CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

ACM/EE 116

Problem Set #8

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Note: In the upper corner of the *second* page of your problem set, please put the number of hours that you spent on this problem set (including reading & office hours).

1. (G&S, 4.9.3) Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ have the $N(\boldsymbol{\mu}, \mathbf{V})$ distribution. Show that $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ has the (univariate) $N(\mu, \sigma^2)$ distribution where

$$\mu = \sum_{i=1}^{n} a_i \mathbb{E}(X_i), \qquad \sigma^2 = \sum_{i=1}^{n} a_i \operatorname{var}(X_i) + 2\sum_{i<j} a_i a_j \operatorname{cov}(X_i, X_j)$$

2. (G&S, 4.9.6) Let $\{Y_r : 1 \le r \le n\}$ be independent $N(0,1)$ random variables and define $X_j = \sum_{r=1}^{n} c_{jr} Y_r$, $1 \le r \le n$ for constants c_{jr}. Show that

$$\mathbb{E}(X_j \mid X_k) = \left(\frac{\sum_r c_{jr} c_{kr}}{\sum_r c_{kr}^2} \right) X_k.$$

3. (OBC, A.2) Consider the motion of a particle that is undergoing a random walk in one dimension (i.e., along a line). We model the position of the particle as

$$X_{k+1} = X_k + U_k,$$

where X_k is the position of the particle at time k and U is a (discrete time) white noise process with $\mathbb{E}(U_i) = 0$ and $\mathbb{E}(U_i U_j) = R_u \delta(i-j)$. Show that the expected value of the particle as a function of k is given by

$$\mathbb{E}(X_k) = \mathbb{E}(X_0) + \sum_{i=0}^{k-1} \mathbb{E}(U_i) = \mathbb{E}(X_0) =: \mu_x$$

and the covariance $\mathbb{E}((X_k - \mu_x)^2)$ is given by

$$\mathbb{E}((X_k - \mu_x)^2) = \operatorname{var}(X[0]) + \sum_{i=0}^{k-1} \mathbb{E}(U_i^2) = \operatorname{var}(X[0]) + kR_u$$

4. (OBC, A.3) Consider a second order system with dynamics

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \qquad Y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

that is forced by Gaussian white noise with zero mean and variance σ^2. Assume $a, b > 0$.

   (a) Compute the correlation function $\rho(\tau)$ for the output of the system. Your answer should be an explicit formula in terms of a, b and σ.

(b) Assuming that the input transients have died out, compute the mean and variance of the output.

5. (Friedland 10.1) Consider a system with transfer function

$$H(s) = \frac{1}{(s+\alpha)(s+\beta)}.$$

Assume that the input to the system is white noise with a spectral density of unity.

(a) Find the spectral density $S(\omega)$ of the output.

(b) Find the mean square of the output by computing $\rho(0)$ using the inverse Fourier transform of $S(\omega)$ evaluated at $\tau = 0$.

(c) Find a set of matrices $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$ and $C \in \mathbb{R}^{1\times 2}$ so that

$$H(s) = C(sI - A)^{-1}B$$

[this is called a state space realization for $H(s)$] and compute the mean square of the output Y solving the appropriate Lyapunov equation.