

Introduction to the Dynamics and Stability of Nonholonomic Mechanical Systems

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Some Highlights

- Symmetry *need not* lead to conservation laws (this fact alone has been known for at least a century). However, the geometry of the resulting *momentum equation* plays a significant role in both the control and stability theory for nonholonomic systems. For example, it was important for the understanding of the role of *gaits* in the snakeboard.
- Related to this, *geometric phases (holonomy)* is used in *locomotion, gaits and motion control*—e.g., the snakeboard. The foundation for this for holonomic systems uses the theory of reduction, both Hamiltonian and Lagrangian; for nonholonomic systems this foundation has been laid by Bloch, Krishnaprasad, Marsden, Scheurle and Murray, A. Lewis, Ostrowski, etc., with the introduction of the *non-holonomic connection*, a generalization of the now standard *mechanical connection*.
- Koon and Marsden have linked the Lagrangian and Hamiltonian approaches (e.g., the *Jacobi identity fails* for the “Poisson” description and how it fails is quite interesting and have also worked out the example of the bicycle.

- Koon and Marsden have extended the *falling cat theorem of Montgomery* to the nonholonomic case and to nonzero values of the momentum—the original result says that *the optimal reorientation control for a cat follows the equations for a colored particle in the Yang–Mills field for the mechanical connection*.
- Work of Zenkov, Bloch and JM extends the energy-momentum method to the nonholonomic case. This method goes back to Routh, Poincaré, Arnold, and Smale (and many others) with the recent block diagonalization developments due to Simo, Lewis, Posburgh and Marsden. In the nonholonomic energy momentum method, the *momentum equation plays a critical role*.
- Many other interesting links with work of Ratiu, Cendra, Perlmutter, Misiolek and JM on reduction by stages and the geometry of both Lagrangian and Hamiltonian reduction. Eg, the realization of *cocycles as curvatures of mechanical connections*, the *Euler–Poincaré view of the KdV equation on the Bott–Virasoro Lie algebra* due to Khesin et. al., etc.

The Planar Skater and Phases

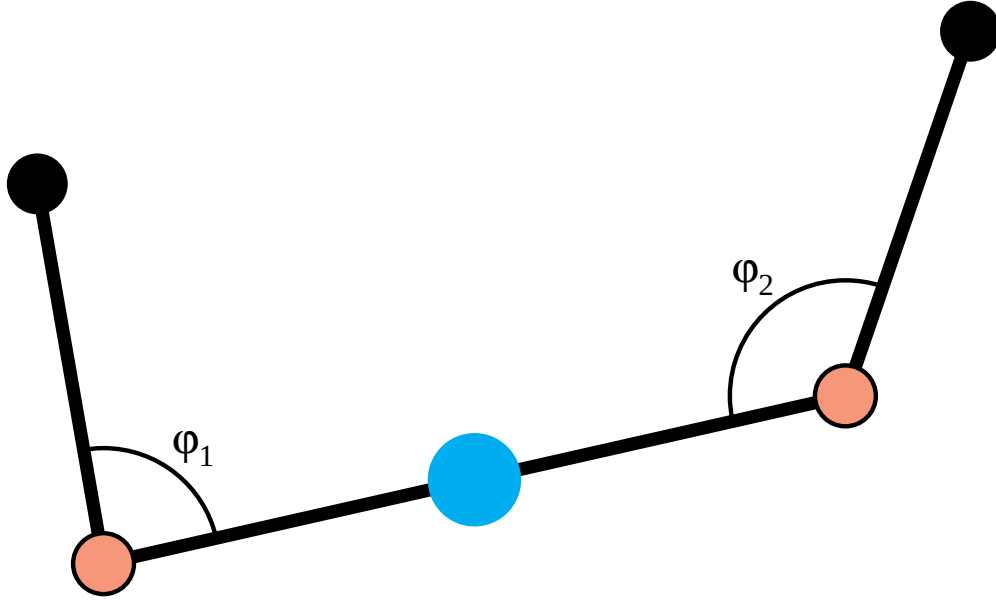


FIGURE 1: The planar skater consists of three interconnected bodies that are free to rotate about their joints.

- For the *planar skater* (and all mechanical systems with Lie group symmetries), conservation of angular momentum (in this case *zero*) takes a simple form. One can think of zero angular momentum (which the controls cannot alter) as a *constraint*.
- g the *group position* and r the *internal shape*:

$$g^{-1}\dot{g} = -A(r)\dot{r},$$

which defines the *mechanical connection*.

- For nonzero momentum p , then we use

$$g^{-1}\dot{g} = -A(r)\dot{r} + (I(r, g))^{-1}p,$$

where I is the *locked inertia tensor*, and p the (constant) total angular momentum.

- *Effect of phases*. The cyclic inputs to this system are shown as a base input curve, while the actual trajectory of the motion is shown lifted above the input curve.

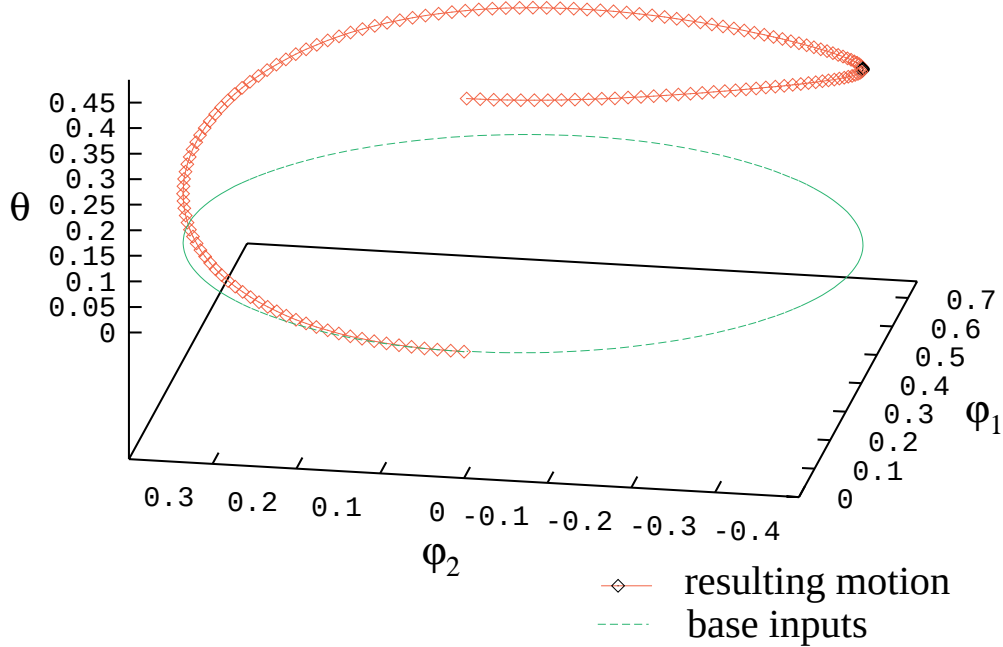


FIGURE 2: Input and output motions for the planar skater.

