Chapter 7

Nonholonomic Behavior in Robotic Systems

In this chapter, we study the effect of nonholonomic constraints on the behavior of robotic systems. These constraints arise in systems such as multifingered robot hands and wheeled mobile robots, where rolling contact is involved, as well as in systems where angular momentum is conserved. We discuss the problem of determining when constraints on the velocities of the configuration variables of a robotic system are integrable, and illustrate the problem in a variety of different situations. The emphasis of this chapter is on the basic tools needed to analyze nonholonomic systems and the application of those tools to problems in robotic manipulation. These tools are drawn both from some basic theorems in differential geometry and from nonlinear control theory.

1 Introduction

In the preceding chapter, we derived the equations of motion for a robotic system with kinematic constraints. We restricted ourselves to Pfaffian constraints which had the general form

$$J(\theta, x)\dot{\theta} = G^T(\theta, x)\dot{x}, \tag{7.1}$$

where $q = (\theta, x) \in \mathbb{R}^n$ is the configuration of the system. As we saw, equations of this form could be used to model a large number of robotic systems, including multifingered hands, robots in contact with their environment, and redundant manipulators.

By shifting our notation slightly, we can write the preceding constraints in the form

$$\omega_i(q)\dot{q} = 0 \qquad i = 1, \ldots, k, \tag{7.2}$$

where the $\omega_i(q)$ are row vectors. We assume that the ω_i are linearly independent at each point $q \in \mathbb{R}^n$, since if they are not, the dependent constraints may be eliminated. Each ω_i describes one constraint on the directions in which $\dot{q}$ is permitted to take values.

Recall from Chapter 6 that a constraint is said to be *holonomic* if it restricts the motion of a system to a smooth hypersurface of the configuration space. It will be convenient to adopt some language and notation from differential geometry, so we call this smooth hypersurface a *manifold*. Locally, a holonomic constraint can be represented as a set of algebraic constraints on the configuration space,

$$h_i(q) = 0, \qquad i = 1, \ldots, k. \tag{7.3}$$

The dimension of the manifold on which the motion of the system evolves is $n - k$.

We say that a set of k Pfaffian constraints of the form in equation (7.2) is *integrable* if there exist functions $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$ such that

$$h_i(q(t)) = 0 \quad \Longleftrightarrow \quad \omega_i(q)\dot{q} = 0 \qquad i = 1, \ldots, k.$$

Thus, a set of Pfaffian constraints is integrable if it is equivalent to a set of holonomic constraints. We often call an integrable Pfaffian constraint a holonomic constraint, although strictly speaking the former is described by a set of velocity constraints and the latter by a set of functions. A set of Pfaffian constraints is said to be *nonholonomic* if it is not equivalent to a set of holonomic constraints.

As we saw in Chapter 6, the presence of nonholonomic constraints requires special care in deriving the equations of motion for the system. The point of view taken in this chapter is somewhat different. Here, we will try to understand when we can exploit the nonholonomy of the constraints to achieve motion between configurations. In particular, we will be interested in answering the following question: given two points q_0 and q_f, when does there exist a path $q(t)$ which satisfies the constraints in equation (7.2) at all times and connects q_0 to q_f? The set of all points which can be connected to q_0 via a path which satisfies the constraints is called the *reachable set* associated with q_0. Thus, we wish to understand under what conditions the reachable set will be the entire configuration space. This is intimately related to the nonholonomy of the constraints, since if the constraints are holonomic, then the motion of the system is restricted to the level sets given by $h_i(q) = h_i(q_0)$, $i = 1, \ldots, k$. Hence, for holonomic constraints the reachable set is some subset of the configuration space which lies in the level set $h_i(q) = h_i(q_0)$, and we cannot move freely between configurations on different level sets.

A good example of the type of behavior which we wish to exploit is that of an automobile. The kinematics of an automobile are constrained because the front and rear wheels are only allowed to roll and spin, but not to slide sideways. As a consequence, the car itself is not capable of sliding sideways, or rotating in place. Despite this, we know from our own experience that we can park an automobile at any position and orientation. Thus, the constraints are not holonomic since the motion of the system is unrestricted. Finding an actual path between two given configurations is an example of a *nonholonomic motion planning problem* and is the subject of the next chapter.

Checking to see if a constraint is holonomic or nonholonomic is neither easy nor obvious. Consider first the case in which there is a single velocity constraint,

$$\omega(q)\dot{q} = \sum_{j=1}^{n} \omega_j(q)\dot{q}_j = 0.$$

This constraint is integrable if there exists a function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$\omega(q)\dot{q} = 0 \qquad \Longleftrightarrow \qquad h(q) = 0.$$

It follows by differentiating $h(q) = 0$ with respect to time that if the Pfaffian constraint is holonomic then

$$\sum_{j=1}^{n} \omega_j(q)\dot{q}_j = 0 \qquad \Longrightarrow \qquad \sum_{j=1}^{n} \frac{\partial h}{\partial q_j}\dot{q}_j = 0.$$

In turn, this implies that there exists some function $\alpha(q)$, called an *integrating factor*, such that

$$\alpha(q)\omega_j(q) = \frac{\partial h}{\partial q_j}(q) \qquad j = 1, \ldots, n. \tag{7.4}$$

Thus, a single Pfaffian constraint is holonomic if and only if there exists an integrating factor $\alpha(q)$ such that $\alpha(q)\omega(q)$ is the derivative of some function h.

Equation (7.4) is not very constructive from the point of view of checking integrability since it involves the unknown function $h(q)$. This situation may be remedied by using the fact that

$$\frac{\partial^2 h}{\partial q_i \partial q_j} = \frac{\partial^2 h}{\partial q_j \partial q_i}$$

to get

$$\frac{\partial(\alpha\omega_j)}{\partial q_i} = \frac{\partial(\alpha\omega_i)}{\partial q_j} \qquad i, j = 1, \ldots, n. \tag{7.5}$$

Equation (7.5) states that the constraint is equivalent to $h(q) = 0$ if there exists some integrating factor $\alpha(q)$ for which the equation (7.5) is true. This should really not be a surprise since

$$\omega(q)\dot{q} = 0 \qquad \Longrightarrow \qquad \alpha(q)\omega(q)\dot{q} = 0$$

for all choices of smooth functions $\alpha(q)$. However, one still has to find a function α which satisfies equation (7.5).

The question of integrability becomes much more difficult in the presence of multiple Pfaffian constraints. Given a set of k constraints of the form of equation (7.2), not only does one need to check whether each one of the k constraints is integrable, but also which independent linear combinations of these,

$$\sum_{i=1}^{k} \alpha_i(q)\omega_i(q)\dot{q},$$

are integrable. That is, even if the given constraints are not individually integrable, they may contain a set of integrable constraints. Thus, there may exist functions h_i for $i = 1, \ldots, p$ with $p \leq k$ such that

$$\operatorname{span}\{\frac{\partial h_1}{\partial q}(q), \ldots, \frac{\partial h_p}{\partial q}(q)\} \quad \subset \quad \operatorname{span}\{\omega_1(q), \ldots, \omega_k(q)\}$$

for all q. If it is possible to find these functions, the motion of the system is restricted to level surfaces of h, namely to sets of the form

$$\{q : h_1(q) = c_1, \ldots, h_p(q) = c_p\}.$$

If $p = k$, then the constraints are holonomic. In the case that $p < k$, the constraints are not holonomic (since they are not completely equivalent to a set of holonomic constraints) but the reachable points of the system are still restricted. Thus the constraints are "partially holonomic." We will be primarily interested in the case in which the constraints do not restrict the reachable configurations. We refer to this situation as being *completely nonholonomic*.

