Lecture 6
Verification of Hybrid Systems

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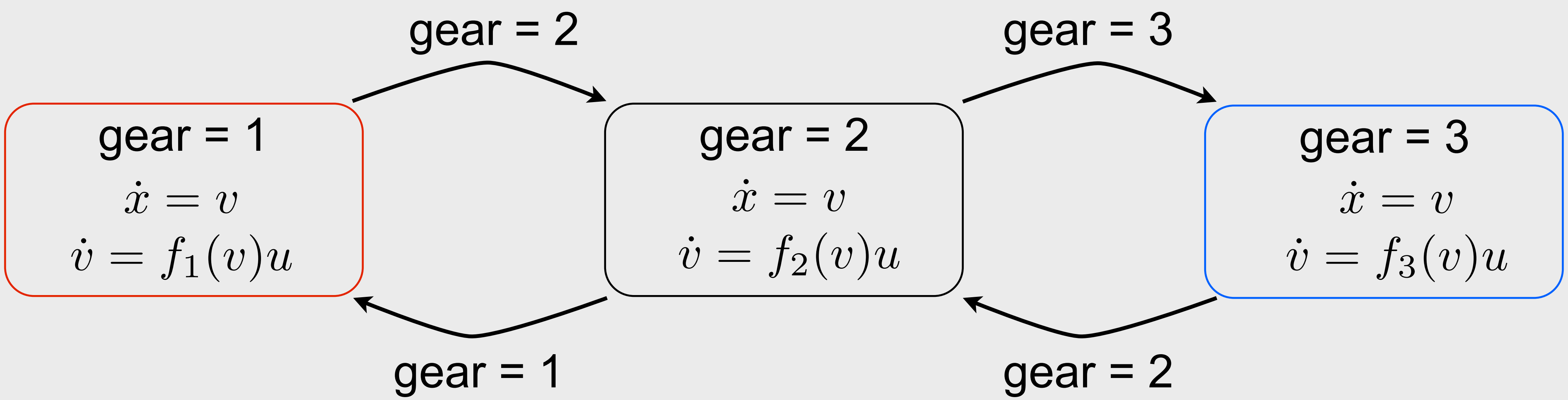
AFRL, 25 April 2012

Outline:

- A hybrid system model
- Finite-state abstractions and use of model checking
- Deductive verification and optimization-based construction of certificates
- Approximate bisimulation functions

Hybrid systems: example



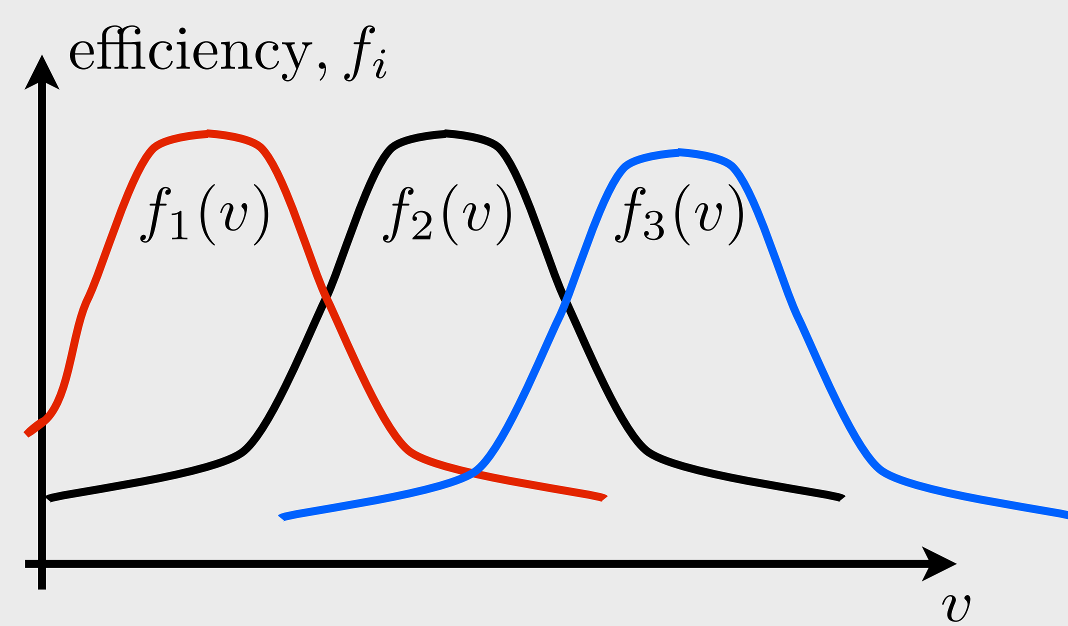
Model and tools so far (in the course) help reason about discrete evolution of systems:

- does there exist a control sequence for which φ holds, or
- do all control sequences lead to executions for which φ holds with

$$\varphi = \Box\,(\text{gear} = 1 \rightarrow \Diamond \text{gear} = 3)\,?$$

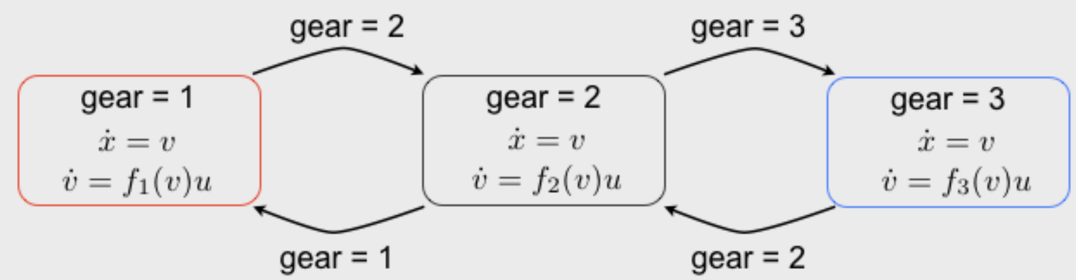
Need to modify to capture continuous evolution:

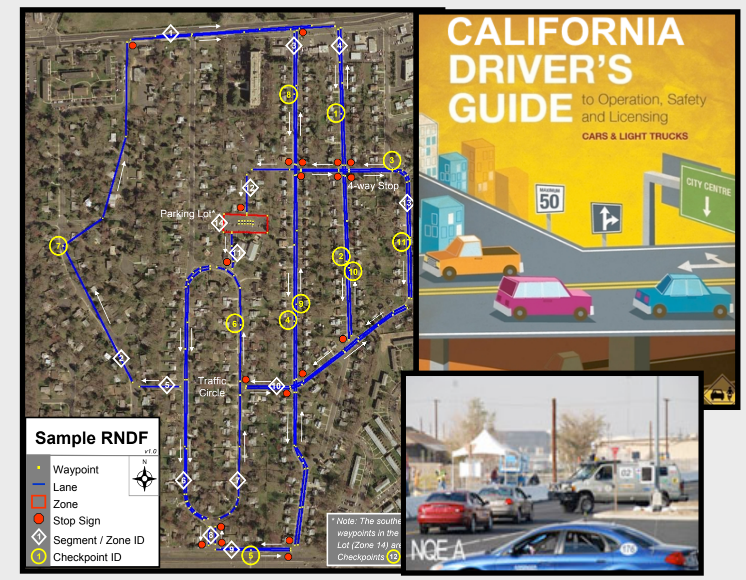
- Can the car come to a stop from any (reasonable) speed within x meters?
- How to efficiently accelerate?

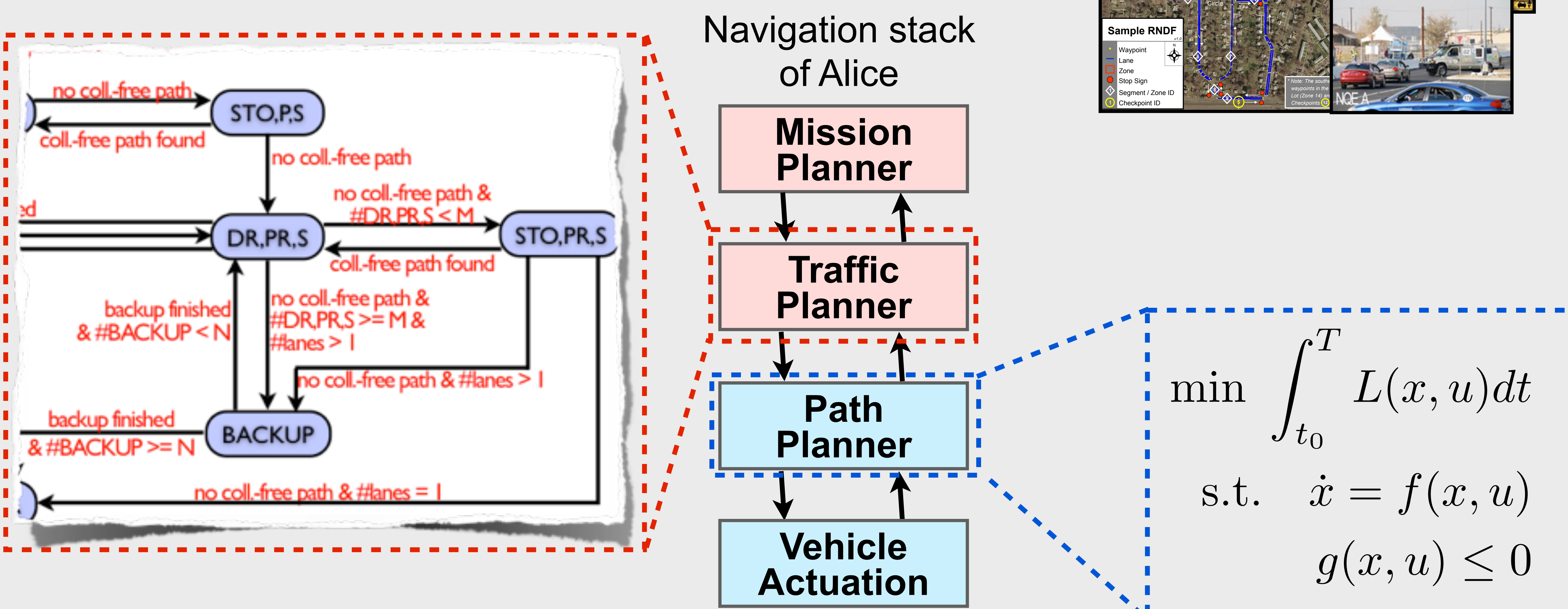


Why to use hybrid system models?