- After completing one cycle of internal shape changes, the skater has undergone a net rotation (change in θ). The *area*: enclosed by the base inputs is proportional to the rotation. This is the geometric phase, or *holonomy*, associated with the cyclic shape inputs.
- This holonomy can be illustrated using parallel trans-

port on the sphere (the same geometry used to understand the Foucault pendulum).

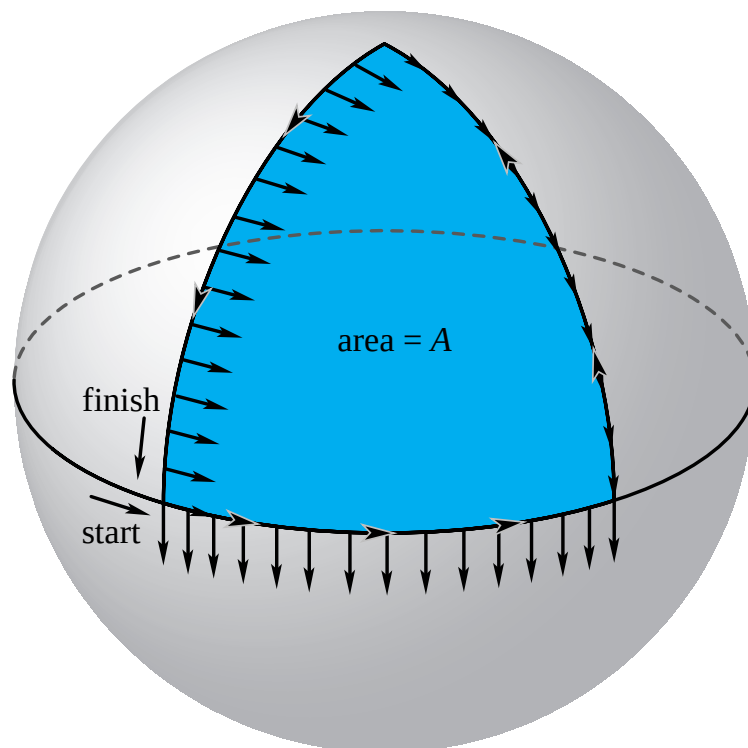


FIGURE 3: A parallel movement of your thumb around a spherical triangle produces a phase shift.

- The general notion of *holonomy* involves the splitting of the tangent space (more generally a distribution).

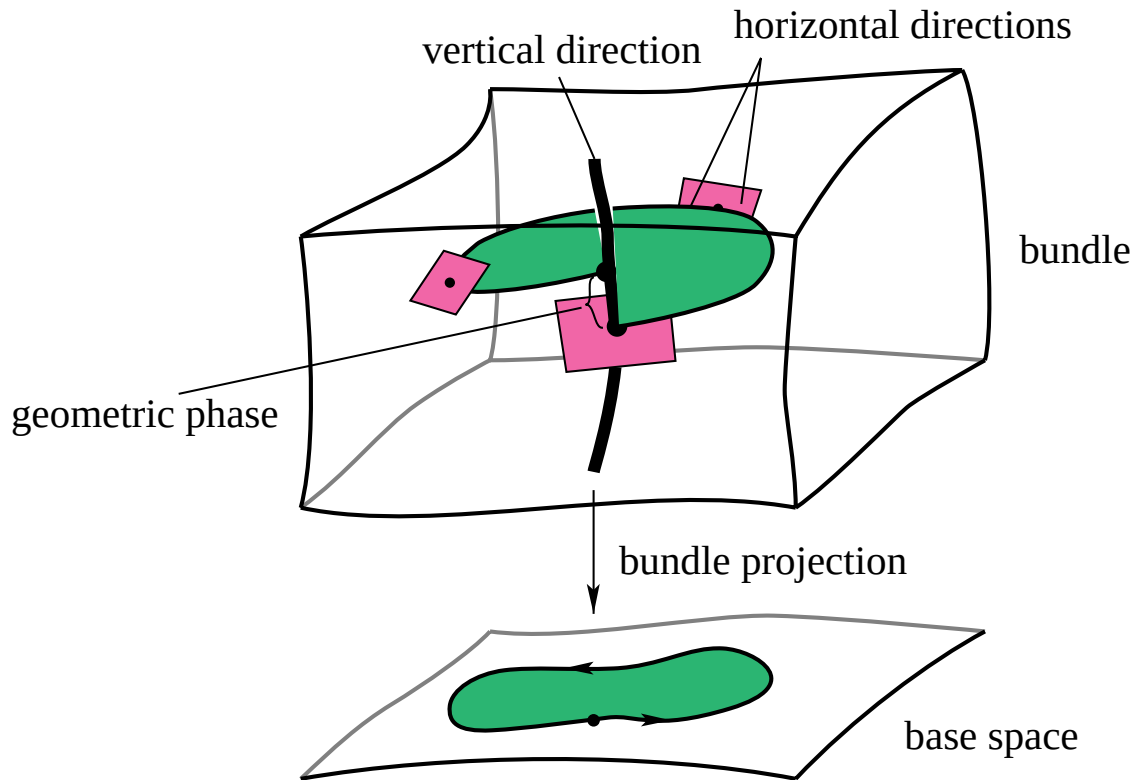


FIGURE 4: A connection divides the space into vertical and horizontal directions.

Systems with Rolling Constraints

- For systems with *rolling constraints or non-holonomic systems* one finds the equations of motion and properties of the solutions (such as the fate of conservation laws) using the *Lagrange–d’Alembert principle*.
- These systems are *not variational* but the basic

mechanics still comes down to $F = ma$.