It will be convenient for us to convert problems with nonholonomic constraints into another form. Roughly speaking, we would like to examine the systems not from the point of view of the constraints (namely, the directions that we *cannot* move), but rather from the viewpoint of the directions in which we are *free* to move. We begin by choosing a basis for the right null space of the constraints, denoted by $g_j(q) \in \mathbb{R}^n$, $i = 1, \ldots, n - k =: m$. By construction, this basis satisfies

$$\omega_i(q)g_j(q) = 0 \qquad \begin{array}{l} i = 1, \ldots, k \\ j = 1, \ldots, n - k, \end{array}$$

and the allowable trajectories of the system can thus be written as the possible solutions of the control system

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m. \tag{7.6}$$

That is, $q(t)$ is a feasible trajectory for the system if and only if $q(t)$ satisfies equation (7.6) for some choice of controls $u(t) \in \mathbb{R}^m$.

In this context, a constraint is completely nonholonomic if the corresponding control system can be steered between any two points. Thus the reachable configurations of the system are not restricted. Conversely, if a constraint is holonomic, then all motions of the system must lie on an appropriate constraint surface and the corresponding control system can only be steered between points on the given manifold. Hence, we can study the nature of Pfaffian constraints by studying the controllability properties of equation (7.6).

Nonholonomic constraints arise in a variety of applications. Besides rolling constraints on multifingered hands, nonholonomic constraints play an important role in the study of mobile robot systems and space-based robotic systems (in which conservation of angular momentum plays the role of a nonholonomic constraint). For these applications the primary question is that of reachability: when can we find a path between two arbitrary configurations and how do we go about computing such a path?

The outline of this chapter is as follows: in Section 2 we develop some tools from differential geometry and nonlinear control. Section 3 gives examples of systems with velocity constraints. In Section 4 the structure of nonholonomic systems is explored and the examples of Section 3 are analyzed. In the next chapter, we will develop methods for planning paths compatible with nonholonomic constraints.

Both this chapter and Chapter 8 are slightly more advanced in flavor than the previous chapters and represent some of the recent research in the robotics literature. Nonholonomic behavior also plays a strong role in many problems in geometric mechanics, which we touch on only briefly in the examples and exercises. In classical mechanics, nonholonomic behavior is closely related to the geometric phase associated with a group symmetry in a Hamiltonian or Lagrangian system. A good introduction to these concepts can be found in the lecture notes by Marsden [67].

2 Controllability and Frobenius' Theorem

In the previous section, we saw the difficulties in trying to determine whether or not constraints on a system were holonomic (or integrable). Further, if they are not holonomic, it is not completely clear as to when they are completely nonholonomic. In this section, we will develop the machinery needed for analyzing nonholonomic systems, in particular for answering the question of when a set of Pfaffian constraints is holonomic.

The tools we develop are based on a variety of results from differential geometry and nonlinear control theory, more specifically Frobenius' theorem and nonlinear controllability. To keep the mathematical prerequisites to a minimum, we do all the calculations in $\mathbb{R}^n$ and restrict ourselves to drift-free control systems (i.e., control systems whose state remains fixed when the input is turned off). Many of the proofs in this section rely on some properties of manifolds which we have omitted from the discussion; they can be skipped without loss of continuity. A good introduction to nonlinear control theory which includes many of the necessary differential geometric concepts can be found in Isidori [43] or Nijmeijer and van der Schaft [83].

2.1 Vector fields and flows

We restrict our attention to $\mathbb{R}^n$. We choose to make a distinction, however, between the space and its tangent space at a given point. A point of contact with Chapter 2 is our insistence there on making a distinction between points and vectors in $\mathbb{R}^3$ and enforcing the distinction by augmenting points by 1 and vectors by 0. Denote by $T_q\mathbb{R}^n$ the tangent space to $\mathbb{R}^n$ at a point $q \in \mathbb{R}^n$. A *vector field* on $\mathbb{R}^n$ is a smooth map which assigns to each point $q \in \mathbb{R}^n$ a tangent vector $f(q) \in T_q\mathbb{R}^n$. In local coordinates, we represent f as a column vector whose elements depend on q,

$$f(q) = \begin{bmatrix} f_1(q) \\ \vdots \\ f_n(q) \end{bmatrix}.$$

A vector field is smooth if each $f_i(q)$ is smooth.

Vector fields are to be thought of as right-hand sides of differential equations:

$$\dot{q} = f(q). \tag{7.7}$$

The rate of change of a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ along the flow of f is given by

$$\dot{V} = \frac{\partial V}{\partial q} f(q) = \sum_{i=1}^{n} \frac{\partial V}{\partial q_i} f_i.$$

The time derivative of V along the flow of f is referred to as the *Lie derivative* of V along f and is denoted $L_f V$:

$$L_f V := \frac{\partial V}{\partial q} f(q).$$

Associated with a vector field, we define the *flow* of a vector field to represent the solution of the differential equation (7.7). Specifically, $\phi_t^f(q)$ represents the state of the differential equation at time t starting from q at time 0. Thus $\phi_t^f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\frac{d}{dt}\phi_t^f(q) = f(\phi_t^f(q)) \qquad q \in \mathbb{R}^n.$$

A vector field is said to be *complete* if its flow is defined for all t. By the existence and uniqueness theorem of ordinary differential equations, for

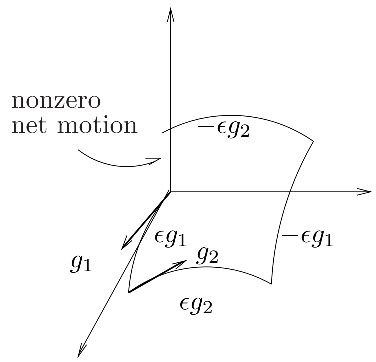


Figure 7.1: A Lie bracket motion.

each fixed t, ϕ_t^f is a local diffeomorphism of $\mathbb{R}^n$ onto itself. Further, it satisfies the following group property:

$$\phi_t^f \circ \phi_s^f = \phi_{t+s}^f,$$

for all t and s, where $\circ$ stands for the composition of the two flows, namely $\phi_t^f(\phi_s^f(q))$.

2.2 Lie brackets and Frobenius' theorem

Given two vector fields g_1 and g_2, the map $\phi_t^{g_1} \circ \phi_s^{g_2}$ stands for the composition of the flow of g_2 for s seconds with the flow of g_1 for t seconds. In general, this quantity is different from the map $\phi_s^{g_2} \circ \phi_t^{g_1}$, which stands for the composition in reverse order. Indeed, consider the flow depicted in Figure 7.1 starting from q_0. It consists of a flow along g_1 for ϵ seconds followed by a flow along g_2 for ϵ seconds, $-g_1$ for ϵ seconds, and $-g_2$ for ϵ seconds. For ϵ small, we may evaluate the Taylor series in ϵ for the state of the differential equation as

$$\begin{aligned} q(\epsilon) &= \phi_\epsilon^{g_1}(q(0)) \\ &= q(0) + \epsilon \dot{q}(0) + \frac{1}{2}\epsilon^2 \ddot{q}(0) + O(\epsilon^3) \\ &= q_0 + \epsilon g_1(q_0) + \frac{1}{2}\epsilon^2 \frac{\partial g_1}{\partial q} g_1(q_0) + O(\epsilon^3), \end{aligned}$$

where the notation $O(\epsilon^k)$ represents terms of order ϵ^k and the partial derivative of g_1 is evaluated at q_0.