- Continuous systems with multiple modes
- Discrete logic controlling continuous systems
- Continuous systems with “hybrid” specifications



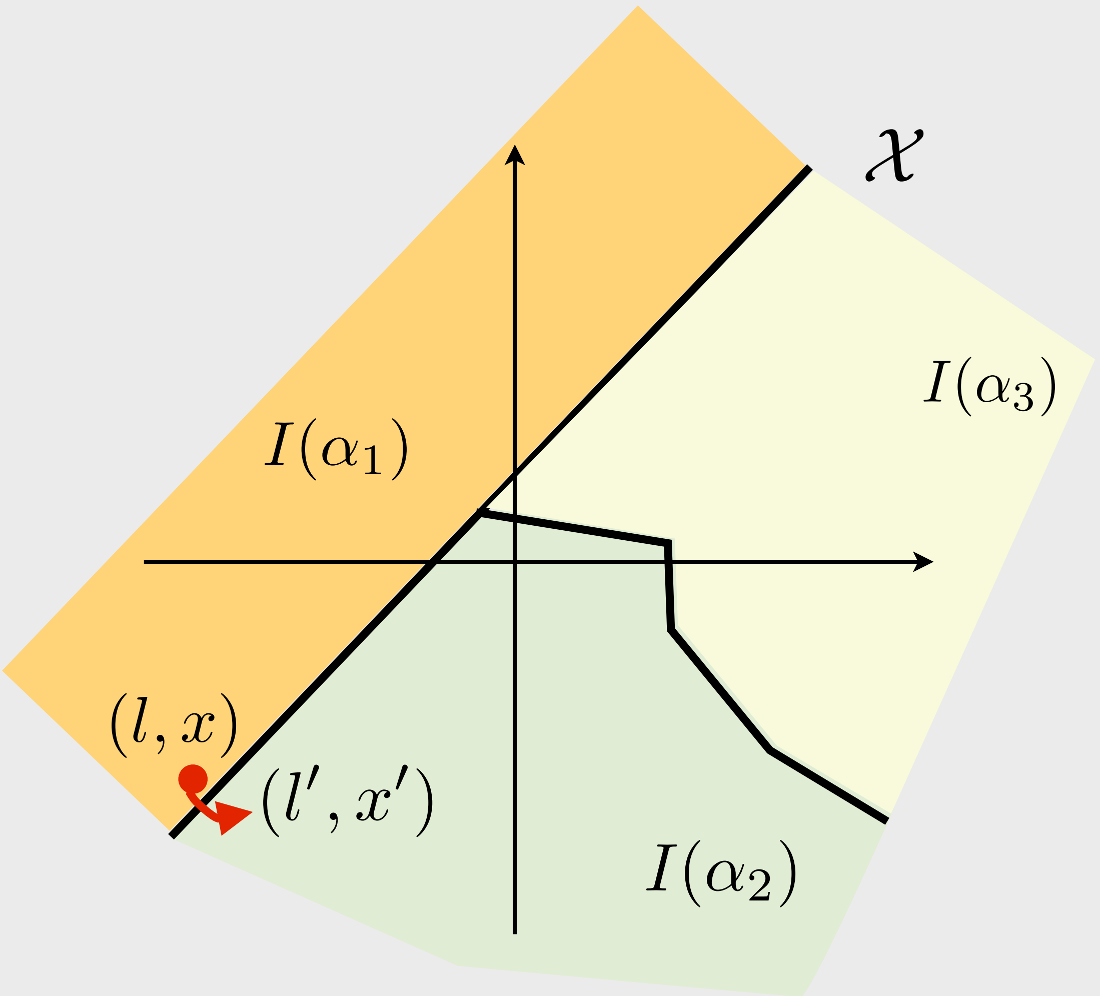




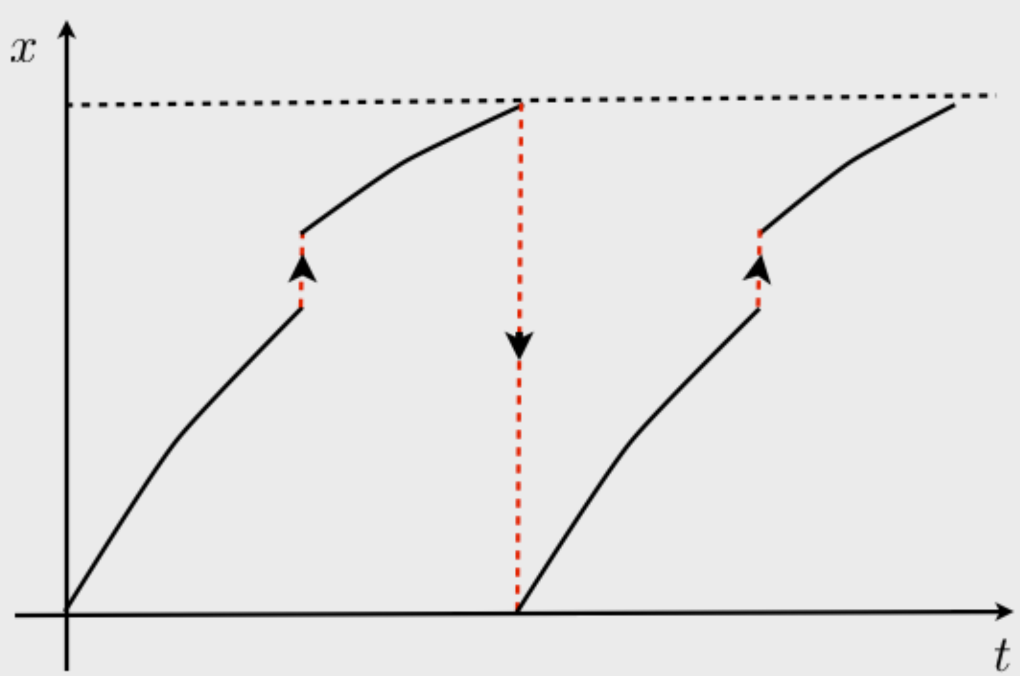
A (simple) hybrid system model

Hybrid system: $H = (\mathcal{X}, L, X_0, I, F, T)$ with

- $\mathcal{X}$, continuous state space;
- L, finite set of locations (modes);
- Overall state space $X = \mathcal{X} \times L$;
- $X_0 \subseteq X$, set of initial states;
- $I : L \to 2^{\mathcal{X}}$, *invariant* that maps $l \in L$ to the set of possible continuous states while in location l;
- $F : X \to 2^{\mathbb{R}^n}$, set of vector fields, i.e., $\dot{x} \in F(l, x)$;
- $T \subseteq X \times X$, relation capturing discrete transitions between locations.



$L = \{\alpha_1, \alpha_2, \alpha_3\}$



Specifications

Given: $H = (\mathcal{X}, L, X_0, I, F, T)$

Solution at time *t* with the initial condition $x_0 \in \mathcal{X}_0$: $\phi(t; x_0)$

- With the simple model *H*, specifying the initial state also specifies the initial mode.

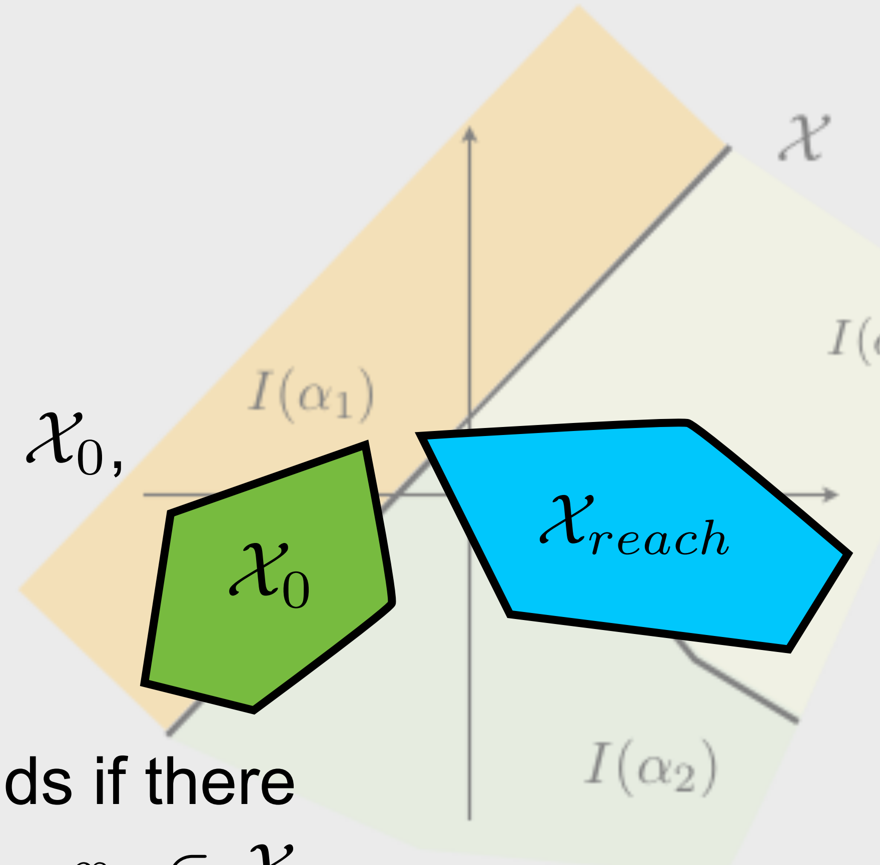
Sample temporal properties:

- *Stability*: Given equilibrium $x_e \in \mathcal{X}$, for all $x_0 \in \mathcal{X}_0 \subseteq \mathcal{X}$,

  $\phi(t; x_0) \in \mathcal{X}, \ \forall t$ and $\phi(t; x_0) \to x_e, \ t \to \infty$

- *Safety*: Given $\mathcal{X}_{unsafe} \subseteq \mathcal{X}$, safety property holds if there exists no t_{unsafe} and trajectory with initial condition $x_0 \in \mathcal{X}_0$,

  $$\phi(t_{unsafe}; x_0) \in \mathcal{X}_{unsafe}$$
  $$\phi(t; x_0) \in \mathcal{X}, \ \forall t \in [0, t_{unsafe}]$$



- *Reachability*: Given $\mathcal{X}_{reach} \subseteq \mathcal{X}$, reachability property holds if there exists finite $t_{reach} \geq 0$ and a trajectory with initial condition $x_0 \in \mathcal{X}_0$,

  $\phi(t_{reach}; x_0) \in \mathcal{X}_{reach}$ and $\phi(t; x_0) \in \mathcal{X}, \ \forall t \in [0, t_{reach}]$

