CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

ACM/EE 116

R. M. Murray, Y. Chen

Problem Set #4

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Note: In the upper corner of the *second* page of your problem set, please put the number of hours that you spent on this problem set (including reading & office hours).

1. (G&S 5.1.9) Let G_1 and G_2 be probability generating functions, and suppose that $0 \leq \alpha \leq 1$. Show that $G_1 G_2$, and $\alpha G_1 + (1-\alpha)G_2$ are probability generating functions. Is $G(\alpha s)/G(\alpha)$ necessarily a probability generating function?

2. (G&S 5.2.3)

   (a) Let X have the Poisson distribution with parameter Y, where Y has the Poisson distribution with parameter μ. Show that $G_{X+Y}(x) = e^{\mu(xe^{x-1}-1)}$.

   (b) Let $X_1, X_2, \ldots$ be i.i.d. random variables with the logarithmic probability mass function

   $$f(k) = \frac{(1-p)^k}{k \log(1/p)}, k \geq 1,$$

   where $0 < p < 1$. If N is independent of the X_i and has the Poisson distribution with parameter μ, show that $Y = \sum_{i=1}^{N} X_i$ has a negative binomial distribution.

3. (random walk) Let $S_n = S_0 + \sum X_i$ be a symmetric random walk in which $X_1, X_2, \ldots$ are i.i.d., ± 1-valued random variables with $P(X_i = \pm 1) = 1/2$.

   (a) Suppose that $S_0 = 0$. Find the mean and variance of S_n (as a function of n).

   (b) Suppose that $|S_0| \sim \text{exponential}(\lambda)$ (so S_0 can be either positive or negative). Find the mean and variance of S_n.

4. (G&S 5.4.5) Each generation of a branching process (with a single progenitor) is augmented by a random number of immigrants who are indistinguishable from the other members of the population. Suppose that the numbers of immigrants in different generations are independent of each other and of the past history of the branching process, each such number having probability generating function $H(s)$. Show that the probability generating function G_n of the size of the nth generation satisfies $G_{n+1}(s) = G_n(G(s))H(s)$, where G is the probability generating function of a typical family of offspring.

5. Let X_1 and X_2 be independent, normal random variables with mean μ_i and variance σ_i^2, $i = 1, 2$. Show that $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Optional exercises: For students interested in a more analytical (and more challenging) set of problems, the following exercises may be substituted for the problems above (if you do more than the required number of problems, we'll drop problems with the lowest scores):

6. G&S Section 5.10, Exercise 2 instead of Problem 4
7. G&S Section 5.7, Exercise 7 instead of Problem 5
8. G&S Section 5.12, Exercise 48 instead of Problem 3