Now evaluating at time 2ϵ,

$$\begin{aligned}
q(2\epsilon) &= \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(q_0) \\
&= \phi_\epsilon^{g_2}(q_0 + \epsilon g_1(q_0) + \frac{\epsilon^2}{2}\frac{\partial g_1}{\partial q} g_1(q_0) + O(\epsilon^3)) \\
&= q_0 + \epsilon g_1(q_0) + \frac{1}{2}\epsilon^2 \frac{\partial g_1}{\partial q} g_1(q_0) \\
&\qquad + \epsilon g_2(q_0 + \epsilon g_1(q_0)) + \frac{\epsilon^2}{2}\frac{\partial g_2}{\partial q} g_2(q_0) + O(\epsilon^3) \\
&= q_0 + \epsilon(g_1(q_0) + g_2(q_0)) \\
&\qquad + \frac{1}{2}\epsilon^2 (\frac{\partial g_1}{\partial q} g_1(q_0) + \frac{\partial g_2}{\partial q} g_2(q_0) + 2\frac{\partial g_2}{\partial q} g_1(q_0)) + O(\epsilon^3).
\end{aligned}$$

Here, we have used the Taylor series expansion for $g_2(q_0 + \epsilon g_1(q)) = g_2(q_0) + \epsilon \frac{\partial g_2}{\partial q} g_1(q_0) + O(\epsilon^2)$. At the next step (we invite the reader to verify this), we get

$$\begin{aligned}
q(3\epsilon) &= \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(q_0) \\
&= q_0 + \epsilon g_2(q_0) \\
&\qquad + \frac{\epsilon^2}{2}(\frac{\partial g_2}{\partial q} g_2(q_0) + 2\frac{\partial g_2}{\partial q} g_1(q_0) - 2\frac{\partial g_1}{\partial q} g_2(q_0)) + O(\epsilon^3).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
q(4\epsilon) &= \phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(q_0) \\
&= q_0 + \epsilon^2(\frac{\partial g_2}{\partial q} g_1(q_0) - \frac{\partial g_1}{\partial q} g_2(q_0)) + O(\epsilon^3).
\end{aligned} \tag{7.8}$$

Motivated by this calculation, we define the *Lie bracket* of two vector fields f and g as

$$[f, g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q).$$

The Lie bracket is thus the infinitesimal motion (actually of order ϵ^2) that results from flowing around a square defined by two vector fields f and g. If $[f, g] = 0$ then it can be shown that the right hand side of equation (7.8) is identically equal to q_0 and f and g are said to *commute*. A *Lie product* is a nested set of Lie brackets, for example,

$$[[f, g], [f, [f, g]]].$$

Example 7.1. Lie brackets of linear vector fields
Consider two linear vector fields given by $f(q) = Aq$ and $g(q) = Bq$. Then the Lie bracket of the two linear vector fields is a linear vector field given by

$$[f, g](q) = (BA - AB)q,$$

that is, it is the *commutator* of the two matrices A, B.

The following properties of Lie brackets follow from the definition. Their proof is left as an exercise.

Proposition 7.1. Properties of Lie brackets
Given vector fields f, g, h *on* $\mathbb{R}^n$ *and smooth functions* $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$*, the Lie bracket satisfies the following properties:*

1. Skew-symmetry:
$$[f, g] = -[g, f]$$

2. Jacobi identity:
$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0$$

3. Chain rule:
$$[\alpha f, \beta g] = \alpha\beta[f, g] + \alpha(L_f \beta)g - \beta(L_g \alpha)f,$$

where $L_f\beta$ *and* $L_g\alpha$ *stand for the Lie derivatives of* β *and* α *along the vector fields* f *and* g *respectively.*

An alternative method of defining the Lie bracket of two vector fields f and g is to require that it satisfies for all smooth functions $\alpha : \mathbb{R}^n \to \mathbb{R}$:

$$L_{[f,g]}\alpha = L_f(L_g\alpha) - L_g(L_f\alpha).$$

The reader should carefully parse the previous equation and convince herself of this fact.

A *distribution* assigns a subspace of the tangent space to each point in $\mathbb{R}^n$ in a smooth way. A special case is a distribution defined by a set of smooth vector fields, $g_1, \ldots, g_m$. In this case we define the distribution as
$$\Delta = \text{span}\{g_1, \ldots, g_m\},$$
where we take the span over the set of smooth real-valued functions on $\mathbb{R}^n$. Evaluated at any point $q \in \mathbb{R}^n$, the distribution defines a linear subspace of the tangent space

$$\Delta_q = \text{span}\{g_1(q), \ldots, g_m(q)\} \subset T_q\mathbb{R}^n.$$

The distribution is said to be *regular* if the dimension of the subspace Δ_q does not vary with q. A distribution is *involutive* if it is closed under the Lie bracket, i.e.,

$$\Delta \text{ involutive} \quad \Longleftrightarrow \quad \forall f, g \in \Delta, \quad [f, g] \in \Delta.$$

For a finite dimensional distribution it suffices to check that the Lie brackets of the basis elements are contained in the distribution. The *involutive closure* of a distribution, denoted $\overline{\Delta}$, is the closure of Δ under bracketing; that is, $\overline{\Delta}$ is the smallest distribution containing Δ such that if $f, g \in \overline{\Delta}$ then $[f, g] \in \overline{\Delta}$.

Definition 7.1. Lie algebra
A vector space V (over $\mathbb{R}$) is a *Lie algebra* if there exists a bilinear operation $V \times V \to V$, denoted $[\ ,\ ]$, satisfying (i) skew-symmetry and (ii) the Jacobi identity.

The set of smooth vector fields on $\mathbb{R}^n$ with the Lie bracket is a Lie algebra, and is denoted $\mathfrak{X}(\mathbb{R}^n)$. Let $g_1, \dots, g_m$ be a set of smooth vector fields, Δ the distribution defined by $g_1, \dots, g_m$ and, $\overline{\Delta}$ the involutive closure of Δ. Then, $\overline{\Delta}$ is a Lie algebra (in fact the smallest Lie algebra containing $g_1, \dots, g_m$). It is called the Lie algebra generated by $g_1, \dots, g_m$ and is often denoted $\mathcal{L}(\}_{\infty}, \dots, \}_{\Updownarrow})$. Elements of $\mathcal{L}(\}_{\infty}, \dots, \}_{\Updownarrow})$ are obtained by taking all linear combinations of elements of $g_1, \dots, g_m$, taking Lie brackets of these, taking all linear combinations of these, and so on. We define the rank of $\mathcal{L}(\}_{\infty}, \dots, \}_{\Updownarrow})$ at a point $q \in \mathbb{R}^n$ to be the dimension of $\overline{\Delta}_q$ as a distribution.

A distribution Δ of constant dimension k is said to be *integrable* if for every point $q \in \mathbb{R}^n$, there exists a set of smooth functions $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n-k$ such that the row vectors $\frac{\partial h_i}{\partial q}$ are linearly independent at q, and for every $f \in \Delta$

$$L_f h_i = \frac{\partial h_i}{\partial q} f(q) = 0 \qquad i = 1, \dots, n-k. \tag{7.9}$$

The hypersurfaces defined by the level sets

$$\{q : h_1(q) = c_1, \dots, h_{n-k}(q) = c_{n-k}\}$$

are called *integral manifolds* for the distribution. If we regard an integral manifold as a smooth surface in $\mathbb{R}^n$, then equation (7.9) requires that the distribution be equal to the tangent space of that surface at the point q.

Integral manifolds are related to involutive distributions by the following celebrated theorem.