- *Eventuality*: reachable from every initial condition

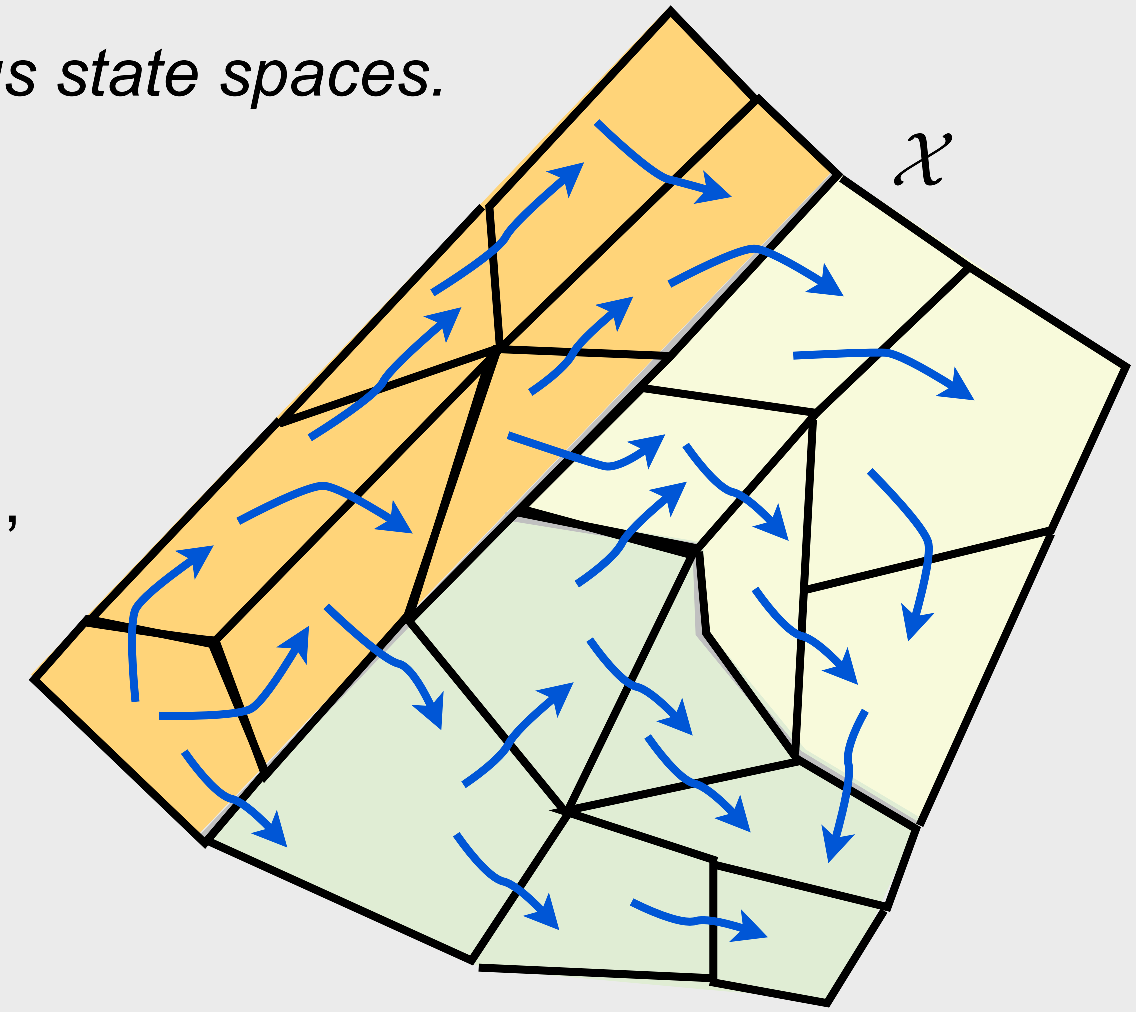
- Combinations of the above, e.g., starting in X_A, reach both X_B and X_C, but X_B will not be reached before X_C is reached while staying safe.

Verification of hybrid systems: Overview

Why not directly use model checking?

- Model checking applied to finite transitions systems
- exhaustively search for counterexamples....
  - if found, property does not hold.
  - if there is no counterexample in all possible executions, the property is verified.

Exhaustive search is not possible over continuous state spaces.



Approaches for hybrid system verification:

1. Construct finite-state approximations and apply model checking
   - preserve the meaning of the properties, i.e., proposition preserving partitions
   - use “over”- or “under”-approximations
2. Deductive verification
   - Construct Lyapunov-type certificates
   - Account for the discrete jumps in the construction of the certificate
3. Explicitly construct the set of reachable states
   - Limited classes of temporal properties (e.g., reachability and safety)
   - Not covered in this lecture

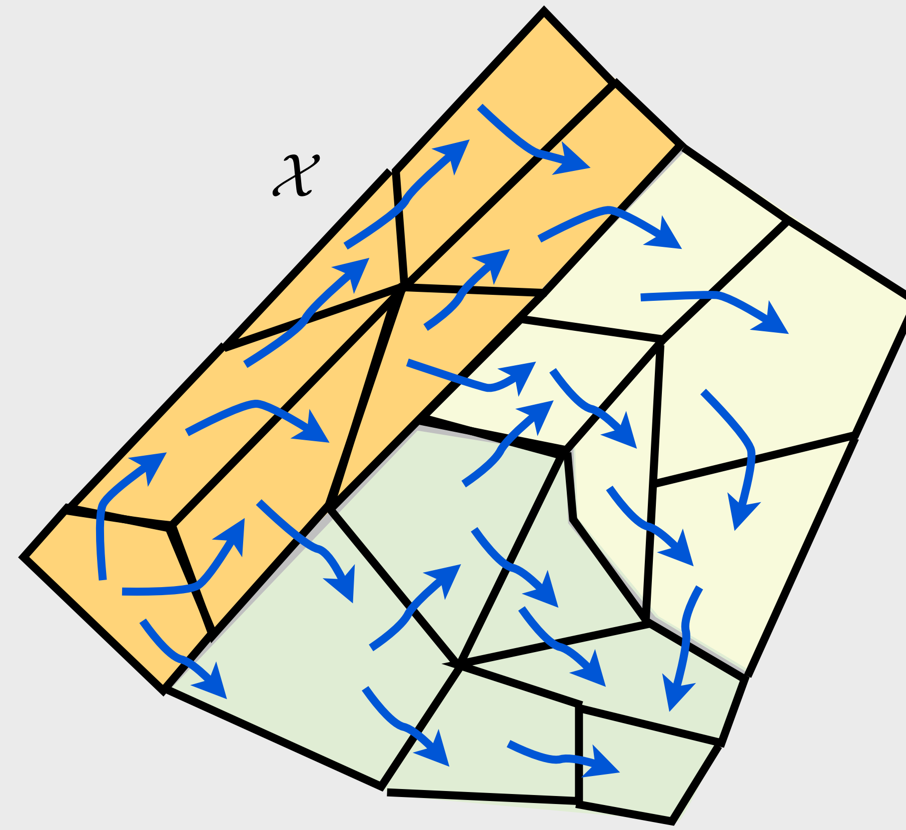
Finite-state, under- and over-approximations

Hybrid system: $H = (\mathcal{X}, L, X_0, I, F, T)$

Finite-transition system: $TS = (Q, \rightarrow, Q_0)$

Define the map $T : Q \rightarrow 2^{\mathcal{X}}$

For discrete state q, $T^{-1}(q)$ is the corresponding cell in $\mathcal{X}$.



Under-approximation: *TS* is an under-approximation of *H* if the following two statements hold.

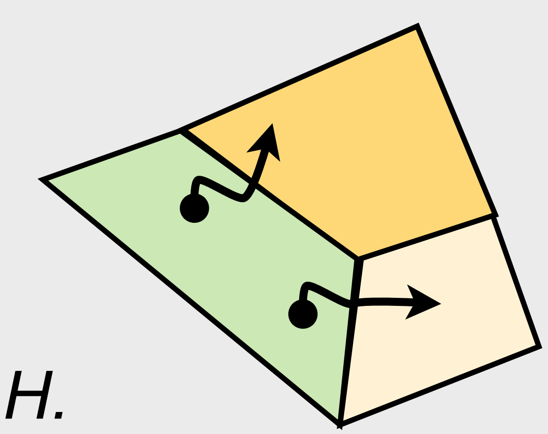
- Given $q, q' \in Q$ with $q \neq q'$, if $q \rightarrow q'$, then for all $x_0 \in T^{-1}(q)$, there exists finite $\tau > 0$ such that

$$\phi(\tau; x_0) \in T^{-1}(q'), \quad \phi(t; x_0) \in T^{-1}(q) \cup T^{-1}(q'), \quad \forall t \in [0, \tau]$$

- If $q \rightarrow q$, then $T^{-1}(q)$ is positively-invariant.

In other words:

- Every discrete trajectory in an under-approximation *TS* can be implemented by *H*.
- *TS* "simulates" *H*.



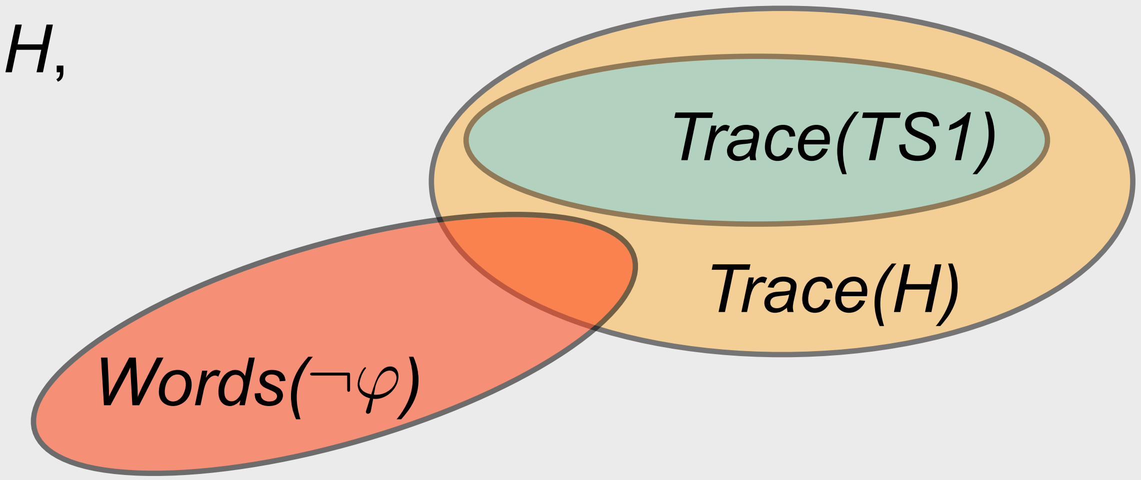
Over-approximation: *TS* is an over-approximation of *H,* if for each discrete transition in TS, there is a "possibility" to be implemented by *H.*

- Possibility induced by the coarseness of the partition.

Use of under-approximations

Let the following be given.

- A hybrid system *H*,
- a finite-state, under-approximation *TS1* for *H*,



Verification

- Let an LTL specification φ be given.
- Question: $H \models \varphi?$
- Model check "$TS1 \models \varphi?$"

$$\begin{array}{c}\mathrm{Words}(\neg\varphi) \cap \mathrm{Trace}(TS1) \text{ is nonempty} \\ \Downarrow \\ \mathrm{Words}(\neg\varphi) \cap \mathrm{Trace}(H) \text{ is nonempty}\end{array}$$

H cannot satisfy the specification.

$$\begin{array}{c} TS1 \not\models \phi \\ \Downarrow \\ H \not\models \phi \end{array}$$

$\mathrm{Words}(\neg\varphi) \cap \mathrm{Trace}(TS1)$ is empty

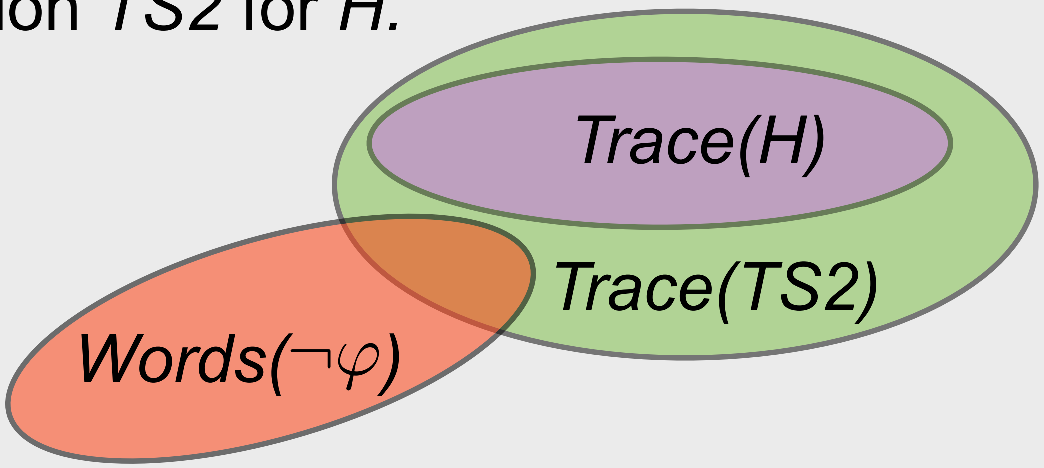
Inconclusive

Logic synthesis:

- If $\mathrm{Words}(\varphi) \cap \mathrm{Trace}(TS1)$ is nonempty, there exists a trajectory of *TS1* which satisfies φ and can be implemented by *H*.
- Otherwise, inconclusive.