- Consider a **configuration space** Q and a **distribution** $\mathcal{D}$ that describes the kinematic constraints; $\mathcal{D}$ is a collection of linear subspaces: $\mathcal{D}_q \subset T_q Q$, for $q \in Q$.
- $q(t) \in Q$ **satisfies the constraints**: $\dot{q}(t) \in \mathcal{D}_{q(t)}$.
- This distribution is, in general, **nonintegrable**; i.e., the constraints can be **nonholonomic**. Anholonomy is measured by the **curvature** of $\mathcal{D}$.
- A simple example of a nonholonomic system is the **rolling disk**. Here, the constraints of rolling define the distribution $\mathcal{D}$:

$$\begin{aligned}\dot{x} &= -\dot{\psi} R \cos \phi \\ \dot{y} &= -\dot{\psi} R \sin \phi,\end{aligned}$$

- The system dynamics is determined by a **Lagrangian** $L : TQ \rightarrow \mathbb{R}$, usually the kinetic minus the potential energy.

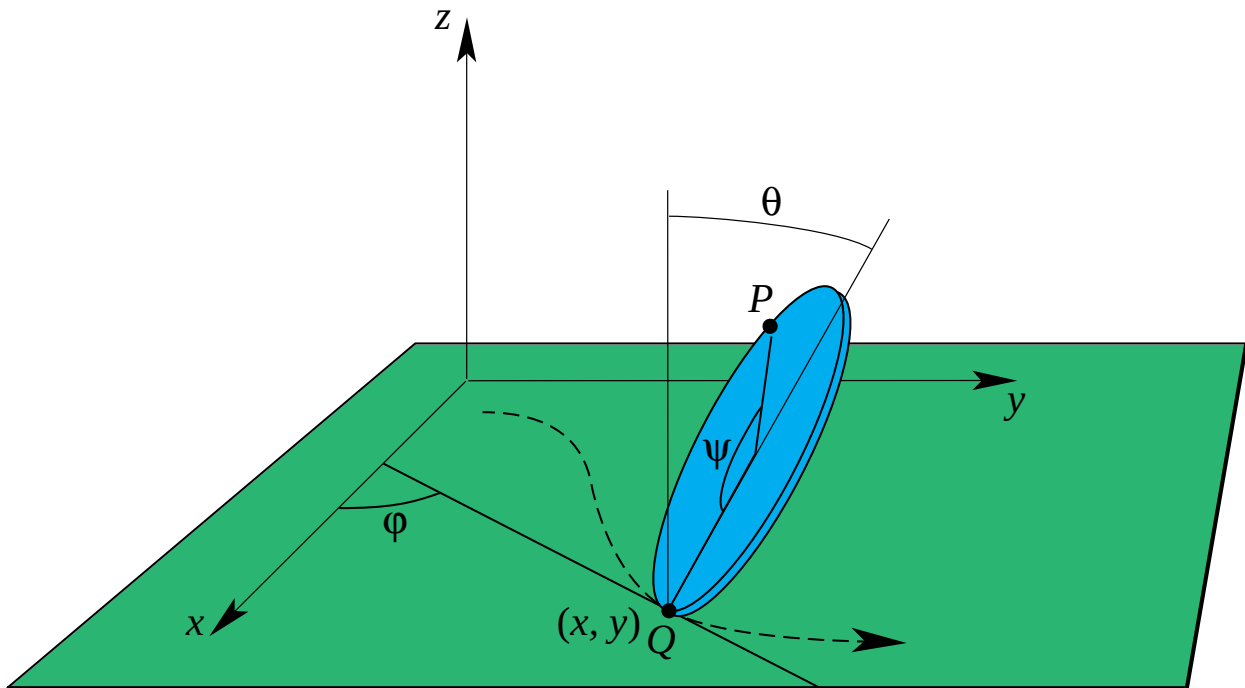


FIGURE 5: The rolling disk.

Lagrange–d’Alembert Principle

- Statement:

$$\delta \int_a^b L(q, \dot{q}) dt = 0,$$

where $\delta q(t) \in \mathcal{D}_q(t)$ for each $t, a \leq t \leq b$.

- This is **not a variational principle** (this issue was “put to rest” by Korteweg in 1899). A similar principle also governs the **Euler–Poincaré equations**—we recall these below.
- The Lagrange d’Alembert principle is supplemented by the condition that the curve itself satisfies the con-

straints.

- Another example is the *roller racer*—it is a wheeled vehicle with two segments connected by a rotational joint.

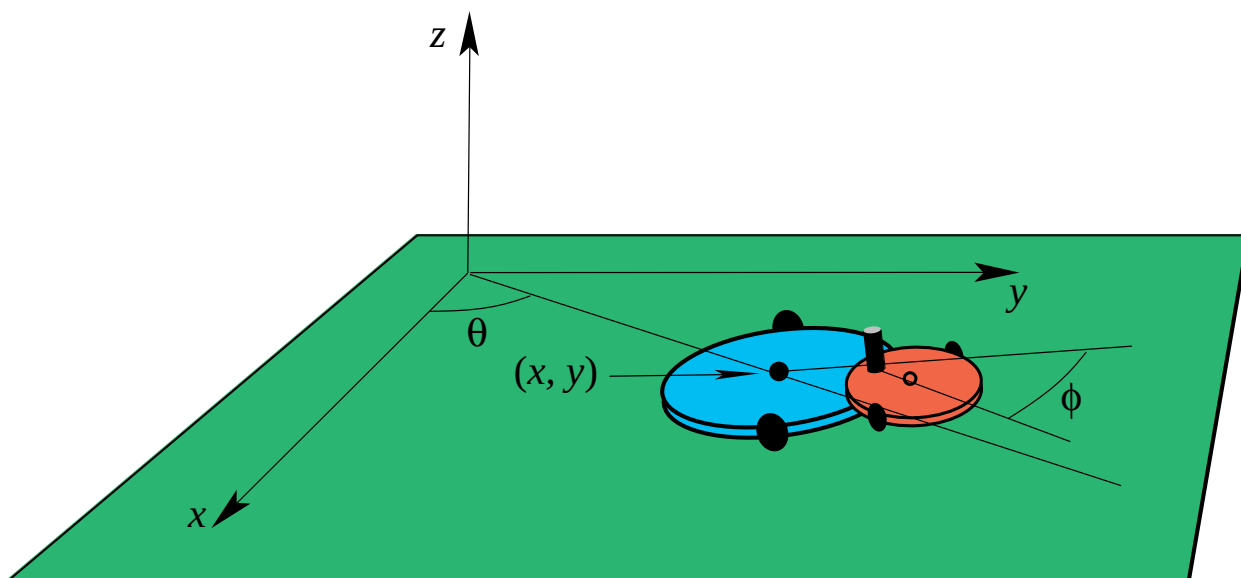


FIGURE 6: The roller racer—Tennessee racer.