Theorem 7.2 (Frobenius)**.** *A regular distribution is integrable if and only if it is involutive.*

Thus, if Δ is an k-dimensional involutive distribution, then locally there exist $n-k$ functions $h_i \colon \mathbb{R}^n \to \mathbb{R}$ such that integral manifolds of Δ are given by the level surfaces of $h = (h_1, \dots, h_{n-k})$. These level surfaces form a *foliation* of $\mathbb{R}^n$. A single level surface is called a *leaf of the foliation*.

Associated with the tangent space $T_q\mathbb{R}^n$ is the dual space $T_q^*\mathbb{R}^n$, the set of linear functions on $T_q\mathbb{R}^n$. Just as we defined vector fields on $\mathbb{R}^n$, we define a *one-form* as a map which assigns to each point $q \in \mathbb{R}^n$ a *covector* $\omega(q) \in T_q^*\mathbb{R}^n$. In local coordinates we represent a smooth one-form as a row vector

$$\omega(q) = \begin{bmatrix} \omega_1(q) & \omega_2(q) & \cdots & \omega_n(q) \end{bmatrix}.$$

Differentials of smooth functions are good examples of one-forms. For example, if $\beta : \mathbb{R}^n \to \mathbb{R}$, then the one-form $d\beta$ is given by

$$d\beta = \begin{bmatrix} \frac{\partial \beta}{\partial q_1} & \frac{\partial \beta}{\partial q_2} & \cdots & \frac{\partial \beta}{\partial q_n} \end{bmatrix}.$$

Note, however, that all one-forms are not necessarily the differentials of smooth functions (a one-form which does happen to be the derivative of a function is said to be *exact*).

A one-form acts on a vector field to give a real-valued function on $\mathbb{R}^n$ by taking the inner product between the row vector ω and the column vector f:

$$\omega \cdot f = \sum_i \omega_i f_i.$$

A *codistribution* assigns a subspace of $T_q^*\mathbb{R}^n$ smoothly to each $q \in \mathbb{R}^n$. A special case is a codistribution obtained as a span of a set of one-forms,

$$\Omega = \operatorname{span}\{\omega_1, \ldots, \omega_m\},$$

where the span is over the set of smooth functions. As before, the rank of the codistribution is the dimension of Ω_q. The codistribution Ω is said to be regular if its rank is constant.

To begin our study of motion planning for nonholonomic systems, our first task is to convert the specified constraints given as one-forms into an equivalent control system. To this end, consider the problem of constructing a path $q(t) \in \mathbb{R}^n$ between a given q_0 and q_f subject to the constraints

$$\omega_i(q)\dot{q} = 0 \qquad i = 1, \ldots, k.$$

The ω_i's are linear functions on the tangent spaces of $\mathbb{R}^n$, i.e., one-forms. We assume that the ω_i's are smooth and linearly independent over the set of smooth functions. The following proposition is a formalization of the discussion of the introduction.

Proposition 7.3. Distribution annihilating constraints
Given a set of one-forms $\omega_i(q)$, $i = 1, \ldots, k$, there exist smooth, linearly independent vector fields $g_j(q), j = 1, \ldots, n-k$ such that $\omega_i(q) \cdot g_j(q) = 0$ for all i and j.

Proof. The ω_i's form a codistribution of dimension k in $\mathbb{R}^n$. We can choose local coordinates such that the set of one-forms is given by

$$\tilde{\omega}_i = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 & \alpha_{i,k+1} & \cdots & \alpha_{in} \end{bmatrix},$$

where the 1 in the preceding equation is in the ith entry, and the functions

$\alpha_{il} : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Define

$$g_j := \begin{bmatrix} -\alpha_{1,(j+k)} \\ \vdots \\ -\alpha_{k,(j+k)} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

where the 1 is in the j +kth entry. The g_j's are linearly independent and annihilate the constraints since

$$\tilde{\omega}_i \cdot g_j = \alpha_{i(j+k)} - \alpha_{i(j+k)} = 0.$$

This shows that $\omega_i \cdot g_j = 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n-k$. □

In the language of distributions and codistributions, the results of this proposition are expressed by defining the codistribution

$$\Omega = \text{span}\{\omega_1, \ldots, \omega_k\}$$

and the distribution

$$\Delta = \text{span}\{g_1, \ldots, g_{n-k}\}$$

and stating that

$$\Delta = \Omega^\perp.$$

We say that the distribution Δ annihilates the codistribution Ω. The control system associated with the distribution Δ is of the form

$$\dot{q} = g_1(q)u_1 + \cdots + g_{n-k}(q)u_{n-k},$$

with the controls u_i to be freely specified.

These results of this section can be used to determine if a set of Pfaffian constraints is holonomic:

Proposition 7.4. Integrability of Pfaffian constraints
A set of smooth Pfaffian constraints is integrable if and only if the distribution which annihilates the constraints is involutive.

2.3 Nonlinear controllability

In view of Proposition 7.3, which yields a set of vector fields orthogonal to a given set of one-forms, it is clear that the motion planning problem is equivalent to steering a control system. Thus, we will now restrict our attention to control systems of the form

$$\Sigma : \qquad \dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m \qquad \begin{array}{l} q \in \mathbb{R}^n \\ u \in U \subset \mathbb{R}^m. \end{array} \tag{7.10}$$

This system is said to be *drift-free*, meaning to say that when the controls are set to zero the state of the system does not drift. We assume that the g_j are smooth, linearly independent vector fields on $\mathbb{R}^n$ and that their flows are defined for all time (i.e., the g_j are complete). We wish to determine conditions under which we can steer from $q_0 \in \mathbb{R}^n$ to an arbitrary $q_f \in \mathbb{R}^n$ by appropriate choice of $u(\cdot)$.

A system Σ is controllable if for any $q_0, q_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u\colon [0,T] \to U$ such that Σ satisfies $q(0) = q_0$ and $q(T) = q_f$. A system is said to be *small-time locally controllable* at q_0 if we can reach nearby points in arbitrarily small amounts of time and stay near to q_0 at all times. Given an open set $V \subseteq \mathbb{R}^n$, define $\mathcal{R}^V(q_0, T)$ to be the set of states q such that there exists $u\colon [0,T] \to U$ that steers Σ from $q(0) = q_0$ to $q(T) = q_f$ and satisfies $q(t) \in V$ for $0 \le t \le T$. We also define

$$\mathcal{R}^V(q_0, \le T) = \bigcup_{0 < \tau \le T} \mathcal{R}^V(q_0, \tau)$$

to be the set of states reachable up to time T. A system is small-time locally controllable (*locally controllable* for brevity) if $\mathcal{R}^V(q_0, \le T)$ contains a neighborhood of q_0 for all neighborhoods V of q_0 and $T > 0$.

Let $\overline{\Delta} = \mathcal{L}(\}_\infty, \ldots, \}_\Updownarrow)$ be the Lie algebra generated by $g_1, \ldots, g_m$. It is referred to as the the *controllability Lie algebra*. From the construction involved in the definition of the Lie bracket in the previous subsection, we saw that by using an input sequence of

$$\begin{array}{lll} u_1 = +1 & u_2 = 0 & \text{for } 0 \le t < \epsilon \\ u_1 = 0 & u_2 = +1 & \text{for } \epsilon \le t < 2\epsilon \\ u_1 = -1 & u_2 = 0 & \text{for } 2\epsilon \le t < 3\epsilon \\ u_1 = 0 & u_2 = -1 & \text{for } 3\epsilon \le t < 4\epsilon, \end{array}$$

we get motion in the direction of the Lie bracket $[g_1, g_2]$. If we were to iterate on this sequence, it should be possible to generate motion along directions given by all the other Lie products associated with the g_i. Thus, it is not surprising that it is possible to steer the system along all of the directions represented in $\mathcal{L}(\}_\infty, \ldots, \}_\Updownarrow)$. This is made precise by the following theorem, which was originally proved by W.-L. Chow (in somewhat different form) in the 1940s.