Use of over-approximations

Hybrid system *H* and a finite-state, over-approximation *TS2* for *H.*



Verification

$\text{Words}(\varphi) \cap \text{Trace}(TS2)$ is nonempty — Inconclusive

$$\text{Words}(\neg\varphi) \cap \text{Trace}(TS2) \text{ is empty} \\ \Downarrow \\ \text{Words}(\neg\varphi) \cap \text{Trace}(H) \text{ is empty}$$

H satisfies the specification.

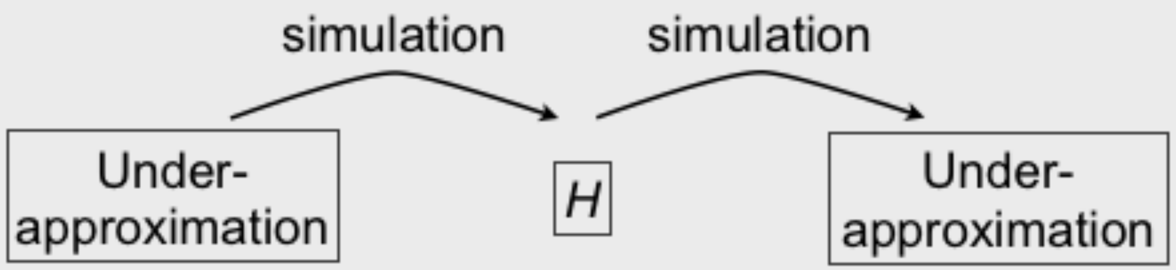
$$TS2 \models \varphi \\ \Downarrow \\ H \models \varphi$$

Logic synthesis:

- If $\text{Words}(\varphi) \cap \text{Trace}(TS2)$ is empty, no valid trajectories for *TS2* or *H*.
- Otherwise, inconclusive.

Remarks:

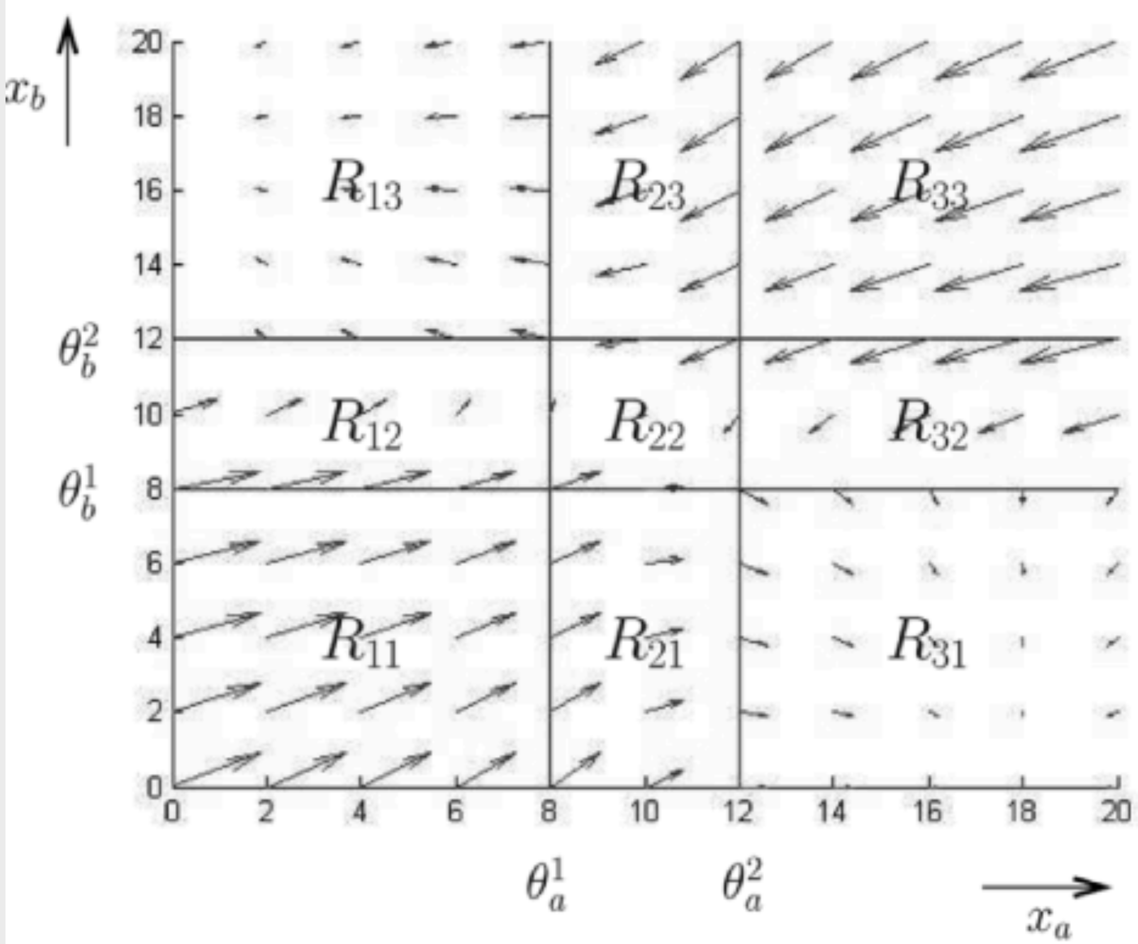
- Under- and over-approximations give partial results.
- Potential remedies:
  - Finer approximations
  - Bisimulations



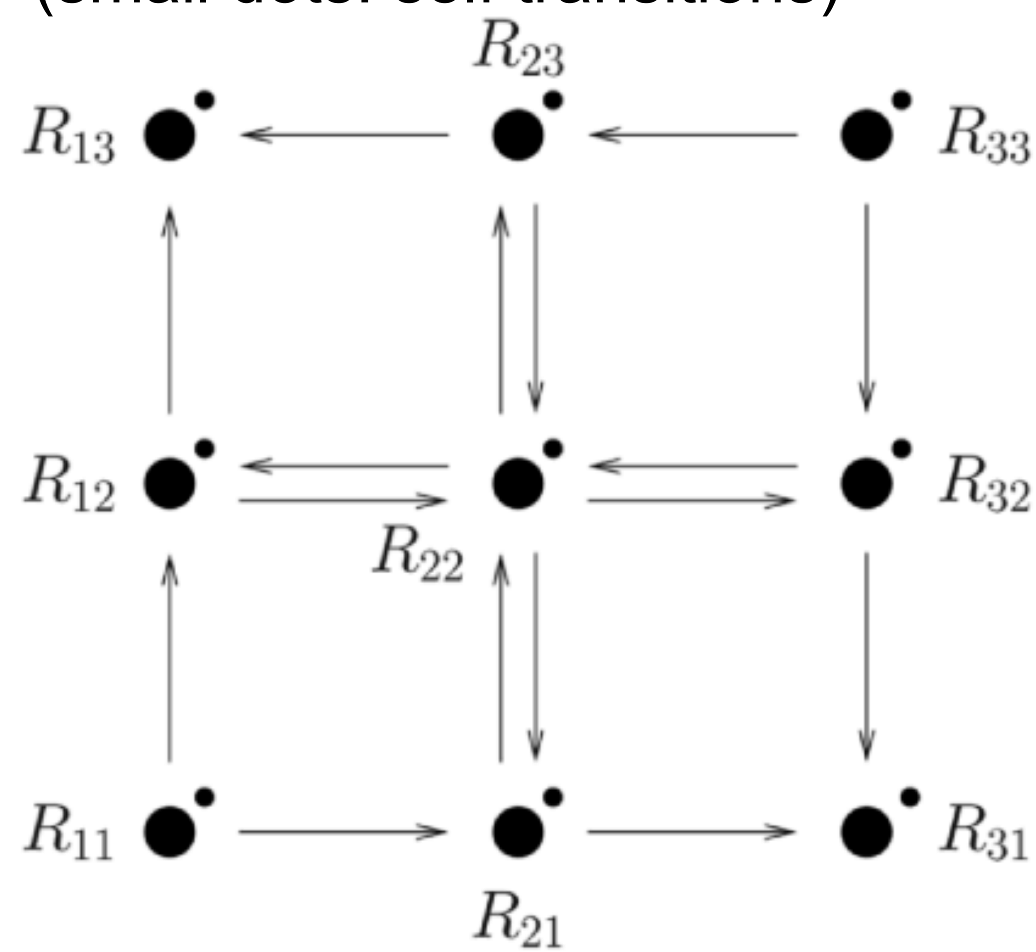
Example: verification

System models:

Continuous vector field:



Discrete over-approximation:
(small dots: self transitions)



Specifications:

$$\phi_1 = \left(x_a < \theta_a^1 \wedge x_b > \theta_b^2 \rightarrow \Box \left(x_a < \theta_a^1 \wedge x_b > \theta_b^2\right)\right) \wedge \left(x_b < \theta_b^1 \wedge x_a > \theta_a^2 \rightarrow \Box \left(x_b < \theta_b^1 \wedge x_a > \theta_a^2\right)\right)$$

$$\phi_2 = \Diamond \left(x_a < \theta_a^2 \vee x_b < \theta_b^2\right)$$

Both hold for the over-approximation; hence, they hold for the actual system.

Example from "Temporal logic analysis of gene networks under parametric uncertainty," Batt, Belta, Weiss, Joint special issue of IEEE TAC & Trans on Circuits, 2008.

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What does deductive verification mean?

Example with continuous, nonlinear dynamics:

$$\dot{x}(t) = f(x(t))$$

where $x(t) \in \mathbb{R}^n$, $f(0) = 0, x = 0$ is an asymptotically stable equilibrium.