- The roller racer is interesting because it *generates locomotion* similar to the snakeboard. *If you climb aboard and wiggle the joint, the vehicle moves!* (See <http://www.isr.umd.edu/krishna/> for interesting movies of this).

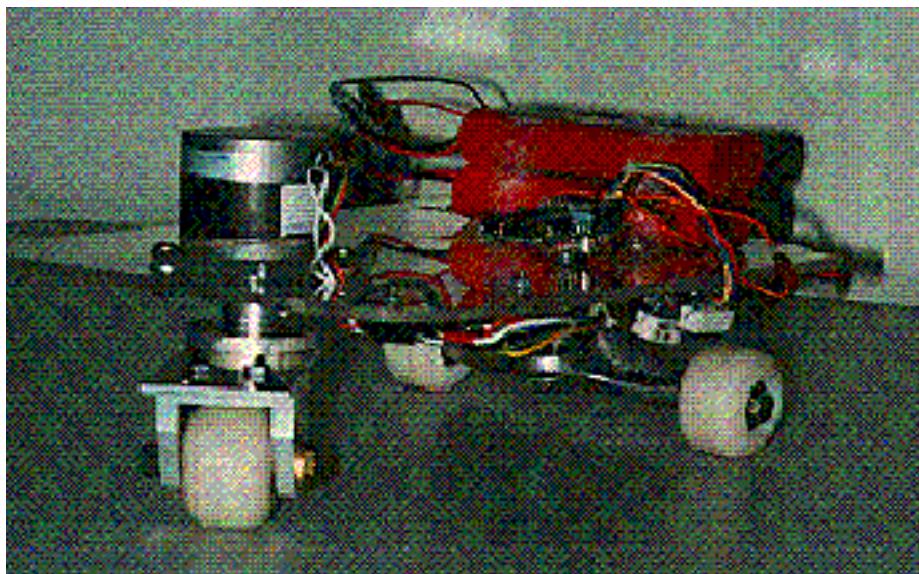


FIGURE 7: The controlled roller racer

- The *rattleback* is another famous example, illustrating the *lack of conservation of angular momentum*—it gets replaced by the *momentum equation*.

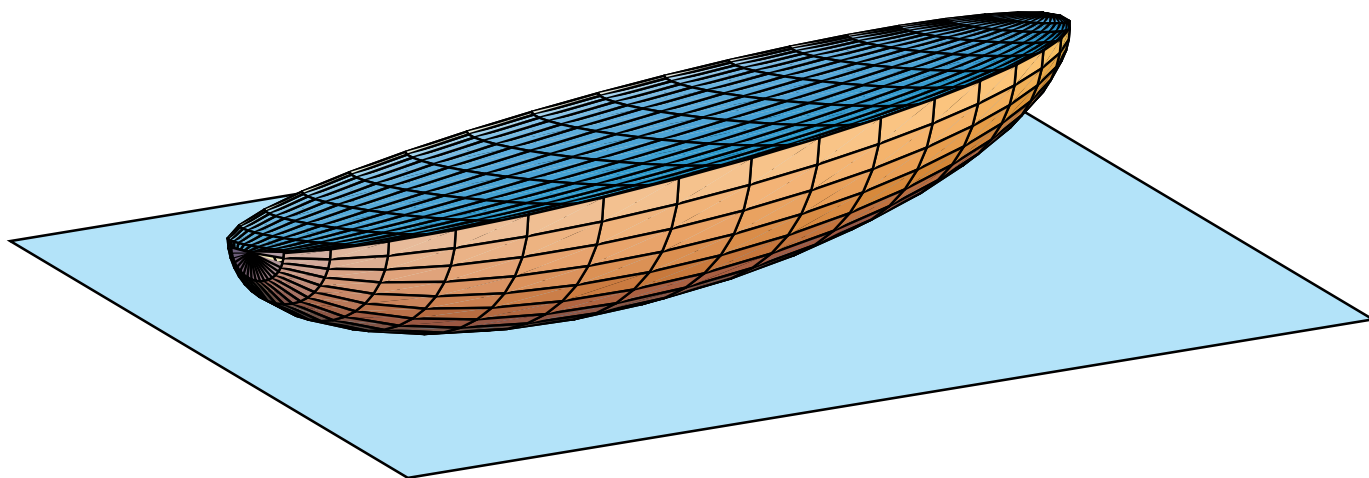


FIGURE 8: The rattleback.

Summary

Special Features of Nonholonomic Mechanics

- symmetry *need not lead to conservation laws*
- equilibria can be *asymptotically stable*
- energy is *still conserved*
- Jacobi's identity for Poisson brackets *can fail*

Lagrangian Reduction

- The idea is to pass the Lagrange d'Alembert principle to $\mathcal{D}/G$; similar to the reduction of Hamilton's principle (e.g., giving the *Routhian*, etc.).
- A simple example of reduction is the *free rigid body*; the Euler equations ($I\dot{\Omega} = I\Omega \times \Omega$) are *not variational*, but they satisfy a Lagrange–d'Alembert type of principle that is obtained by reducing Hamilton's principle from $\mathrm{SO}(3)$.

Aside on Euler–Poincaré (Holm, Ratiu, JM).

- Assume there is a *left* representation of Lie group G on the vector space V ; then G acts on $TG \times V^*$.
- Assume $L : TG \times V^* \rightarrow \mathbb{R}$ is left G -invariant.

- For $a_0 \in V^*$, define $L_{a_0} : TG \rightarrow \mathbb{R}$ by

$$L_{a_0}(v_g) = L(a_0, v_g).$$

- Define $l : \mathfrak{g} \times V^* \rightarrow \mathbb{R}$ by

$$l(g^{-1}v_g, g^{-1}a) = L(v_g, a).$$

- For a curve $g(t) \in G$, let

$$\xi(t) := g(t)^{-1}\dot{g}(t)$$

and define the curve $a(t)$ as the unique solution of the equation

$$\dot{a}(t) = -\xi(t)a(t)$$

with initial condition $a(0) = a_0$; *i.e.*, $a(t) = g(t)^{-1}a_0$.

Euler–Poincaré Equations

The Euler–Poincaré equations are the Lagrangian analogue of ***Hamiltonian systems on semidirect products*** (with its own long and distinguished history), such as the heavy top, compressible flow, MHD, etc. Of course it includes the ***pure Euler case*** as a special instance—just drop the last term in the Euler–Poincaré equations that follow.

The following are equivalent:

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1. With a_0 fixed, the standard *Hamilton principle* holds:

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t)) dt = 0$$

for variations with fixed endpoints.

