CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

ACM/EE 116

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Problem Set #2

Fall 2011

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Note: In the upper corner of the *second* page of your problem set, please put the number of hours that you spent on this problem set (including reading & office hours).

1. (G&S 3.11.1a, b)

   (a) Let X and Y be independent discrete random variables and let $g, h : \mathbb{R} \to \mathbb{R}$. Show that $g(X)$ and $h(Y)$ are independent.

   (b) Show that two discrete random variables X and Y are indepedent if and only if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all $x, y \in \mathbb{R}$

2. (G&S 3.11.6) Let X and Y be independent Poisson variables with respective parameters λ and μ. Show that:

   (a) $X + Y$ is Poisson, parameter $\lambda + \mu$,

   (b) the conditional distribution of X, given $X + Y = n$, is binomial, and find its parameters.

3. (G&S 3.11.16) Let X and Y be independent Bernoulli random variables with parameter $\frac{1}{2}$. Show that $X + Y$ and $|X - Y|$ are dependent though uncorrelated.

4. (Coin collecting) The U.S. Mint has produced quarters commemorating each of the 50 states, and I want to collect them all. Every day, I buy lunch from Ernie's and receive a quarter in change. Assume that Ernie only gives me state quarters, and on each day I am equally likely to receive any one of the 50.

   (a) Let the random variable T_i be the time it takes (in days) for me to collect the i-th distinct quarter *after* the $(i-1)$-th quarter has been collected. What is the distribution of T_i? What is the expectation of T_i?

   (b) Are T_i and T_{i-1} independent? Explain your reasoning.

   (c) Show that $\mathbb{P}(T_i = n + m | T_i > n) = \mathbb{P}(T_i = m)$. Explain in plain English what this means.

   (d) What is $\mathbb{E}[T_i | T_i > n]$?

   (e) Let the random variable τ be the time (in days) it takes for me to collect all 50 state quarters. Show that

   $$\mathbb{E}[\tau] = 50 \sum_{i=1}^{50} \frac{1}{i}.$$

   (f) Use a computer to calculate this expectation. How many days should I expect to eat at Ernie's before completing my collection?

5. (Zipf's distribution) In 1935, the linguist George Kingsley Zipf observed a curious fact about languages. He found that the most frequently used word is used roughly twice as often as the second most frequent word. It is also used three times as often as the third most frequent word, four times the fourth, etc. In linguistics we call this *Zipf's Law.* This kind of relationship is known as a rank-frequency or rank-size relationship. In particular, the frequency of usage is inversely proportional to the rank. This kind of rank-size relationship has also been observed in the population of cities and number of citations garnered by academic papers. In this problem, we will explore the probability distribution that describes such phenomena.

   (a) Mathematically we write $f(r) \propto r^{-s}$ to say that the relative frequency f is inversely proportional to the rank r (where $s > 0$). What does this relationship look like on a log-log plot?

   (b) Now we wish to model this relationship as a probability distribution. Imagine that we have collected a large catalog of words (called a corpus) from a variety of texts. For each of the N distinct words in the corpus, we have an associated rank r which is an integer between 1 and N. By convention, we say that the word with rank 1 is the most common word (and rank N the least common). What is the probability mass function that predicts the frequency of the rank r-th word out of the population of N words?

   (c) The distribution you have derived is called *Zipf's distribution.* When $s = 1$ we reproduce Zipf's Law as described above. What is the expectation of Zipf's Law? What happens to the expectation as $N \to \infty$?

   (d) The distribution that results when we take $N \to \infty$ in Zipf's distribution is called the *zeta distribution.* If X has the zeta distribution, what values of the exponent s result in a finite expectation of X? What about $\mathbb{E}[X^n]$?