Theorem 7.5 (Chow)**.** *The control system* (7.10) *is locally controllable at $q \in \mathbb{R}^n$ if $\overline{\Delta}_q = T_q\mathbb{R}^n$.*

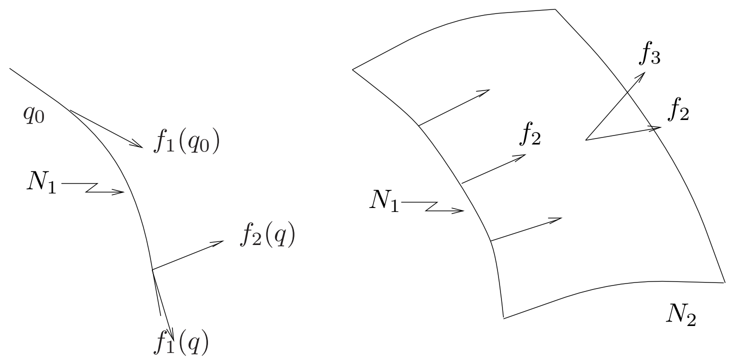


Figure 7.2: Proof of local controllability. At each step we can find a vector field which is not in N_k.

This result asserts that the drift-free system Σ is controllable if the rank of the controllability Lie algebra is n. The condition of Chow's theorem consists of checking the rank of the controllability Lie algebra and is hence referred to as the *controllability rank condition.*

To prove Chow's theorem, we prove the following pair of implications for a given system Σ in a neighborhood of a point q:

$$\overline{\Delta}_q = T_q\mathbb{R}^n \implies \operatorname{int}\mathcal{R}^V(q, \leq T) \neq \{\} \iff \Sigma \text{ is locally controllable},$$

where $\overline{\Delta} = \mathcal{L}(\}_{\infty}, \ldots, \}_{\Updownarrow})$ and $\{\}$ stands for the empty set.

Proposition 7.6. Controllability rank condition
If $\overline{\Delta}_q = T_q\mathbb{R}^n$ for all q in some neighborhood of q_0, then for any $T > 0$ and neighborhood V of q_0, $\operatorname{int}\mathcal{R}^V(q_0, \leq T)$ is non-empty.

Proof. The proof is by recursion. Choose $f_1 \in \Delta$. For $\epsilon_1 > 0$ sufficiently small,

$$N_1 = \{\phi_{t_1}^{f_1}(q_0) : 0 < t_1 < \epsilon_1\}$$

is a smooth surface (manifold) of dimension one which contains points arbitrarily close to q_0. Without loss of generality, take $N_1 \subset V$. Assume $N_k \subset V$ is a k-dimensional manifold. If $k < n$, there exists $q \in N_k$ and $f_{k+1} \in \overline{\Delta}$ such that $f_{k+1} \notin T_qN_k$. If this were not so then $\overline{\Delta}_q \subset T_qN_k$ for any q in some open set $W \subset N_k$, which would imply $\overline{\Delta}|_W \subset TN_k$. This cannot be true since $\dim \overline{\Delta}_q = n > \dim N_k$. For ϵ_{k+1} sufficiently small

$$N_{k+1} = \{\phi_{t_{k+1}}^{f_{k+1}} \circ \cdots \circ \phi_{t_1}^{f_1}(q_0) : 0 < t_i < \epsilon_i,\, i = 1, \cdots, k+1\}$$

is a $k+1$ dimensional manifold. Since ϵ can be made arbitrarily small, we can assume $N_{k+1} \subset V$.

If $k = n$, $N_k \subset V$ is an n-dimensional manifold and by construction $N_k \subset \mathcal{R}^V(q_0, \leq \epsilon_1 + \cdots + \epsilon_n)$. Hence $\mathcal{R}^V(q_0, \epsilon)$ contains an open set. By

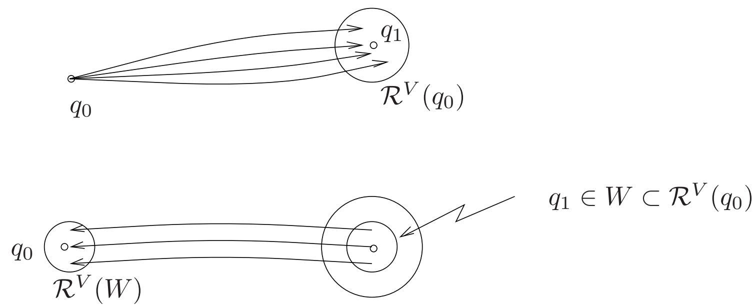


Figure 7.3: Proof of local controllability. To show $\mathcal{R}^V(q_0)$ contains a neighborhood of the origin, we move to any point q_f and map a neighborhood of q_f to a neighborhood of q_0 by reversing our original path.

restricting each $\epsilon_i \leq T/n$, we can find such an open set for any $T > 0$. This proof is illustrated in Figure 7.2. □

Having established conditions under which the set $\operatorname{int} \mathcal{R}^V(q, \leq T)$ is not empty, we would like to determine if the set can be chosen so as to have q_0 in its interior. This is the subject of the next proposition:

Proposition 7.7. Local controllability
The interior of the set $\mathcal{R}^V(q_0, \leq T)$ is non-empty for all neighborhoods V of q_0 and $T > 0$ if and only if Σ is locally controllable at q_0.

Proof. The sufficiency follows from the definition of locally controllable. To prove necessity, we need to show that $\mathcal{R}^V(q_0, \leq T)$ contains a neighborhood of q_0. Choose a piecewise constant $u\colon [0, T/2] \to U$ such that u steers q_0 to some $q_f \in \mathcal{R}^V(q_0, \leq T/2)$ and $q(t) \in V$. Let ϕ_t^u be the flow corresponding to this input (as given in the proof of the previous theorem). Since Σ is drift-free, we can flow backwards from q_f to q_0 using $u'(t) = -u(T/2 - t)$, $t \in [0, T/2]$. The flow corresponding to u' is $(\phi_t^u)^{-1}$. By continuity of the flow, there exists $W \subset \mathcal{R}^V(q_0, T/2)$ such that $q_f \in W$ and $(\phi_t^u)^{-1}(W) \subset V$ for all t. Furthermore, $(\phi_{T/2}^u)^{-1}(W)$ is a neighborhood of q_0. It follows that $\mathcal{R}^V(q_0, \leq T)$ contains a neighborhood of q_0 since we can concatenate the inputs which steer q_0 to $q_f \in W$ with u' to obtain an open set containing q_0. This is illustrated in Figure 7.3. □

In principle, we now have a recipe for solving the motion planning problem for systems which meet the controllability rank condition. Given an initial point q_0 and final point q_f, find finitely many intermediate via points $q_1, q_2, \ldots, q_p \in \mathbb{R}^n$ and neighborhoods V_i such that

$$\bigcup_{i=1}^{p} \mathcal{R}^{V_i}(q_i, \leq T)$$

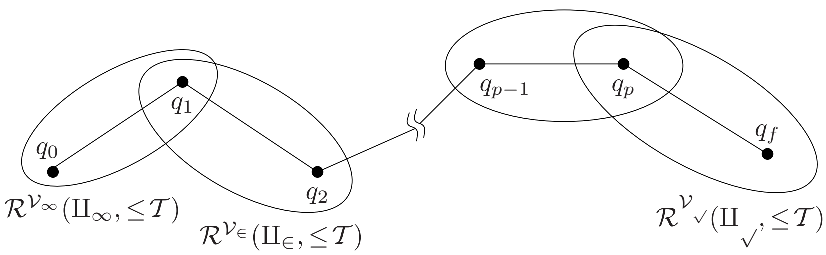


Figure 7.4: Steering between q_0 and q_f.

contains the straight line segment connecting q_0 to q_f, as shown in Figure 7.4. Then there exists a control law of p segments which steers from q_0 to q_f. The difficulty with this procedure and the preceding theorems in this section is that they are *non-constructive*. It is in principle possible to solve the motion planning problem for a given set of constraints of the form

$$\omega_i(q)\dot{q} = 0 \qquad i = 1, \ldots, k$$

for arbitrary given q_0 and q_f, provided that the associated control system

$$\dot{q} = g_1(q)u_1 + \cdots + g_{n-k}(q)u_{n-k}$$

has a full rank controllability Lie algebra. However, the preceding theorems do not give a constructive procedure for generating paths for the system joining q_0 and q_f. This *constructive controllability* is the goal of the next chapter.