2. The curve $g(t)$ satisfies the *Euler–Lagrange equations* for L_{a_0} .
3. The *Lagrange–d’Alembert-type principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t)) dt = 0$$

holds on $\mathfrak{g}$, using variations of ξ and a of the form

$$\delta \xi = \dot{\eta} + [\xi, \eta], \quad \delta a = -\eta a,$$

where $\eta(t)$ is a curve in $\mathfrak{g}$ vanishing at the endpoints.

4. The *Euler–Poincaré* equations hold on $\mathfrak{g} \times V^*$

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}_\xi^* \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a.$$

where $\rho_v : \mathfrak{g} \rightarrow V$ is given by $\xi \mapsto \xi v$, the infinitesimal action, $\rho_v^* : V^* \rightarrow \mathfrak{g}^*$ is its dual and $v \diamond a = \rho_v^* a$.

Nonholonomic Reduced Equations

- **Form of the reduced equations:**

$$\begin{aligned}g^{-1}\dot{g} &= -A(r)\dot{r} + B(r)p, \\ \dot{p} &= \dot{r}^T \alpha(r)\dot{r} + \dot{r}^T \beta(r)p + p^T \gamma(r)p \\ M(r)\ddot{r} &= -C(r, \dot{r}) + N(r, \dot{r}, p) + \tau\end{aligned}$$

- **first equation:** describes the *motion in the group variables* as the flow of a left-invariant vector field determined by the internal shape r , the velocity $\dot{r}$, as well as the nonholonomic momentum p , the component of momentum in the symmetry directions compatible with the constraints.
- **second equation: *momentum equation*.** The momentum equation has terms that are
 - quadratic in $\dot{r}$,
 - linear in $\dot{r}$ and p and
 - quadratic in p .

The coefficients $\beta(r)$ define a *connection*. This term is called the *transport part* of the momentum equation. The *curvature* of this connection plays an important role later on.

- **third equation:** describes the *motion in the shape variables* r . $M(r)$ is the *mass matrix*

of the system, C is the *Coriolis term*. τ are any external or control forces.

- This framework has been useful for *controllability, gait selection, and locomotion* of systems like the snakeboard.

Optimality: the falling cat theorem

- The ideas above of reducing variational principles aids in *optimality* of gaits, using a nonholonomic version of the *falling cat theorem* (Montgomery, Koon, and Marsden).
- The *mechanical connection* plays a role; this is the connection whose horizontal space is the kinetic energy metric orthogonal to the group orbits (just as in Kaluza–Klein theory).
- This is part of the *gauge theory of mechanics*.
- Since there is a connection, it makes sense to speak of a particle in the base moving in the associated *Yang–Mills field* (these are *Wong’s equations*, a generalization of the equations for a charged particle).
- The technique of Koon and Marsden is to understand this using the ideas of Lagrangian reduction (reduction of variational or Lagrange d’Alembert principles).

- Flavor of main result: *Optimal control paths in the base space (shape space) are solutions of Wong’s equations.*
- **Example:** Optimal trajectory selection (Ostrowski, Desai, Kumar). Numerics: a *variational optimal control algorithm* (a version of collocation adapted to mechanics).

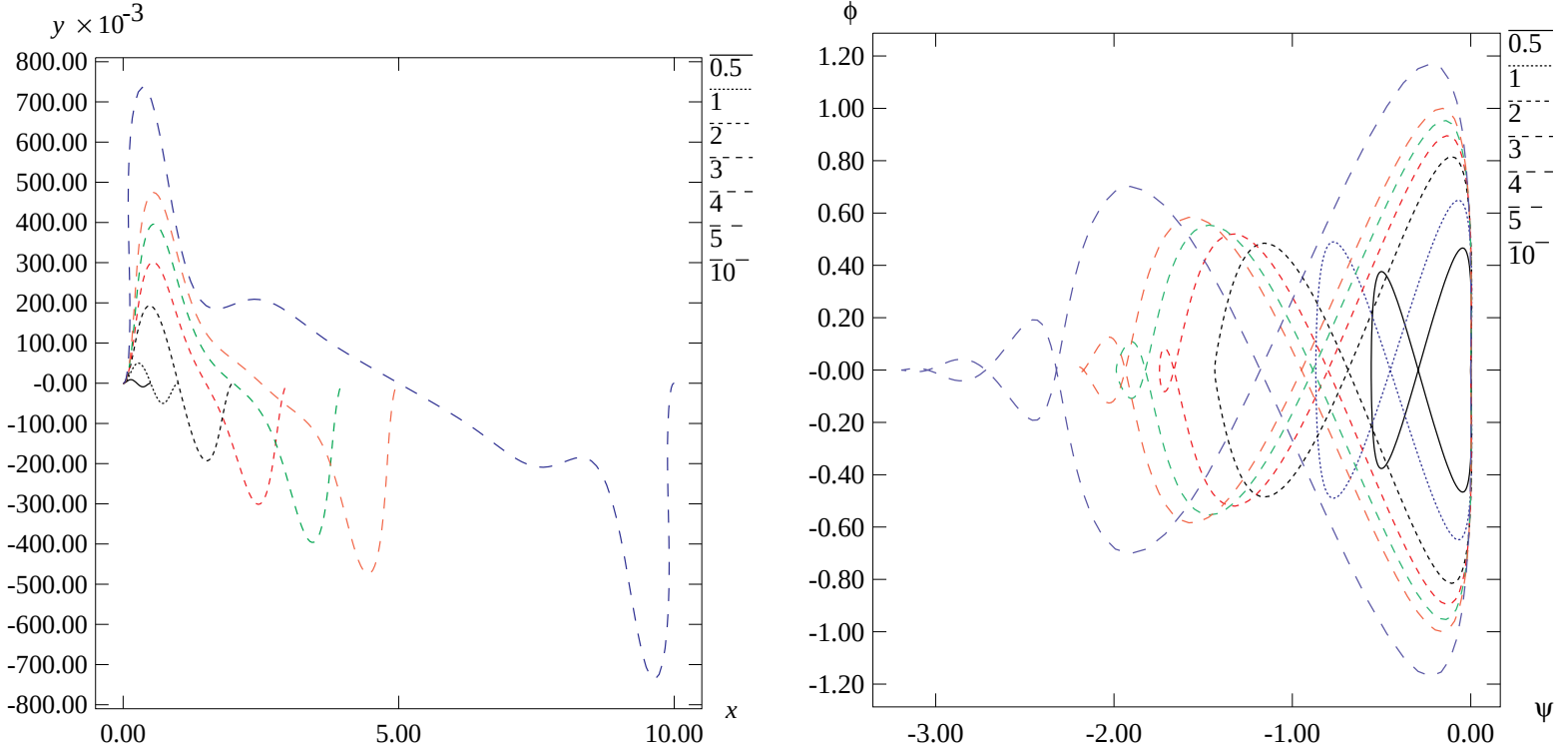


FIGURE 9: Optimal trajectories for the snakeboard “drive” gait. Shown are (left) the center of mass positions (x, y) and (right) the motion of the shape variables, ϕ and ψ .

