CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101/110
Homework Set #3

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Note: In the upper left hand corner of the first page of your homework set, please put the class you are taking (CDS 101, CDS 110) and the number of hours that you spent on this homework set (including reading).

All students should complete the following problems:

1. For each of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable, or unstable. To determine stability, you can either use a phase portrait (if appropriate) or simulate the system by perturbing the initial condition slightly from the equilibrium point and then seeing how the state evolves. (Note: if you know how to check stability through the linearization, you can also use this approach.)

   (a) *Duffing equation.* The Duffing equation is a model for a nonlinear mass spring system:

   $$m\ddot{x} = k(x + ax^3) - c\dot{x},$$

   where $m = 1000$ kg is the mass, $k = 250$ N/sec^2 is the nominal spring constant, $a = 10$ represents the nonlinearity of the spring, and $c = 1$ N/sec is the damping coefficient. Note that this is very similar to the mass spring systems we have studied in class, except for the nonlinearity.

   (b) *Modified Predator-Prey ODE.* In class we saw an ODE model for the predator-prey problem:

   $$\dot{x}_1 = b_r x_1 - a x_1 x_2 - b x_1^2$$
   $$\dot{x}_2 = a x_1 x_2 - d_f x_2 - b x_2^2$$

   Use the following parameters: $b_r = 0.7$, $d_f = 0.5$, $a = 0.007$, $b = 0.0005$. (Note: in the plot shown in Lecture 3.1, two equilibrium points were missing.)

   (c) *Pendulum.* The equations of motion for a single inverted pendulum are given by

   $$ml^2\ddot{\theta} = -b\dot{\theta} - mgl\sin(\theta)$$

   where θ is the angle of the pendulum ($\theta = 0$ rad corresponds to pointing down), $m = 1$ kg is the mass of the pendulum (assumed concentrated at the end), $l = 0.5$ m is the length of the pendulum, $b = 0.25$ N·m·sec is the damping coefficient, and $g = 9.8$ m/sec^2 is the gravitational constant.

2. (MATLAB/SIMULINK) Consider the cruise control system from Homework Set #1, problem 1. Set the gains of the system to their default values ($K_i = 100$, $K_p = 500$).

   (a) Plot the step response of the system (from 55 mph to 65 mph) and measure the rise time, overshoot, settling time, and steady state error.

   (b) Modify the block diagram to allow a sinusoidal reference signal superimposed on top of a commanded reference (so that you get something that oscillates around the nominal speed of 55 m/s). Plot the response of the system to a commanded reference speed that varies sinusoidally between 50 m/s and 60 m/s at a frequency of 1 Hz (about 6 rad/sec). Measure the relative amplitude and phase of the velocity with respect to the commanded input. Your answer should be the ratio of the output amplitude to the input amplitude (after subtracting off the means) and the number of radians of phase “lead” or “lag” between the sinusoids.

(c) Plot the frequency response for the cruise control system, showing the gain (relative amplitude) and phase at the following frequencies (all in rad/sec): 0.01 0.03 0.07 0.1 0.3 0.7 1 3 7 10. Your answer should be in the form of two plots: the relative amplitude (gain) versus frequency and the relative phase versus frequency. Use a logarithmic scale for the frequency and amplitude, and a linear scale for the phase.

(d) *Optional:* The system model given in Homework Set #1 originally had a saturation function on the input. This version of the model is called `hw1cruise_sat.mdl` and you can see the saturation by clicking into the vehicle block. Using that model, show that if we increase the amplitude of the desired oscillations sufficiently high, that the response of the system is no longer a pure sinusoid at the desired frequency.

You should use `hw1cruise.mdl` to solve this problem, available on the course web page.

Only CDS 110a/ChE 105 students need to complete the following additional problems:

3. Consider a second order system of the form

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u(t)$$

with initial conditions $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$.

(a) Compute the homogenous solution to this equation ($u(t) = 0$) with initial condition $y_0 = 1$, $\dot{y}_0 = 0$. This is the "impulse response" for this system. Plot the impulse response as a function of time for $\omega_n = 1$, $\zeta = 0.5$.

(b) Compute the response of the system to a sinusoidal input $u(t) = A\sin(\omega t)$. Your result should be analytical (a formula, like the ones I gave in lecture) and you should make sure to keep the effects of the initial conditions. Now assuming that the initial conditions have died out (i.e., ignoring the homogeneous part of the solution), plot the frequency response of the system on a Bode plot, labelling all relevant points. Note: you can find this solution worked out in many textbooks. You are encouraged to look for the solution, but make sure that you provide a derivation of your results and that you understand them. (Pretend that this might be the type of thing you were asked on a closed book section of the midterm.)

(c) Suppose that we now implement a feedback control law of the form

$$u(t) = k_1(y - v(t)) + k_2\dot{y},$$

which is intended to allow us to track a new input $v(t)$ (just like the cruise control example). Compute the frequency response of the closed loop and show that we can set the closed loop natural frequency ω_n' and damping ratio ζ' to arbitrary values by adjusting the gains k_1 and k_2. Give formulas for the gains in terms of the desired ω_n' and ζ'.

(d) *Optional:* Use the results from this problem to design a cruise control law for the system in problem #2 of last week's homework that has a settling time of 1 second and no overshoot.

4. For each of the following systems, use a quadratic Lyapunov function to show that the origin is asymptotically stable. Then investigate whether the origin is exponentially stable and/or globally asymptotically stable.

$$\text{(a)} \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned} \qquad\qquad \text{(b)} \quad \begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{aligned}$$

Hint: you might want to generate a phase plot of the system to get a better understanding of the dynamics. Time domain simulations may also give you insight about whether the system is exponentially stable or not. However, in all cases you should *prove* your results by finding a Lyapunov function and checking the conditions.