3 Examples of Nonholonomic Systems

We now present a set of examples of systems with nonholonomic constraints which we will use repeatedly throughout this chapter and the next to illustrate the different concepts. Nonholonomic constraints arise in two kinds of situations:

1. Bodies in contact with each other which *roll without slipping*
2. *Conservation of angular momentum* in a multibody system

An example of the first kind can be found in the problem of dextrous manipulation with a multifingered robot hand. Here the nonholonomic constraint arises from the fingers rolling without slipping on the surface of a grasped object. Other such examples arise in path planning problems for mobile robots or automobiles, where the wheels roll without slipping. For examples of the second kind, we have motion of a satellite with robotic

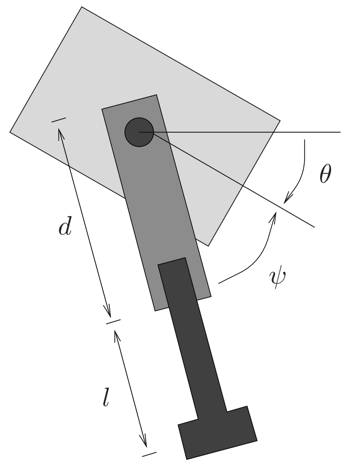


Figure 7.5: A simple hopping robot.

appendages moving in space, where angular momentum is conserved, or a diver or gymnast in mid-air maneuvers.

In the sequel, we will give a description of several nonholonomic systems. The proof of their nonholonomy (that is the impossibility of finding functions of the configuration variables which are "integrals" of the constraints) is deferred to Section 4.

Example 7.2. Hopping robot in flight
As our first example, we consider the dynamics of a hopping robot in the flight phase, as shown in Figure 7.5. This robot consists of a body with an actuated leg that can rotate and extend; the "constraint" on the system is conservation of angular momentum.

The configuration $q = (\psi, l, \theta)$ consists of the leg angle, the leg extension, and the body angle of the robot. We denote the moment of inertia of the body by I and concentrate the mass of the leg, m, at the foot. The upper leg length is taken to be d, with l representing the extension of the leg past this point. The total angular momentum of the robot is given by

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\psi}). \tag{7.11}$$

Assume that the angular momentum of the robot is initially zero. Equation (7.11) is a single Pfaffian constraint in the three velocities $\dot{\psi}$, $\dot{l}$, and $\dot{\theta}$. Thus, the associated control system has two inputs—three configuration variables minus one constraint. As a basis for the 2-dimensional right null space of the constraint, we choose one vector field corresponding to controlling the leg angle ψ, and the other corresponding to controlling

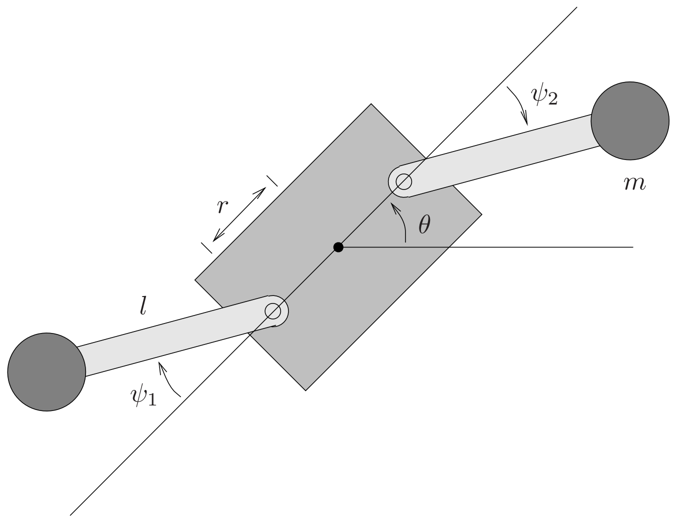


Figure 7.6: A simplified model of a planar space robot.

the leg extension l; i.e., set $\dot{\psi} = u_1$ and $\dot{l} = u_2$. Then, we have

$$g_1(q) = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{I+m(l+d)^2} \end{bmatrix} \qquad g_2(q) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and the equivalent control system is given by

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2.$$

Example 7.3. Planar space robot
Figure 7.6 shows a simplified model of a planar robot consisting of two arms connected to a central body via revolute joints. If the robot is free floating, then the law of conservation of angular momentum implies that moving the arms causes the central body to rotate. In the case that the angular momentum is zero, this conservation law can be viewed as a Pfaffian constraint on the system.

Let M and I represent the mass and inertia of the central body and let m represent the mass of the arms, which we take to be concentrated at the tips. The revolute joints are located a distance r from the middle of the central body and the links attached to these joints have length l. We let (x_1, y_1) and (x_2, y_2) represent the position of the ends of each of the arms (in terms of θ, ψ_1, and ψ_2). Assuming that the body is free floating in space and that friction is negligible, we can derive the constraints arising from conservation of angular momentum.

Let θ be the angle of the central body with respect to the horizontal, ψ_1 and ψ_2 the angles of the left arm and right arms with respect to the central body, and $p \in \mathbb{R}^2$ the location of a point on the central body (say the center of mass). The kinetic energy of the system has the form

$$\begin{aligned} K &= \frac{1}{2}(M+2m)\|\dot{p}\|^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2+\dot{y}_2^2) \\ &= \frac{1}{2}(M+2m)\|\dot{p}\|^2 + \frac{1}{2}\begin{bmatrix}\dot{\psi}_1\\ \dot{\psi}_2\\ \dot{\theta}\end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{23} & a_{33}\end{bmatrix}\begin{bmatrix}\dot{\psi}_1\\ \dot{\psi}_2\\ \dot{\theta}\end{bmatrix}, \end{aligned}$$

where a_{ij} can be calculated as

$$\begin{aligned} a_{11} &= a_{22} = ml^2 \\ a_{12} &= 0 \\ a_{13} &= ml^2 + mr\cos\psi_1 \\ a_{23} &= ml^2 + mr\cos\psi_2 \\ a_{33} &= I + 2ml^2 + 2mr^2 + 2mrl\cos\psi_1 + 2mrl\cos\psi_2. \end{aligned}$$

Note that the kinetic energy of the system is independent of the variable θ. It therefore follows from Lagrange's equations that in the absence of external forces,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0.$$

Thus the quantity $\frac{\partial L}{\partial \dot{\theta}}$ is a constant of the motion. This is precisely the angular momentum, μ, of the system:

$$\mu = \frac{\partial L}{\partial \dot{\theta}} = a_{13}\dot{\psi}_1 + a_{23}\dot{\psi}_2 + a_{33}\dot{\theta}.$$