Stabilization

- *Satellite stabilization*

Bloch, Krishnaprasad, Marsden, and Sánchez de Alvarez [1992], found a feedback control that stabilizes rigid body dynamics undergoing steady rotation about its middle (unstable) axis using an internal rotor aligned with the long axis. Stability was determined by the Arnold (energy-Casimir) method.

- Applicability to other systems requires a *systematic and structured approach*—recent work of Bloch, Marsden, and Leonard achieves that.
- We show that the feedback stabilized system can be realized as the Euler–Lagrange equations of a *controlled Lagrangian*; this new Lagrangian is obtained by a *Kaluza–Klein type of construction*.
- We expect that such controls can be combined with the symmetry breaking controls that can be applied to problems of *tracking*.
- The technique allows one to make use of the *energy-momentum method*. Geometric phases are also very important here.

- **Examples:** *inverted pendulum on a cart*, *satellites* and *underwater vehicles with internal rotors*, the *inverted spherical pendulum on a “hockey puck”*.
- This approach is done within the context of mechanics; one can understand the stabilization in terms of the effective creation of an energy extremum.
- **Many questions left:** the role of damping, the swing-up problem, the efficiency and energy consumption of the method, etc.
- For problems with nonholonomic constraints (like a *bicycle*) we hope that a similar construction will work.

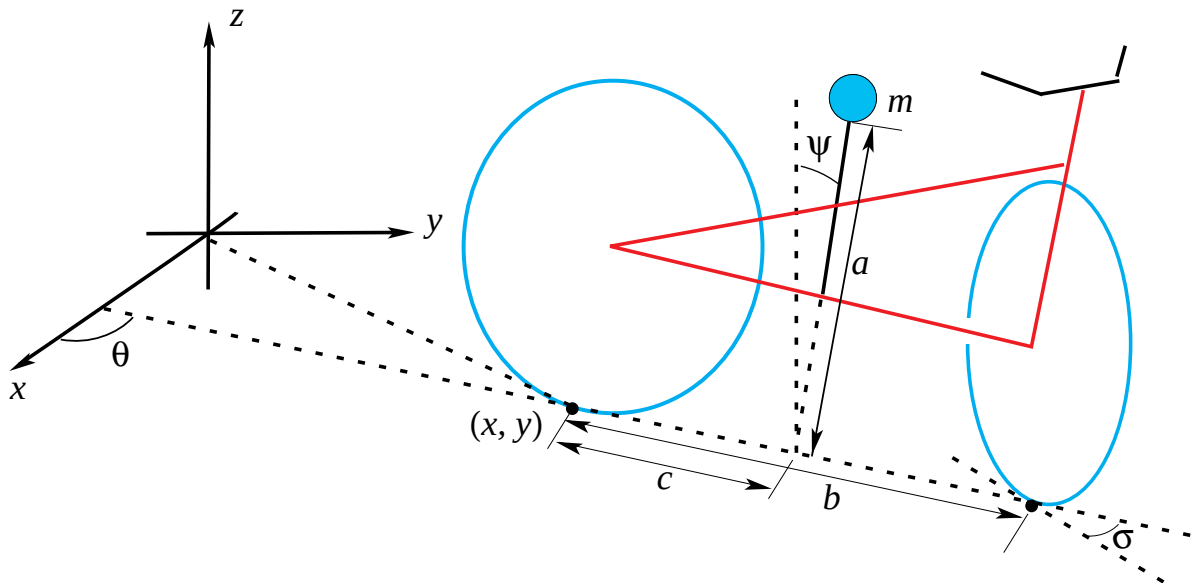


FIGURE 10: The mechanics of a bicycle (from Koon and Marsden [1996]).

A Rigid Body with a Symmetric Rotor

System: a carrier rigid body with a rotor aligned along the third principal axis. The rotor spins under the influence of a torque u .

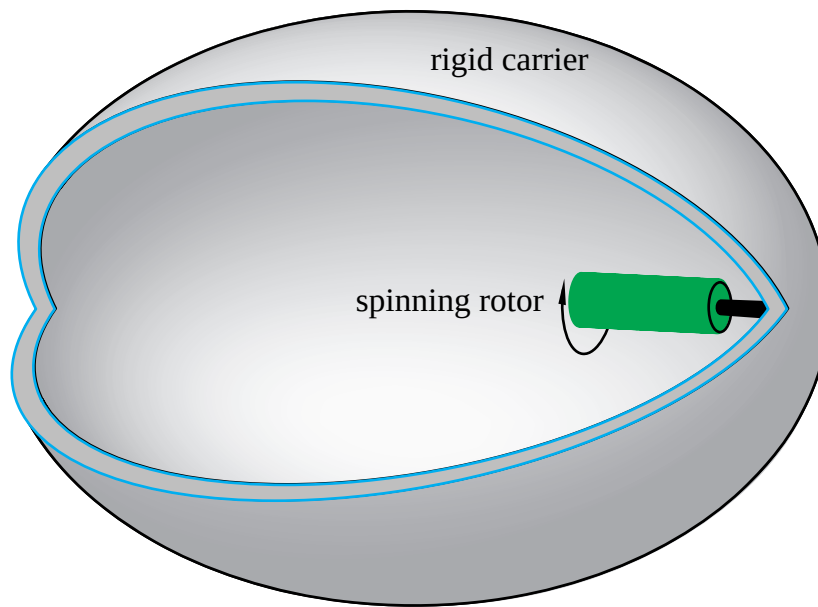


FIGURE 11: A rigid body with a rotor aligned on the long axis

Equations of motion:

$$\dot{\Pi} = \Pi \times \Omega \quad \dot{l} = u$$

where

- $I_1 > I_2 > I_3$, carrier *moments of inertia*,
- $J_1 = J_2$ and J_3 , *rotor moments of inertia*.
- $\Omega = (\Omega_1, \Omega_2, \Omega_3)$, *carrier angular velocity*

