CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

ACM/EE 116

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Note: In the upper corner of the *second* page of your problem set, please put the number of hours that you spent on this problem set (including reading & office hours).

1. (G&S 6.1.2) A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix.
   (a) The largest number X_n shown up to the nth roll.
   (b) The number N_n of sixes in n rolls.
   (c) At time r, the time C_r since the most recent six.
   (d) At time r, the time B_r until the next six.
2. (G&S 6.1.8) Let X and Y be Markov chains on the set $\mathbb{Z}$ of integers. Is the sequence $Z_n = X_n + Y_n$ necessarily a Markov chain?
3. (G&S 6.8.2) Insects land in the soup in the manner of a Poisson process with intensity λ, and each such insect is green with probability p, independently of the colours of all other insects. Show that the arrivals of green insects form a Poisson process with intesity λp.
4. (G&S 8.2.1) Let $\{X_n\}$ be a Markov chain on the state space $S = \{0, 1\}$ with transition matrix
$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix},$$
where $\alpha + \beta > 0$. Find:
   (a) the correlation $\rho(X_m, X_{m+n})$, and its limit as $m \to \infty$ with n remaining fixed;
   (b) $\lim_{n\to\infty} n^{-1} \sum_{r=1}^{n} \mathbb{P}(X_r = 1)$.

   Under what condition is the process strongly stationary?
5. (G&S 8.7.4) Customers arrive in a shop in the manner of a Poisson process with parameter λ. There are infinitely many servers, and each service time is exponentially distributed with parameter μ. Show that the number $Q(t)$ of waiting customers constitutes a birth-death process. Find its stationary distribution.

Optional exercises: The following exercises may be substituted for the problems above (if you do more than the required number of problems, we'll drop problems with the lowest scores):

6. G&S Section 6.1, Exercise 3 instead of Problem 1
7. G&S Section 6.8, Exercise 4 instead of Problem 3
8. G&S Section 8.7, Exercise 2 instead of Problem 5