CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

ACM/EE 116

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Note: In the upper corner of the *second* page of your problem set, please put the number of hours that you spent on this problem set (including reading & office hours).

1. In class we motivated the Poisson distribution as an approximation of the binomial distribution over an appropriate range of parameters. Let $X_n \sim \mathrm{Bin}(n, \frac{\lambda}{n})$ be a sequence of independent random variables, and $X \sim \mathrm{Pois}(\lambda)$. Show that $X_n \xrightarrow{D} X$.

2. (Paley-Zygmund inequality) Establish the following complement to Markov's inequality. Let X be a nonnegative random variable with $0 < \mathbb{E}X < \infty$. Then for all $0 < r < 1$,

$$\mathbb{P}(X > r\mathbb{E}X) \geq (1-r)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)}.$$

3. (G&S 7.3.10). Let $\{X_n : n \geq 1\}$ be independent $N(0,1)$ random variables. Show that

   (a) $\mathbb{P}(X_n \geq a_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_n \mathbb{P}(X_1 > a_n) < \infty \\ 1 & \text{if } \sum_n \mathbb{P}(X_1 > a_n) = \infty \end{cases},$

   (b) $\mathbb{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}\right) = 1.$

   Hint: Mill's ratio for standard normal random variables may be useful. It states that for all $x > 0$,

$$\left(1 - \frac{1}{x^2}\right)\frac{1}{x\sqrt{2\pi}}\exp(-x^2/2) < \mathbb{P}(X > x) < \frac{1}{x\sqrt{2\pi}}\exp(-x^2/2).$$

4. (G&S 7.4.1) Let $X_2, X_3, \ldots$ be independent random variables such that

$$\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2n\log n}, \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log n}.$$

   Show that this sequence obeys the weak law but not the strong law, in the sense that $n^{-1}\sum_{i=2}^n X_i$ converges to 0 in probability but not almost surely.

5. (G&S 7.9.1) Let Y be uniformly distributed on $[-1, 1]$ and let $X = Y^2$.

   (a) find the best predictor of X given Y, and of Y given X.

   (b) Find the best linear predictor of X given Y, and of Y given X. (By linear predictor, we mean that it must have the form $\hat{X} = aY + b$.)

Optional exercises: For students interested in a more analytical (and more challenging) set of problems, the following exercises may be substituted for the problems above (if you do more than the required number of problems, we'll drop problems with the lowest scores):

6. G&S Section 7.3, Exercise 6 instead of Problem 2