- α , *relative angle of the rotor*.
- **Body angular momenta:** with $\lambda_i = I_i + J_i$,

$$\begin{aligned}\Pi_1 &= \lambda_1 \Omega_1 & \Pi_2 &= \lambda_2 \Omega_2 \\ \Pi_3 &= \lambda_3 \Omega_3 + J_3 \dot{\alpha} & l_3 &= J_3(\Omega_3 + \dot{\alpha}).\end{aligned}$$

- **Equations in components:**

$$\begin{aligned}\dot{\Pi}_1 &= \left(\frac{1}{I_3} - \frac{1}{\lambda_2} \right) \Pi_2 \Pi_3 - \frac{l_3 \Pi_2}{I_3} \\ \dot{\Pi}_2 &= \left(\frac{1}{\lambda_1} - \frac{1}{I_3} \right) \Pi_1 \Pi_3 + \frac{l_3 \Pi_1}{I_3} \\ \dot{\Pi}_3 &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \Pi_1 \Pi_2 \\ \dot{l}_3 &= u.\end{aligned}$$

- **Conservation law:** If $u = 0$, then l_3 is a constant of motion and the remaining equations are Hamiltonian (Lie-Poisson) with

$$H = \frac{1}{2} \left(\frac{\Pi_1^2}{\lambda_1} + \frac{\Pi_2^2}{\lambda_2} + \frac{(\Pi_3 - l_3)^2}{I_3} \right) + \frac{1}{2} l_3^2.$$

- **Feedback control law:**

$$u = k \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \Pi_1 \Pi_2,$$

where k is a *gain parameter*.

- System retains the S^1 symmetry and $P_k = l_3 - k\Pi_3$ is a new conserved quantity.
- ***The closed loop equations*** (eliminating the rotor variable) are

$$\begin{aligned}\dot{\Pi}_1 &= \Pi_2 \left(\frac{(1-k)\Pi_3 - P_k}{I_3} \right) - \frac{\Pi_3\Pi_2}{\lambda_2} \\ \dot{\Pi}_2 &= -\Pi_1 \left(\frac{(1-k)\Pi_3 - P_k}{I_3} \right) + \frac{\Pi_1\Pi_3}{\lambda_1} \\ \dot{\Pi}_3 &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \Pi_1\Pi_2.\end{aligned}$$

- These equations are Hamiltonian with

$$H = \frac{1}{2} \left(\frac{\Pi_1^2}{\lambda_1} + \frac{\Pi_2^2}{\lambda_2} + \frac{((1-k)\Pi_3 - P_k)^2}{(1-k)I_3} \right) + \frac{1}{2} \frac{P_k^2}{J_3(1-k)},$$

again using the Lie-Poisson (rigid body) Poisson structure on $\mathfrak{so}(3)^*$.

Noteworthy special cases:

1. $k = 0$, ***uncontrolled case***,
2. $k = J_3/\lambda_3$, the ***driven case***, where $\dot{\alpha} = \text{constant}$.

Stabilization

Use the *Arnold method for stability*.

- Let $P = 0$ and consider the *equilibrium* $(0, M, 0)$.
- **Technique:** Look at $H + C$ where $C = \varphi(\|\Pi\|^2)$.
Pick φ so that $\delta(H + C)|_{(0,M,0)} = 0$,
- Compute definiteness of $\delta^2(H + C)$.

Rigid body phase portrait; uncontrolled case— $k = 0$:

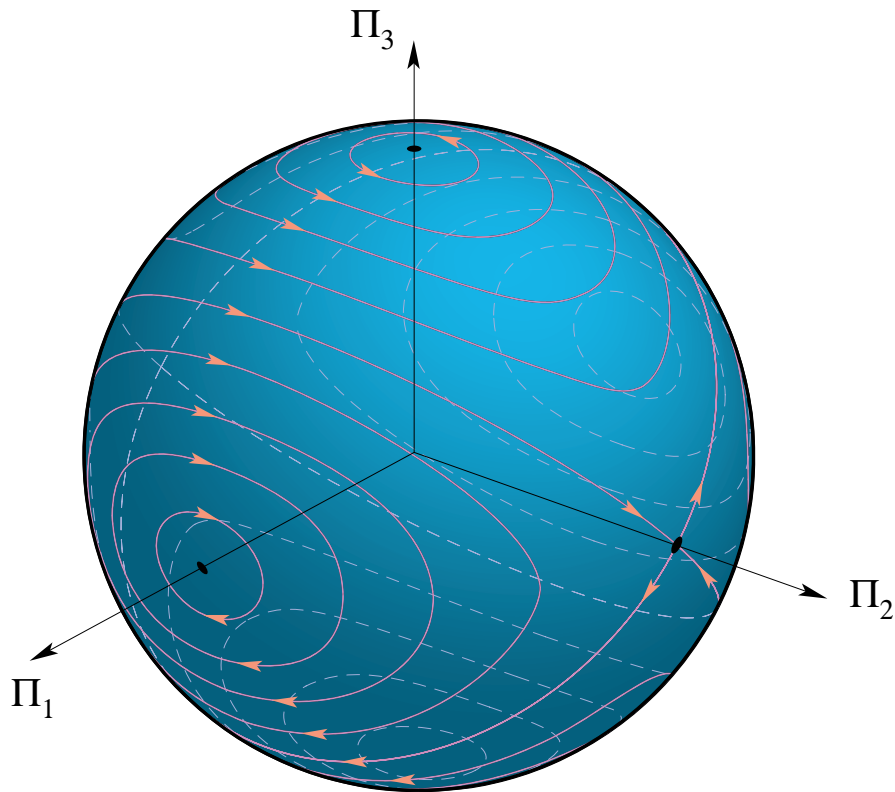


FIGURE 12: The rigid body phase portrait

The feedback control in effect modifies the Lagrangian to *interchange the moments of inertia* of the

system.

The stabilization that takes place as the gain is increased can be viewed in terms of a *modification of the phase portrait of the rigid body*.