If the initial angular momentum is zero, then *conservation of angular momentum* ensures that the angular momentum stays zero, giving the following constraint equation

$$a_{13}(\psi)\dot{\psi}_1 + a_{23}(\psi)\dot{\psi}_2 + a_{33}(\psi)\dot{\theta} = 0. \tag{7.12}$$

Since the variables that are actuated are the hinge angles of the left and right arm, we choose as inputs $u_1 = \dot{\psi}_1$ and $u_2 = \dot{\psi}_2$. Using these in equation (7.12) and setting $q = (\psi_1, \psi_2, \theta)$, we get

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

where

$$g_1(q) = \begin{bmatrix} 1\\ 0\\ \frac{-a_{13}}{a_{33}}\end{bmatrix} \qquad g_2(q) = \begin{bmatrix} 0\\ 1\\ \frac{-a_{23}}{a_{33}}\end{bmatrix}.$$

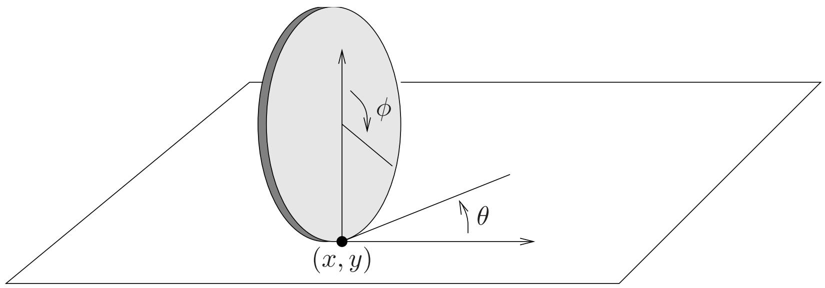


Figure 7.7: Disk rolling on a plane.

Example 7.4. Disk rolling on a plane
Consider the motion of a thin flat disk rolling on a plane shown in Figure 7.7. The configuration space of the system is parameterized by the xy location of the contact point of the disk with the plane, the angle θ that the disk makes with the horizontal line, and the angle ϕ of a fixed line on the disk with respect to the vertical axis. We assume that the disk rolls without slipping. As a consequence we have that

$$\dot{x} - \rho \cos\theta \, \dot{\phi} = 0$$
$$\dot{y} - \rho \sin\theta \, \dot{\phi} = 0,$$

where $\rho > 0$ is the radius of the disk. Writing these equations in the form of Pfaffian constraints with $q = (x, y, \theta, \phi)$ we have

$$\begin{bmatrix} 1 & 0 & 0 & -\rho\cos\theta \\ 0 & 1 & 0 & -\rho\sin\theta \end{bmatrix} \dot{q} = 0.$$

Choosing $\dot{\theta} = u_1$, the rate of rolling, and $\dot{\phi} = u_2$, the rate of turning about the vertical axis, we have the associated control system:

$$\dot{q} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2. \tag{7.13}$$

Example 7.5. Kinematic car
Consider a simple kinematic model for an automobile with front and rear tires, as shown in Figure 7.8. The rear tires are aligned with the car, while the front tires are allowed to spin about the vertical axes. To simplify the derivation, we model the front and rear pairs of wheels as single wheels at the midpoints of the axles. The constraints on the system arise by allowing the wheels to roll and spin, but not slip.

Let (x, y, θ, ϕ) denote the configuration of the car, parameterized by the xy location of the rear wheel(s), the angle of the car body with

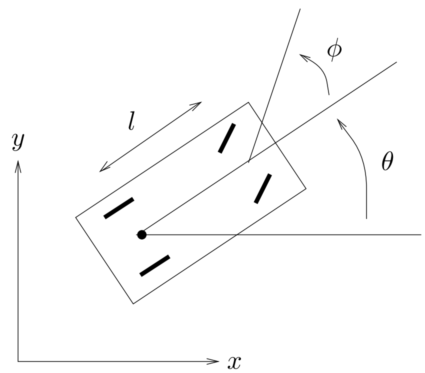


Figure 7.8: Kinematic model of an automobile.

respect to the horizontal, θ, and the steering angle with respect to the car body, ϕ. The constraints for the front and rear wheels are formed by setting the sideways velocity of the wheels to zero. In particular, the velocity of the back wheels perpendicular to their direction is $\sin\theta\dot{x} - \cos\theta\dot{y}$ and the velocity of the front wheels perpendicular to the direction they are pointing is $\sin(\theta+\phi)\dot{x} - \cos(\theta+\phi)\dot{y} - l\dot{\theta}\cos\phi$, so that the Pfaffian constraints on the automobile are:

$$\begin{aligned}\sin(\theta+\phi)\dot{x} - \cos(\theta+\phi)\dot{y} - l\cos\phi\,\dot{\theta} = 0\\ \sin\theta\,\dot{x} - \cos\theta\,\dot{y} = 0.\end{aligned}$$

Converting this to a control system with the inputs chosen as the driving velocity u_1 and the steering velocity u_2 gives

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\\ \frac{1}{l}\tan\phi\\ 0\end{bmatrix} u_1 + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} u_2. \tag{7.14}$$

For this choice of vector fields, u_1 corresponds to the forward velocity of the rear wheels of the car and u_2 corresponds to the velocity of the angle of the steering wheel.

Example 7.6. Fingertip rolling on an object
Let us analyze the motion of a curved fingertip over a curved object. As we discussed in Section 6 of Chapter 5, we parameterize the object surface by $\alpha_o \in \mathbb{R}^2$, the fingertip surface by $\alpha_f \in \mathbb{R}^2$, and the angle of contact by $\psi \in \mathbb{S}^1$, giving a 5-dimensional configuration space. The kinematic

equations of contact are given by

$$\begin{aligned}
\dot{\alpha}_f &= M_f^{-1}(K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix}-\omega_y\\ \omega_x\end{bmatrix} - \tilde{K}_o\begin{bmatrix}v_x\\ v_y\end{bmatrix}\right)\\
\dot{\alpha}_o &= M_o^{-1}R_\psi(K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix}-\omega_y\\ \omega_x\end{bmatrix} + K_f\begin{bmatrix}v_x\\ v_y\end{bmatrix}\right)\\
\dot{\psi} &= \omega_z + T_f M_f \dot{\alpha}_f + T_o M_o \dot{\alpha}_o.
\end{aligned} \tag{7.15}$$

The rolling constraint is obtained by setting the sliding velocity and the velocity of rotation about the contact normal to zero:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = 0 \qquad \omega_z = 0. \tag{7.16}$$

Substituting (7.16) into equation (7.15) yields the following constraints:

$$\begin{aligned}
M_f\dot{\alpha}_f - R_\psi M_o \dot{\alpha}_o &= 0\\
T_f M_f \dot{\alpha}_f + T_o M_o \dot{\alpha}_o - \dot{\psi} &= 0.
\end{aligned} \tag{7.17}$$

If we set $q = (\alpha_f, \alpha_o, \psi) \in \mathbb{R}^5$, then the foregoing set of three constraints is of the form

$$\omega_i(q)\dot{q} = 0 \qquad i = 1, 2, 3.$$

To obtain a control system associated with these constraints, we let $u_1 = \omega_x$ and $u_2 = \omega_y$ in the kinematic equations for rolling contact. After rearranging the results, we have

$$\dot{q} = \begin{bmatrix} M_f^{-1} \\ M_o^{-1}R_\psi \\ T_f + T_o R_\psi \end{bmatrix} (K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}u_1 + \begin{bmatrix}-1\\0\end{bmatrix}u_2\right). \tag{7.18}$$