Region-of-attraction: $\mathcal{R} := \left\{ x : \lim_{t \to \infty} \phi(t; x) = 0 \right\}$

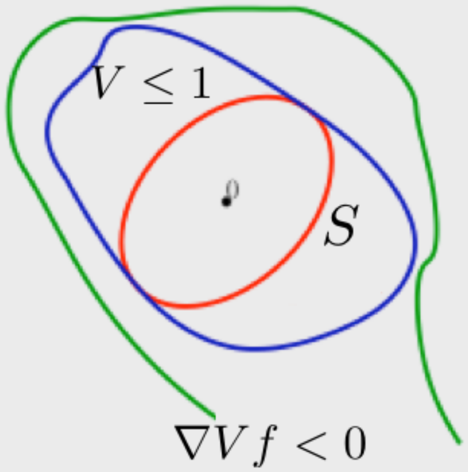
Question 1 (a system analysis question):
Given $S \subset \mathbb{R}^n$, is S invariant and $S \subseteq \mathcal{R}$?

the question we <u>want</u> to answer

the question we <u>attempt</u> to answer

Question 2 (an algebraic question):
Does there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- *V* is positive definite,
- *V(0) = 0*,
- $\Omega := \{x : V(x) \leq 1\} \subset \{x : \nabla V \cdot f(x) < 0\} \cup \{0\}$
- $S \subseteq \Omega$?

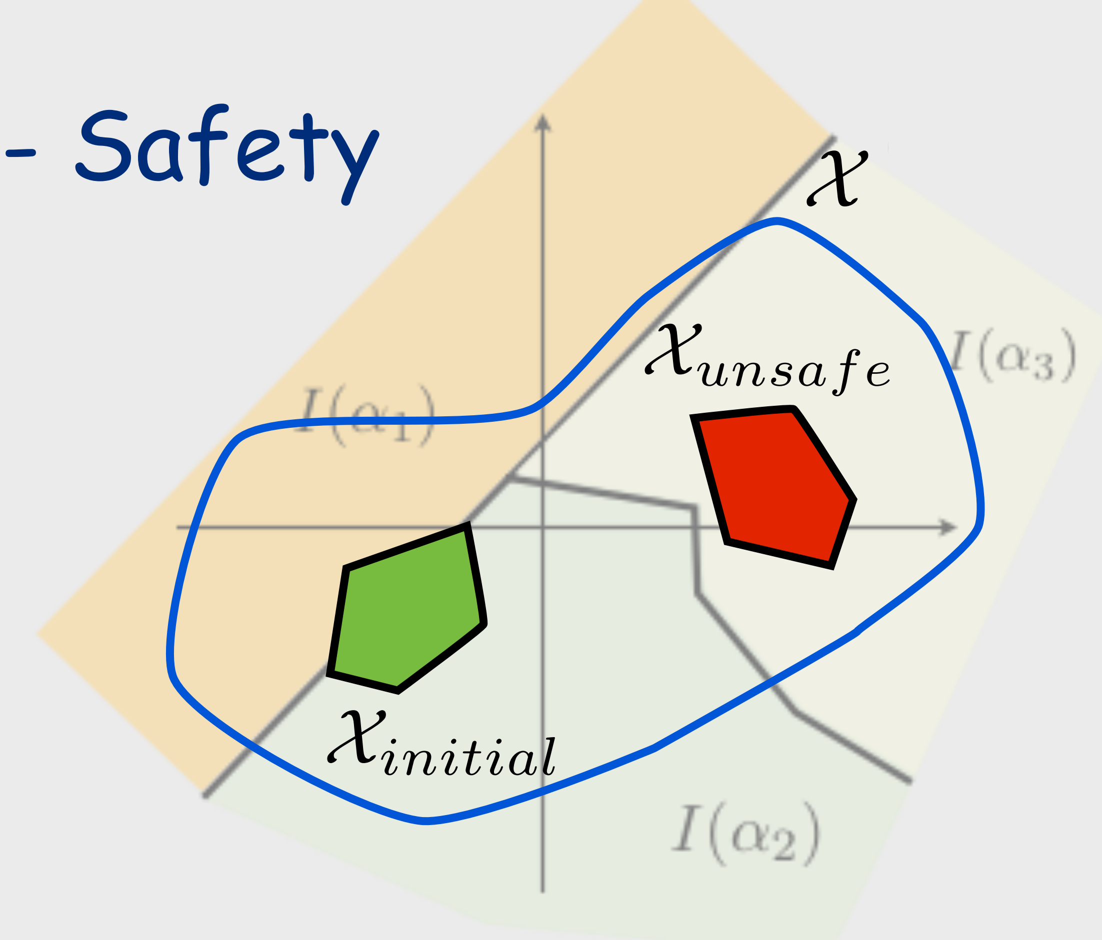


Yes to Question 2 $\rightarrow$ Yes to Question 1.

Barrier Certificates - Safety

Safety property holds if there exists
<u>no</u> $T \geq 0$ and trajectory such that:

$$x = \phi(0; x) \in \mathcal{X}_{initial}$$
$$\phi(T; x) \in \mathcal{X}_{unsafe}$$
$$\phi(t; x) \in \mathcal{X} \ \forall t \in [0, T].$$



Continuous dynamics:

$$\dot{x}(t) = f(x(t))$$

Suppose there exists a differentiable function B such that

$$B(x) \leq 0, \ \forall x \in \mathcal{X}_{initial}$$
$$B(x) > 0, \ \forall x \in \mathcal{X}_{unsafe}$$
$$\frac{\partial B}{\partial x} f(x) \leq 0, \ \forall x \in \mathcal{X}.$$

Then, the safety property holds.

Hybrid dynamics:

$$H = (\mathcal{X}, L, X_0, I, F, \mathcal{T})$$

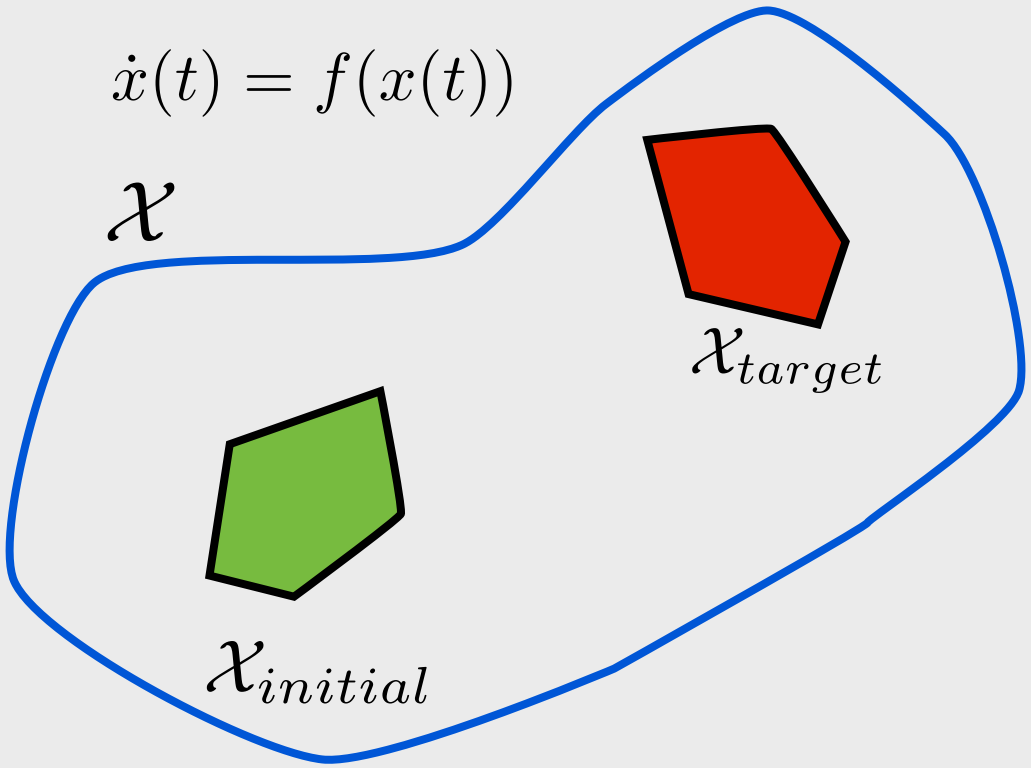
Suppose there exist differentiable functions B_l (for each mode) such that

$$B_l(x) \leq 0, \ \forall x \in I(l) \cap \mathcal{X}_{initial}$$
$$B_l(x) > 0, \ \forall x \in I(l) \cap \mathcal{X}_{unsafe}$$
$$\frac{\partial B_l}{\partial x} F(x) \leq 0, \ \forall x \in I(l)$$
$$B_{l'}(x') - B_l(x) \leq 0, \ \text{for each jump} \ (l, x) \to (l', x')$$

Then, the safety property holds.

Barrier Certificates - Eventuality

$\dot{x}(t) = f(x(t))$



$\mathcal{X},\ \mathcal{X}_{target},\ \mathcal{X}_{initial}$ are bounded

Eventuality property holds if for all $x_0 \in \mathcal{X}_{initial}$,

$$\phi(T; x_0) \in \mathcal{X}_{target}$$

$$\phi(t; x_0) \in \mathcal{X},\ \forall t \in [0, T]$$

for some non-negative *T*.

notation: set closure

Suppose that f is continuously differentiable and there exists a continuously differentiable function *B* such that

$$B(x) \leq 0,\ \forall x \in \mathcal{X}_{initial}$$

don't leave $\mathcal{X}$ before reaching $\mathcal{X}_{target}$

$$B(x) > 0,\ \forall x \in \overline{\partial\mathcal{X} \backslash \partial\mathcal{X}_{target}}$$

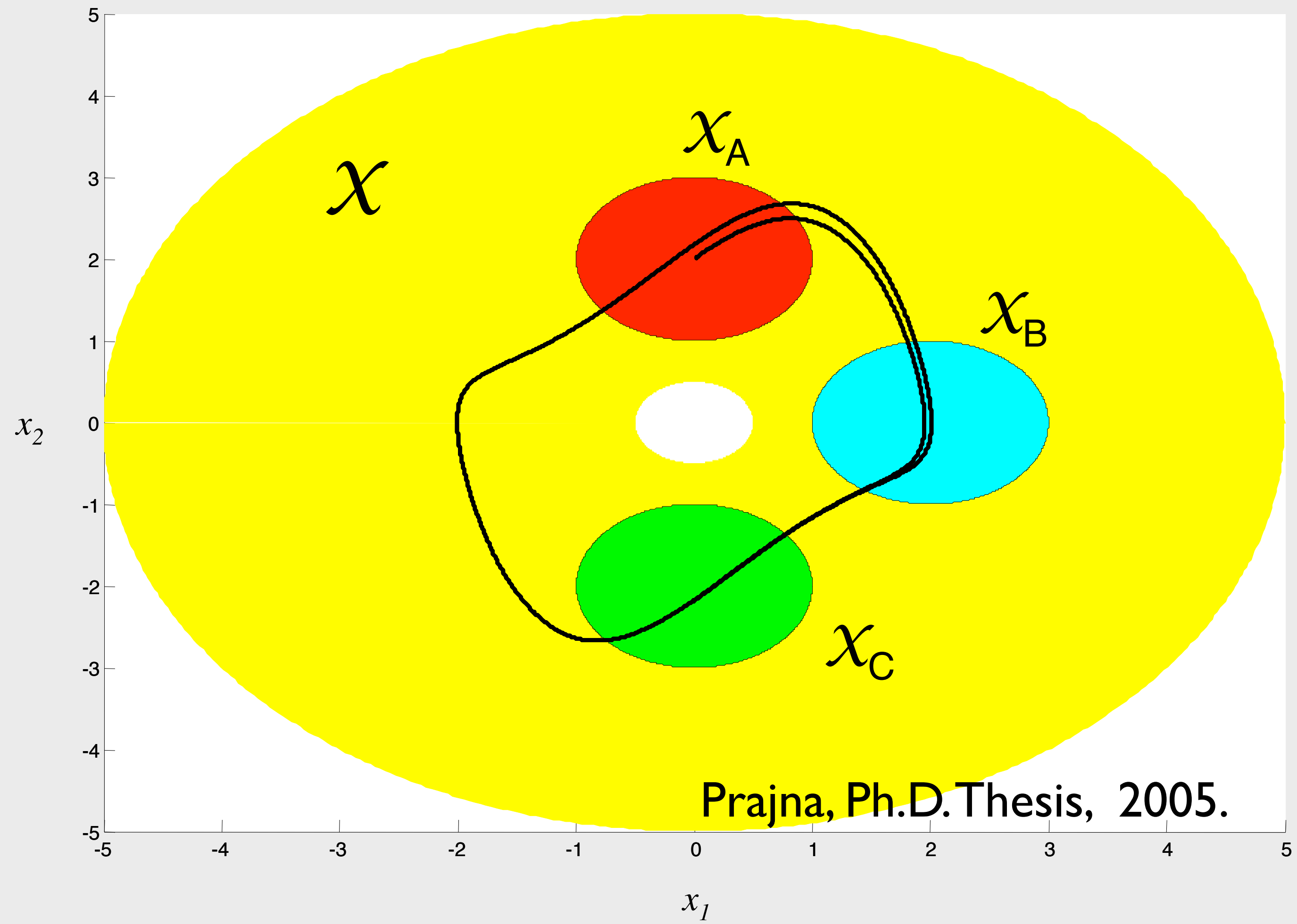
leave $\mathcal{X} \backslash \mathcal{X}_{target}$ in finite time

