve it.)