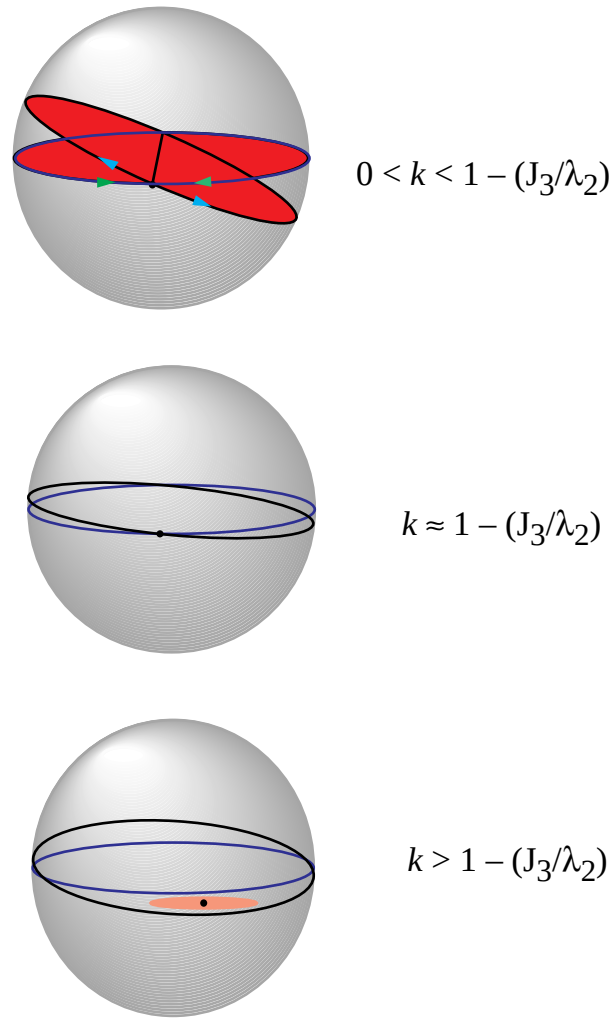


FIGURE 13: Stabilization by feedback

Nonholonomic Energy-Momentum Method

Three principal cases are considered:

1. Pure Transport Case

- In this case, terms quadratic in $\dot{r}$ are not present in the momentum equation and in this case *the momentum equation is an equation of parallel transport* for a certain associated connection.
- When the *curvature of this connection vanishes* the transport equation defines invariant surfaces, which *play the role of the usual momentum map level surfaces* in the standard energy-momentum method. For nonholonomic systems, the invariant surfaces *do not arise from conservation of momentum*.
- One can get stable, but not asymptotically stable, relative equilibria. Examples include the *rolling disk, a body of revolution rolling on a horizontal plane*, and a *sphere rolling inside a surface of revolution—the Routh problem*.

2. Integrable Transport Case

- Here terms quadratic in $\dot{r}$ are present in the momentum equation and thus it is not a pure trans-

port equation. However, there is still a well defined connection and in this case, we assume that the *transport part is integrable*; i.e., the associated connection has zero curvature.

- In this case, relative equilibria may be asymptotically stable. We find a generalization of the energy-momentum method which *gives conditions for asymptotic stability*. An example is the *roller racer*.

3. Nonintegrable Transport Case

- Again, the terms quadratic in $\dot{r}$ are present in the momentum equation and thus it is *not a pure transport equation*. However, the transport part is not integrable.
- We are able to demonstrate asymptotic stability using the *Lyapunov–Malkin Theorem* (a main tool used by, e.g., Karapetyan) which we relate to an *energy-momentum type analysis and the center manifold theorem*. An example is the *rattleback*.

The Roller Racer Again

A *glimpse* into how the energy momentum method works for the roller racer.

- The *Lagrangian* is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\phi})^2$$

- The *constraints* are

$$\begin{aligned}\dot{x} &= \cos \theta \left(\frac{d_1 \cos \phi + d_2}{\sin \phi} \dot{\theta} + \dot{\phi} \right), \\ \dot{y} &= \sin \theta \left(\frac{d_1 \cos \phi + d_2}{\sin \phi} \dot{\theta} + \dot{\phi} \right).\end{aligned}$$

- The *configuration space* is $\text{SE}(2) \times \text{SO}(2)$ and the Lagrangian and the constraints are *invariant under the left action of* $\text{SE}(2)$.
- The *nonholonomic momentum* is (Tsakiris [1995])
$$p = m(d_1 \cos \phi + d_2)(\dot{x} \cos \theta + \dot{y} \sin \theta) + [(I_1 + I_2)\dot{\theta} + I_2\dot{\phi}] \sin \phi.$$

This equals *the angular momentum of the system about the point of intersection of the two axles*.

- The *momentum equation* is

$$\dot{p} = \frac{((I_1 + I_2) \cos \phi - m d_1 (d_1 \cos \phi + d_2))}{m(d_1 \cos \phi + d_2)^2 + (I_1 + I_2) \sin^2 \phi} p \dot{\phi} + \frac{(d_1 + d_2 \cos \phi)(I_2 d_1 \cos \phi - I_1 d_2)}{m(d_1 \cos \phi + d_2)^2 + (I_1 + I_2) \sin^2 \phi} \dot{\phi}^2.$$

- The *curvature* of the connection describing the transport part of this equation is zero (the “base space” is one dimensional in this case).
- Writing the Lagrangian using p instead of $\dot{\theta}$, we obtain the *energy function* for the roller racer:

$$E = \frac{1}{2} g(\phi) \dot{\phi}^2 + \frac{1}{2} I(\phi) p^2,$$

where

$$g(\phi) = I_2 + \frac{m d_2^2}{\sin^2 \phi} - \frac{[m(d_1 \cos \phi + d_2) d_2 + I_2 \sin^2 \phi]^2}{\sin^2 \phi [m(d_1 \cos \phi + d_2)^2 + (I_1 + I_2) \sin^2 \phi]}$$

and

$$I(\phi) = \frac{1}{(d_1 \cos \phi + d_2)^2 + (I_1 + I_2) \sin^2 \phi}.$$

- The *amended potential* is given by

$$U = \frac{p^2}{2[(d_1 \cos \phi + d_2)^2 + (I_1 + I_2) \sin^2 \phi]}.$$

- Computations using the *locked inertia tensor* $I(\phi)$ show that the roller racer has a two dimensional manifold of relative equilibria parametrized by ϕ and p .
- *These relative equilibria are motions of the roller racer in circles about the point of intersection of lines through the axles.*
- For such motions, p is the system momentum about this point and ϕ is the relative angle between the two bodies.
- We apply general energy-momentum stability conditions to obtain *the condition for stability of a relative equilibrium* $\phi = \phi_0$, $p = p_0$ of the roller racer:

$$(d_1 + d_2 \cos \phi_0)(I_2 d_1 \cos \phi_0 - I_1 d_2) p_0 > 0.$$

- This equilibrium is *stable modulo SE(2) and in addition asymptotically stable with respect to $\dot{\phi}$.*

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