$$\frac{\partial B}{\partial x}(x) \cdot f(x) < 0,\ \forall x \in \overline{\mathcal{X} \backslash \mathcal{X}_{target}}$$

Then, the eventuality property holds.

- Straightforward extensions for hybrid dynamics as in safety verification are possible.

Composing Barrier Certificates



incorporating disturbances and uncertainties

If system starts in $\mathcal{X}_A$, then both $\mathcal{X}_B$ and $\mathcal{X}_C$ are reached in finite time, but $\mathcal{X}_C$ will not be reached before system reaches $\mathcal{X}_B$.

$$\begin{cases} B_1(x) \le 0 \quad \forall x \in \mathcal{X}_A, \\ B_1(x) > 0 \quad \forall x \in \partial\mathcal{X} \cup \mathcal{X}_C, \\ \frac{\partial B_1}{\partial x}(x) f(x,d) \le -\epsilon \quad \forall (x,d) \in (\mathcal{X} \setminus \mathcal{X}_B) \times \mathcal{D}, \end{cases}$$

$$\begin{cases} B_2(x) \le 0 \quad \forall x \in \mathcal{X}_A, \\ B_2(x) > 0 \quad \forall x \in \partial\mathcal{X}, \\ \frac{\partial B_2}{\partial x}(x) f(x,d) \le -\epsilon \quad \forall x \in (\mathcal{X} \setminus \mathcal{X}_C) \times \mathcal{D}, \end{cases}$$

How to construct the certificates?

- System properties $\rightarrow$ Algebraic conditions
  - Lyapunov, dissipation inequalities.

$\}$ Problem-dependent!

- Algebraic conditions $\rightarrow$ Numerical optimization problems
  - Restrict the attention to polynomial vector fields, polynomial certificates,...
  - S-procedure like conditions (for set containment constraints)
  - Sum-of-squares (SOS) relaxations for polynomial nonnegativity
  - Pass to semidefinite programming (SDP) that are equivalent of SOS conditions

- Solve the resulting (linear or "bilinear") SDPs
- Construct polynomial certificates

$\}$ Generally taken care of by software packages.

Some preliminaries

- Semidefinite programming problems
- Positive semidefinite polynomials and sum-of- squares (SOS) programming
- Set containment conditions and S-procedure

Linear and bilinear matrix inequalities

Convex, efficient, general-purpose solvers exist

Given matrices $\{F_i\}_{i=0}^{N} \subset \mathcal{S}^{n\times n}$, Linear Matrix Inequality (LMI) is a constraint on $\lambda \in \mathbb{R}^N$ of the form:

$$F_0 + \sum_{k=1}^{N} \lambda_k F_k \succeq 0$$

Non-convex, **no** efficient, general-purpose solvers

Given matrices $\{F_i\}_{i=0}^{N}$, $\{G_j\}_{j=1}^{M}$, and $\{H_{k,j}\}_{k=1}^{N}{}_{j=1}^{M}$ $\subset \mathcal{S}^{n\times n}$, a Bilinear Matrix Inequality (BMI) is a constraint on $\lambda \in \mathbb{R}^N$ and $\gamma \in \mathbb{R}^M$ of the form:

$$F_0 + \sum_{k=1}^{N} \lambda_k F_k + \sum_{j=1}^{M} \gamma_k G_j + \sum_{k=1}^{N}\sum_{j=1}^{M} \lambda_k \gamma_j H_{k,j} \succeq 0$$

Semidefinite program (?)

very roughly speaking, optimization of affine objective subject to LMI or/and BMI constraints

Polynomials and Multipoly Toolbox

Given $\alpha \in \mathbb{N}^n$, a <u>monomial</u> in n variables is a function $m_\alpha : \mathbb{R}^n \to \mathbb{R}$ defined as $m_\alpha(x) := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
The degree of a monomial is defined as $\deg m_\alpha := \sum_{i=1}^n \alpha_i$.

Polynomial: Finite linear combination of monomials.

$$p := \sum_{\alpha \in \mathcal{A}} c_\alpha m_\alpha = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$$ where $\mathcal{A} \subset \mathbb{N}^n$ is a finite set and $c_\alpha \in \mathbb{R}\ \forall \alpha \in \mathcal{A}$.

Multipoly is a Matlab toolbox for the creation and manipulation of polynomials of one or more variables.

Example:

```
pvar x1 x2
p = 2*x1^4 + 2*x1^3*x2 - x1^2*x2^2 + 5*x2^4
q = x1^2
p*q =
    2*x1^6 + 2*x1^5*x2 - x1^4*x2^2 + 5*x1^2*x2^4
jacobian(p, [x1;x2]) =
     [ 8*x1^3 + 6*x1^2*x2 - 2*x1*x2^2 ,
         2*x1^3 - 2*x1^2*x2 + 20*x2^3 ]
```

Positive semidefinite polynomials

$\mathbb{R}[x_1, \ldots, x_n]$ or $\mathbb{R}[x]$ denotes the set of polynomials (with real coefficients) in the variables $\{x_1, \ldots, x_n\}$.

$p \in \mathbb{R}[x]$ is <u>positive semi-definite</u> (PSD) if $p(x) \geq 0 \ \forall x$. The set of PSD polynomials in n variables $\{x_1, \ldots, x_n\}$ will be denoted $\mathcal{P}[x_1, \ldots, x_n]$ or $\mathcal{P}[x]$.

Testing if $p \in \mathcal{P}[x]$ is NP-hard when the polynomial degree is at least four.

How about a quadratic polynomial?

Reference: Parrilo, P., *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, Ph.D. thesis, California Institute of Technology, 2000. (Chapter 4 of this thesis and the reference contained therein summarize the computational issues associated with verifying global non-negativity of functions.)

Sum-of-Squares Polynomials

p is a sum of squares (SOS) if there exist polynomials $\{f_i\}_{i=1}^N$ such that $p = \sum_{i=1}^N f_i^2$.
The set of SOS polynomials in n variables $\{x_1, \ldots, x_n\}$ will be denoted $\Sigma[x_1, \ldots, x_n]$ or $\Sigma[x]$.

If p is a SOS then p is PSD.

For every polynomial p of degree 2d, there exists a symmetric matrix Q such that

$$p(x) = z(x)^T Q z(x)$$

with

$$z(x) := \left[1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^2, \ldots, x_n^d\right]^T$$

p is SOS if and only if there exists $Q \succeq 0$ s.t. $p(x) = z(x)^T Q z(x)$

Given p, can be verified as SDP.

SOS example

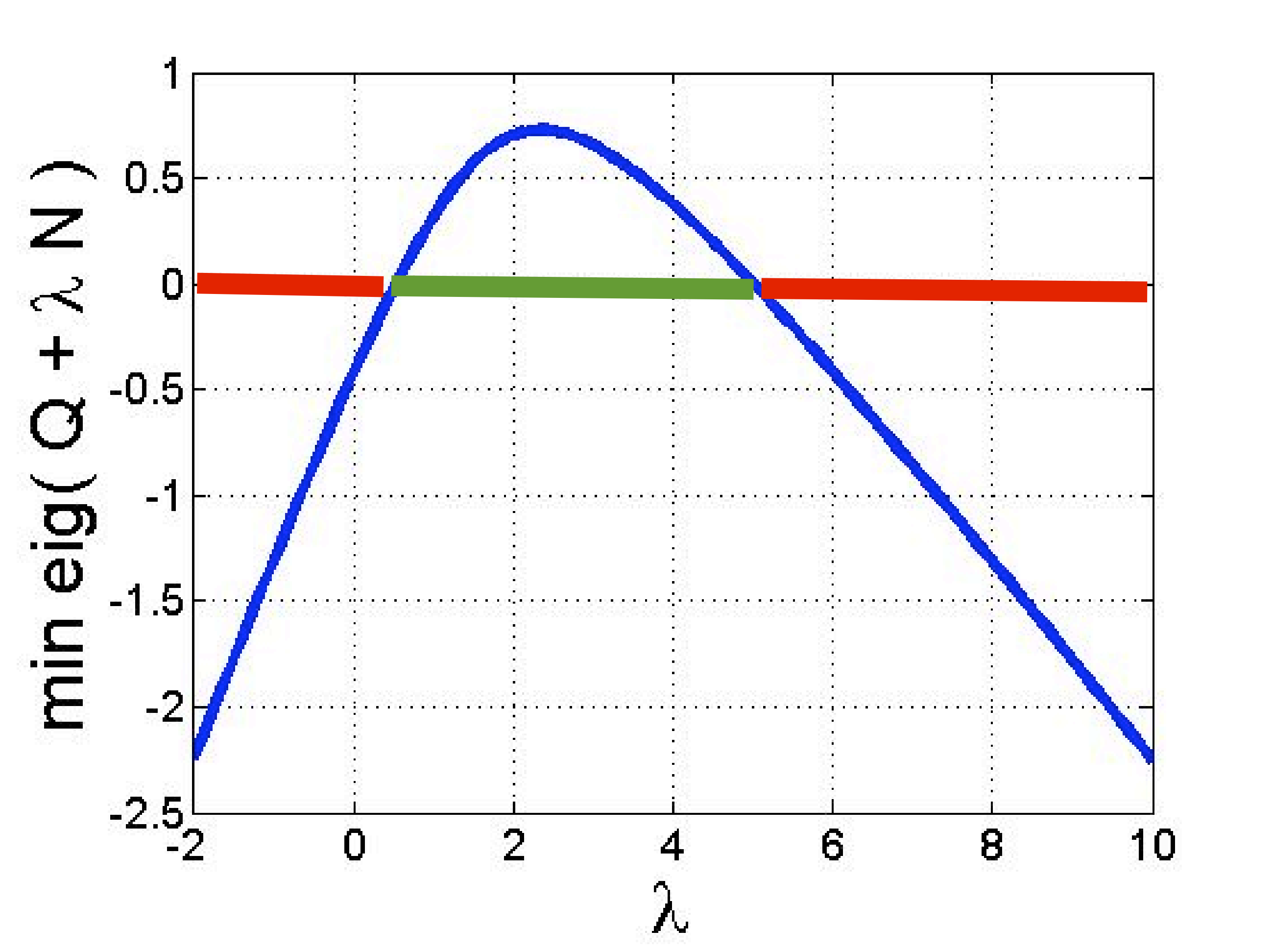
$$p(x) = z(x)^T Q z(x)$$
$$0 = z(x)^T N z(x)$$
$$(x_1 x_2) \cdot (x_1 x_2) = x_1^2 \cdot x_2^2$$

All possible Gram matrix representations of

$$p(x) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$$

are given by $z^T(Q + \lambda N)z$ where:

$$z = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}, \; Q = \begin{bmatrix} 2 & 1 & -0.5 \\ 1 & 0 & 0 \\ -0.5 & 0 & 5 \end{bmatrix}, \; N = \begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}$$



p is SOS iff

$$Q + \lambda N \succeq 0 \longleftarrow \text{LMI}$$

for some $\lambda \in \mathbb{R}$.

SOS programming

<u>SOS Programming</u>: Given $c \in \mathbb{R}^m$ and polynomials $\{f_{j,k}\}_{j=1}^{N_s} {}_{k=0}^{m}$, solve:

$$\min_{\alpha \in \mathbb{R}^m} c^T \alpha$$

subject to:

$$f_{1,0}(x) + f_{1,1}(x)\alpha_1 + \cdots + f_{1,m}(x)\alpha_m \in \Sigma[x]$$

$$\vdots$$

$$f_{N_s,0}(x) + f_{N_s,1}(x)\alpha_1 + \cdots + f_{N_s,m}(x)\alpha_m \in \Sigma[x]$$

There is freely available software (e.g. SOSTOOLS, YALMIP, SOSOPT) that:

1. Converts the SOS program to an SDP
2. Solves the SDP with available SDP codes (e.g. Sedumi)
3. Converts the SDP results back into polynomial solutions

Set containment conditions

Given polynomials g_1 and g_2, define sets S_1 and S_2:

$$S_1 := \{x \in \mathbb{R}^n \ : \ g_1(x) \leq 0\}$$
$$S_2 := \{x \in \mathbb{R}^n \ : \ g_2(x) \leq 0\}$$

Is $S_2 \subseteq S_1$?

Polynomial S-procedure

$$\exists \lambda \in \Sigma[x] \text{ s.t. } -g_1(x) + \lambda(x) g_2(x) \in \Sigma[x]$$
$$\Downarrow$$
$$\exists \lambda \text{ positive semidefinite polynomial s.t. } -g_1(x) + \lambda(x) g_2(x) \geq 0 \ \forall x$$
$$\Downarrow$$
$$\{x : g_2(x) \leq 0\} \subseteq \{x : g_1(x) \leq 0\}$$

Example: $B(x) \leq 0, \quad \forall x \in \mathcal{X}_{initial}$

Suppose $\mathcal{X}_{initial} = \{x : g(x) \leq 0\}$ for some *g*

Sufficient condition: There exists positive semidefinite function *s* such that

$$-B(x) + s(x)g(x) = -B(x) - s(x)(-g(x)) \geq 0, \quad \forall x \in \mathbb{R}^n$$

Global stability theorem

Theorem: Let $l_1, l_2 \in \mathbb{R}[x]$ satisfy $l_i(0) = 0$ and $l_i(x) > 0\ \forall x \neq 0$ for $i = 1, 2$. If there exists $V \in \mathbb{R}[x]$ such that:

- $V(0) = 0$
- $V - l_1 \in \Sigma[x]$
- $-\nabla V \cdot f - l_2 \in \Sigma[x]$

Then $\mathcal{R}_0 = \mathbb{R}^n$.

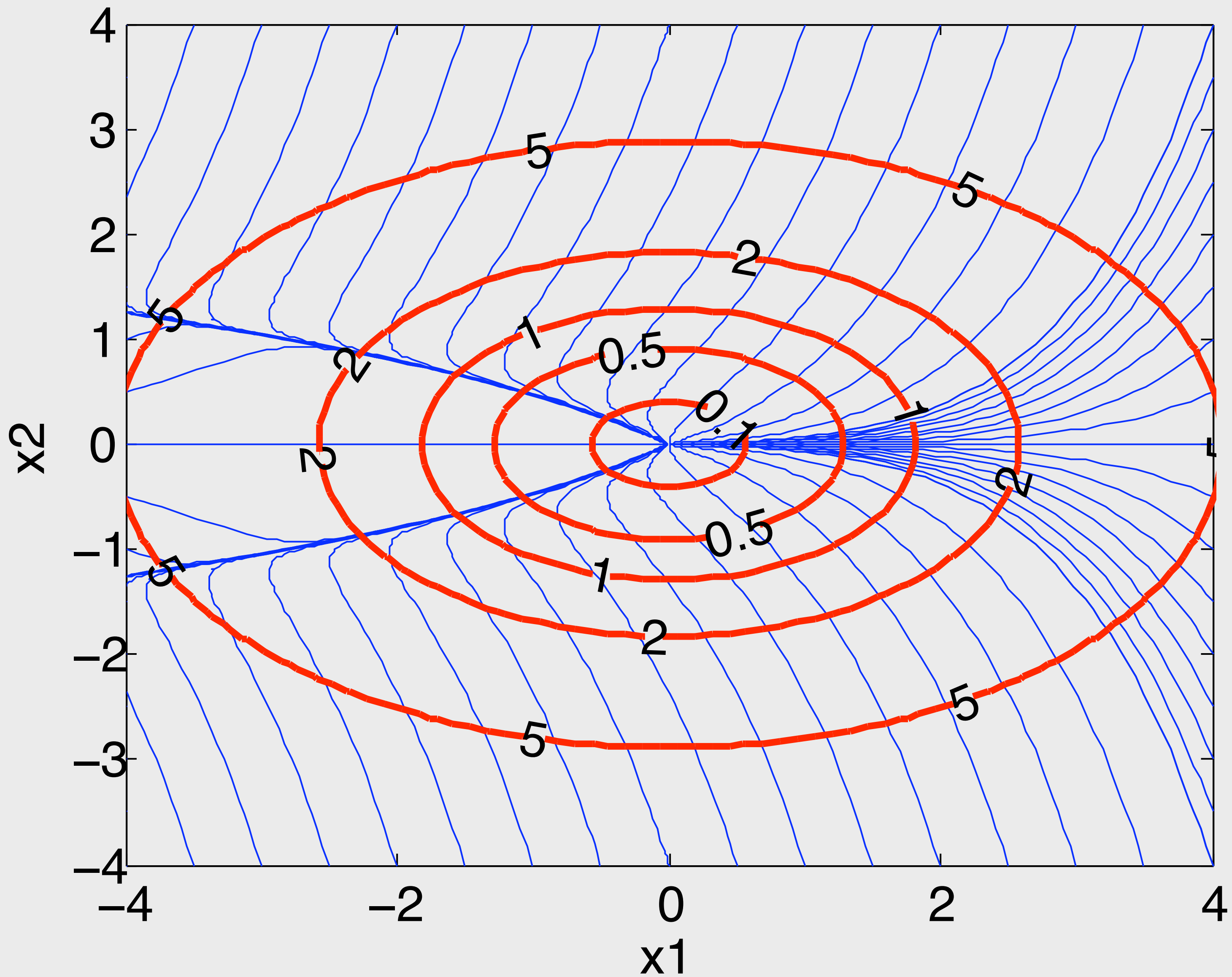
V is positive definite, radially unbounded

V is decreasing along the vector field

Reference: Vidyasagar, M., *Nonlinear Systems Analysis*, SIAM, 2002.
(Refer to Section 5.3 for theorems on Lyapunov's direct method.)

Global stability examples with sosopt

```
% Code from Parrilo1_GlobalStabilityWithVec.m

% Create vector field for dynamics
pvar x1 x2;
x = [x1;x2];
x1dot = -x1 - 2*x2^2;
x2dot = -x2 - x1*x2 - 2*x2^3;
xdot = [x1dot; x2dot];

% Use sosopt to find a Lyapunov function
% that proves x = 0 is GAS

% Define decision variable for quadratic
% Lyapunov function
zV = monomials(x,2);
V = polydecvar('c',zV,'vec');

% Constraint 1 : V(x) - L1 \in SOS
L1 = 1e-6 * ( x1^2 + x2^2 );
sosconstr{1} = V - L1;

% Constraint 2: -Vdot - L2 \in SOS
L2 = 1e-6 * ( x1^2 + x2^2 );
Vdot = jacobian(V,x)*xdot;
sosconstr{2} = -Vdot - L2;

% Solve with feasibility problem
[info,dopt,sossol] = sosopt(sosconstr,x);
Vsol = subs(V,dopt)
Vsol =
  0.30089*x1^2 + 1.8228e-017*x1*x2 + 0.6018*x2^2
```



Approximate bisimulation relations & bisimulation functions

Two systems with $x_i \in \mathbb{R}^{n_i},\ x_i(0) \in I_i \subseteq \mathbb{R}^{n_i},\ u_i(t) \in U_i \subseteq R^{m_i},\ y_i \in \mathbb{R}^p$

$$\Phi_1 : \begin{cases} \dot{x}_1(t) = f_1(x_1(t), u_1(t)) \\ y_1(t) = g_1(x_1(t)) \end{cases} \qquad \Phi_2 : \begin{cases} \dot{x}_2(t) = f_2(x_2(t), u_2(t)) \\ y_2(t) = g_2(x_2(t)) \end{cases}$$

A relation $\mathcal{R}_\delta \in \mathbb{R}^{m_1} \times \mathbb{R}^{n_2}$ is a δ-approximate bisimulation relation between Φ_1 and Φ_2 if for all $(x_1, x_2) \in \mathcal{R}_\delta$:

· $\|g_1(x_1) - g_2(x_2)\| \leq \delta$;

· $\forall T > 0$ and $\forall u_1(\cdot)$, $\exists\ u_2(\cdot)$ s.t. $(\phi_1(t; x_1) - \phi_2(t; x_2)) \in \mathcal{R}_\delta\ \forall t \in [0, T]$;

· $\forall T > 0$ and $\forall u_2(\cdot)$, $\exists\ u_1(\cdot)$ s.t. $(\phi_1(t; x_1) - \phi_2(t; x_2)) \in \mathcal{R}_\delta\ \forall t \in [0, T]$.

If start in relation, stay in relation. Observations are “close.”

A function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^+ \cup \{+\infty\}$ is a bisimulation function between Φ_1 and Φ_2 if for all $\delta \geq 0$:

$$\mathcal{R}_\delta = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : V(x_1, x_2) \leq \delta\}$$

sublevel sets of V induce a relation

is a closed set and a δ-approximate bisimulation relation between Φ_1 and Φ_2.

Approximate bisimulation relations & bisimulation functions

Two systems with $x_i \in \mathbb{R}^{n_i},\ x_i(0) \in I_i \subseteq \mathbb{R}^{n_i},\ u_i(t) \in U_i \subseteq R^{m_i},\ y_i \in \mathbb{R}^p$

$$\Phi_1 : \begin{cases} \dot{x}_1(t) = f_1(x_1(t), u_1(t)) \\ y_1(t) = g_1(x_1(t)) \end{cases} \qquad \Phi_2 : \begin{cases} \dot{x}_2(t) = f_2(x_2(t), u_2(t)) \\ y_2(t) = g_2(x_2(t)) \end{cases}$$

Let $W : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^+$ be a continuously differentiable function. If for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

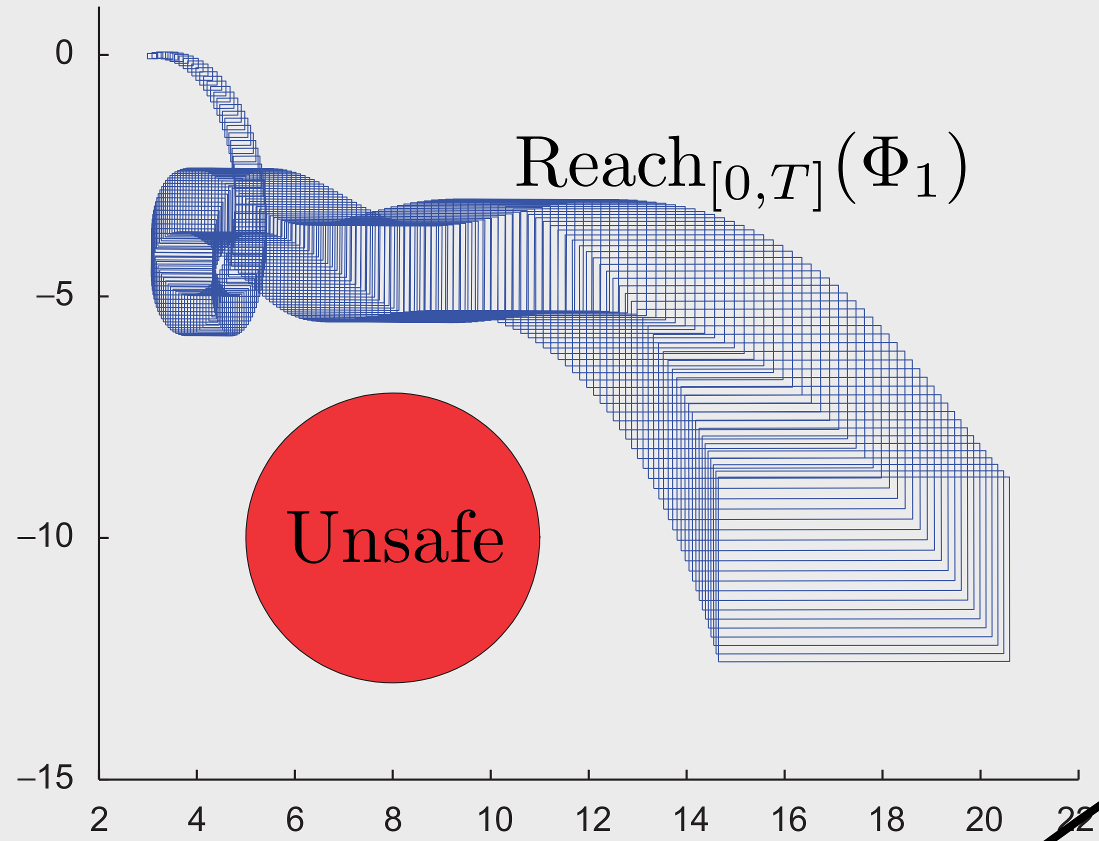
$$W(x_1, x_2) \geq \|g_1(x_1) - g_2(x_2)\|^2$$

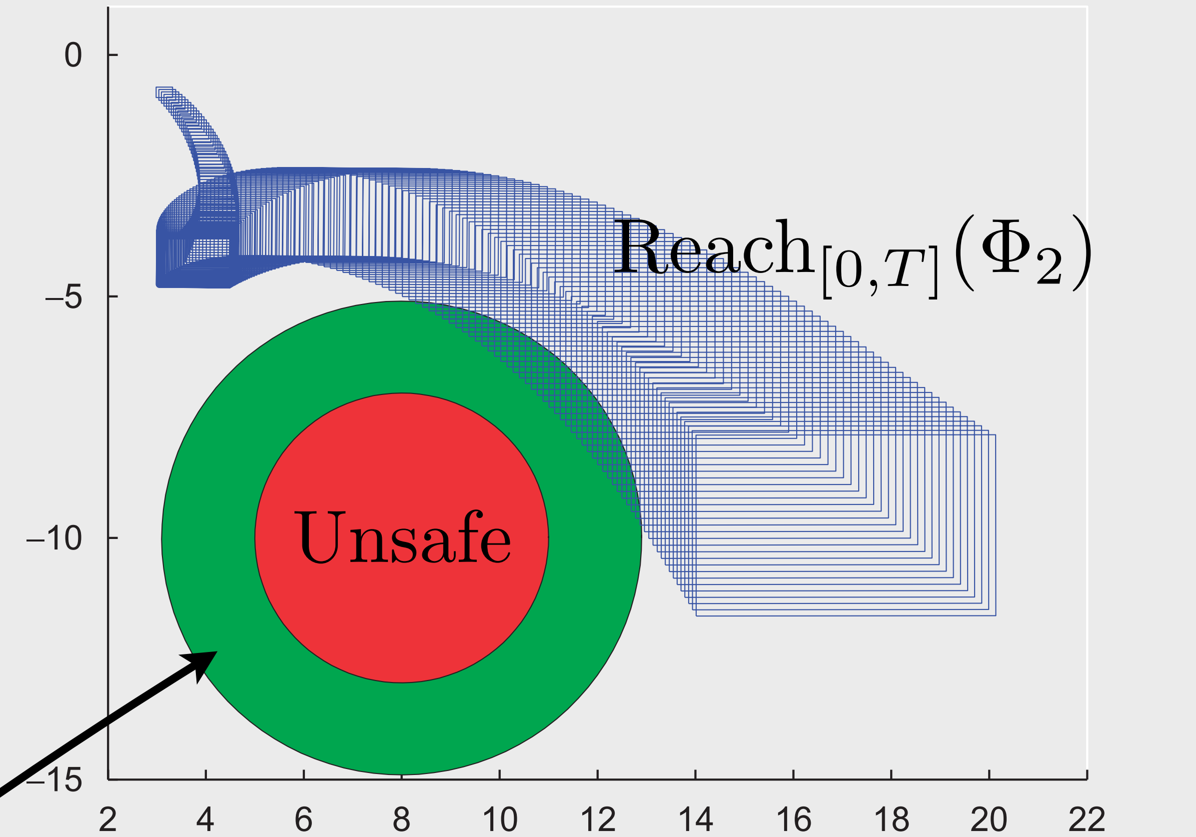
$$\boxed{\frac{\partial W}{\partial x_1} f_1(x_1, u_1) - \frac{\partial W}{\partial x_2} f(x_2, u_2) \leq 0,\ \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},\ u_1 \in \mathbb{R}^{m_1},\ u_2 \in \mathbb{R}^{m_2}}$$

then $V := |\sqrt{W}|$ is a bisimulation function between Φ_1 and Φ_2.

guarantees that no matter what u_1 and u_2 do, the time derivative of W stays non-positive

Approximate bisimulations + safety

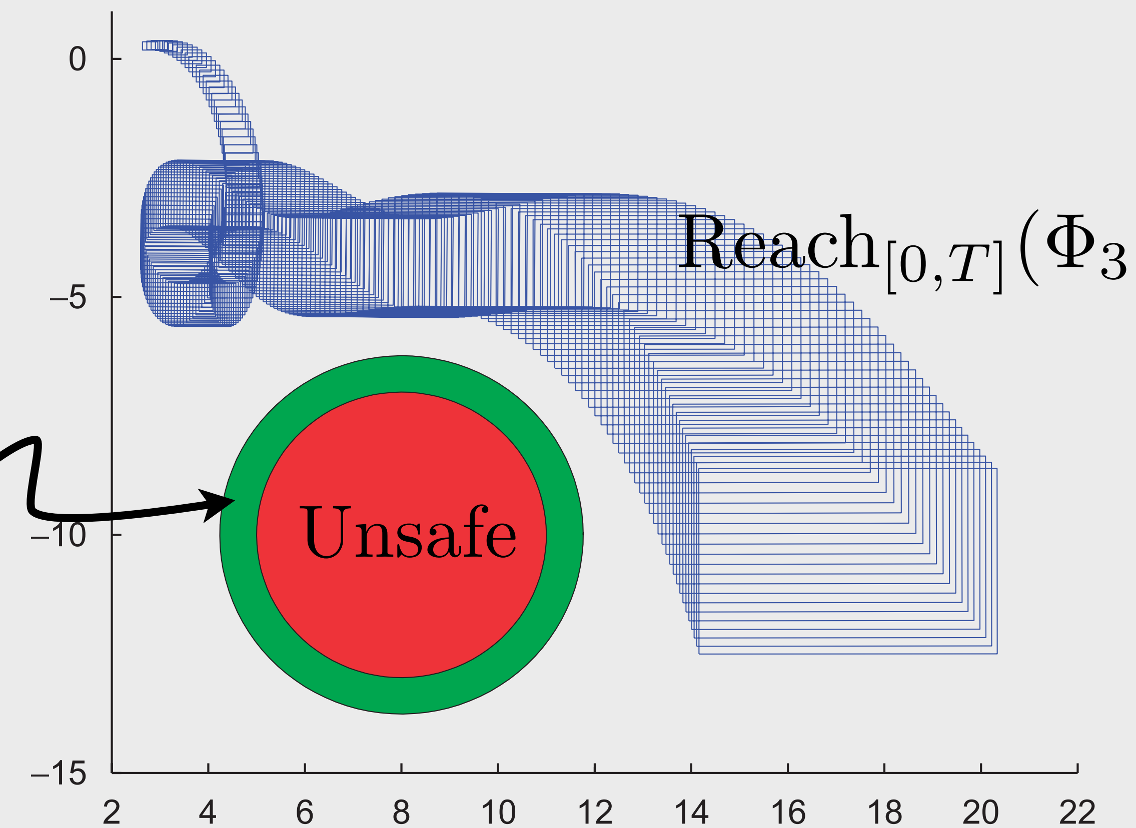




Φ_1 s.t. $x_1 \in \mathbb{R}^{10}$
Φ_2 s.t. $x_2 \in \mathbb{R}^{4}$
Φ_3 s.t. $x_3 \in \mathbb{R}^{6}$

Φ_1 is 1.90-approximate bisimilar to Φ_2

Φ_1 is 0.76-approximate bisimilar to Φ_3



Figures from Girard & Pappas, Automatica, 2007.