We now specialize the example to the case that the object is flat and the fingertip is a sphere of radius one. The curvature forms, metric tensors, and torsions for the fingertip and the object have been derived in Example 5.7 and are reproduced here for convenience:

$$\begin{aligned}
K_o &= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & K_f &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
M_o &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & M_f &= \begin{bmatrix}1 & 0\\ 0 & \cos q_1\end{bmatrix}\\
T_o &= \begin{bmatrix}0 & 0\end{bmatrix} & T_f &= \begin{bmatrix}0 & -\tan q_1\end{bmatrix}.
\end{aligned}$$

Substituting the above results into (7.17) gives

$$\begin{bmatrix} 1 & 0 & -\cos q_5 & \sin q_5 & 0\\ 0 & \cos q_1 & \sin q_5 & \cos q_5 & 0\\ 0 & \sin q_1 & 0 & 0 & 1 \end{bmatrix}\dot{q} = 0.$$

In this case, the formula (7.18) gives, with the inputs being the rates of rolling about the two tangential directions,

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ \dot q_4 \\ \dot q_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \sec q_1 \\ -\sin q_5 \\ -\cos q_5 \\ -\tan q_1 \end{bmatrix} u_1 + \begin{bmatrix} -1 \\ 0 \\ -\cos q_5 \\ \sin q_5 \\ 0 \end{bmatrix} u_2. \tag{7.19}$$

4 Structure of Nonholonomic Systems

We return to the problem of motion planning for systems satisfying linear velocity constraints of the form

$$\omega_i(q)\dot q = 0 \qquad i = 1, \ldots, k.$$

In Section 2 we showed how the problem of finding feasible trajectories in the configuration space could be dualized to one of finding trajectories of the control system

$$\dot q = g_1(q)u_1 + \cdots + g_m(q)u_m, \tag{7.20}$$

with $m = n - k$ and $\omega_i(q)g_j(q) = 0$. From the controllability rank condition, it follows that one can find a trajectory joining an arbitrary starting point and end point if the rank of the Lie algebra generated by $g_1, \ldots, g_m$ is n. If $\overline{\Delta}_q \neq T_q\mathbb{R}^n$ and in addition $\overline{\Delta}_q$ has a constant rank $n - p$ which is less than n, then it follows from Frobenius' theorem that there exist functions $h_i(q) = c_i$, $i = 1, \ldots, p$ such that

$$\omega_i(q)\dot q = 0 \quad i = 1, \ldots, k \qquad \Longleftrightarrow \qquad h_j(q) = c_j \quad j = 1, \ldots, p.$$

Consider this a little further: since the dimension of $\overline{\Delta}$ is greater than or equal to the dimension of Δ, it follows that $p \leq k$. Thus, the number of functions whose level sets are tangential to the given distribution are fewer than the dimension of the distribution. The process of converting from the given constraints, specified as a codistribution, to an equivalent control system and then integrating the involutive closure of this distribution may seem to be convoluted. It is indeed possible to deal directly with a given codistribution and to find the maximal integrable codistribution contained within it, but this involves methods of exterior differential systems which are beyond the scope of this book. Of course, in the event that $\overline{\Delta} = T_q\mathbb{R}^n$ for all q, then $p = 0$, i.e., there are no nontrivial functions which integrate the given constraints. In this case the distribution is said to be *completely nonholonomic*, as was noted earlier.

In this section, we will try to make precise some notation that we will use in dealing with nonholonomic systems and apply it to the examples that we considered in Section 3. Some additional machinery to study the growth of the controllability Lie algebra is discussed at the end of this section.

4.1 Classification of nonholonomic distributions

The complexity of the motion planning problem is related to the order of Lie brackets in its controllability Lie algebra. Here we develop some concepts which allow us to classify nonholonomic systems. Let $\Delta = \text{span}\{g_1, \ldots, g_m\}$ be the distribution associated with the control system (7.20). Define $\Delta_1 = \Delta$ and

$$\Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}],$$

where

$$[\Delta_1, \Delta_{i-1}] = \text{span}\{[g, h] : g \in \Delta_1, h \in \Delta_{i-1}\}.$$

It is clear that $\Delta_i \subset \Delta_{i+1}$. The chain of the distributions Δ_i is defined as the *filtration* associated with the distribution $\Delta = \Delta_1$. Each Δ_i is defined to be spanned by the input vector fields plus the vector fields formed by taking up to $i-1$ Lie brackets of the generators, i.e., elements of Δ_1. The Jacobi identity (see Proposition 7.1, page 325) implies that $[\Delta_i, \Delta_j] \subset [\Delta_1, \Delta_{i+j-1}] \subset \Delta_{i+j}$. The proof of this fact is left as an exercise.

A filtration is said to be *regular* in a neighborhood U of q_0 if

$$\text{rank } \Delta_i(q) = \text{rank } \Delta_i(q_0) \qquad \forall q \in U.$$

We say the control system (7.20) is *regular* if the corresponding filtration is regular. If a filtration is regular, then at each step of its construction, Δ_i either gains dimension or $\Delta_{i+1} = \Delta_i$, so that the construction terminates. If rank $\Delta_{i+1} =$ rank Δ_i, then Δ_i is involutive and hence $\Delta_{i+j} = \Delta_i$ for all $j \geq 0$. Clearly, rank $\Delta_i \leq n$ and hence if a filtration is regular, then there exists an integer $\kappa < n$ such that $\Delta_i = \Delta_\kappa$ for all $i \geq \kappa$.

Definition 7.2. Degree of nonholonomy
Consider a regular filtration $\{\Delta_i\}$ associated with a distribution Δ. The smallest integer κ such that the rank of Δ_κ is equal to that of $\Delta_{\kappa+1}$ is called the *degree of nonholonomy* of the distribution.

We know that rank $\Delta_\kappa \leq n$. In general, let the rank of $\Delta_\kappa = n - p$. Then, by Frobenius' theorem there are p functions h_i for $i = 1, \ldots, p$ whose level surfaces are the integral manifolds of Δ_κ. Thus, the state q of the control system must be confined to the a level set of the h_i's. This, then, is the complete answer to the question we posed ourselves at the beginning of this chapter. The maximum number of functions h_i such that

$$\text{span}\{\frac{\partial h_1}{\partial q}, \ldots, \frac{\partial h_p}{\partial q}\} \quad \subset \quad \text{span}\{\omega_1, \ldots, \omega_k\}$$

is given to be the set of functions such that

$$\text{span}\{\frac{\partial h_1}{\partial q}, \ldots, \frac{\partial h_p}{\partial q}\} = (\overline{\Delta})^\perp$$

by Frobenius' theorem. If $p = 0$, that is rank Δ_κ is equal to n, then there are no nontrivial functions h_i and it is possible to steer between arbitrary given initial and final points. This is Chow's theorem, which was discussed in the previous section. Chow's theorem is actually also valid when the filtration Δ_i is not regular, as long as Δ is smooth and constant rank.

We now give a definition which serves to classify the growth of a filtration:

Definition 7.3. Growth vector, relative growth vector
Consider a regular filtration associated with a given distribution Δ and having degree of nonholonomy κ. For such a system, we define the *growth vector* $r \in \mathbb{Z}^\kappa$ as

$$r_i = \text{rank } \Delta_i.$$

We define the *relative growth vector* $\sigma \in \mathbb{Z}^\kappa$ as $\sigma_i = r_i - r_{i-1}$ and $r_0 := 0$.

The growth vector for a regular filtration is a convenient way to represent complexity information about the associated controllability Lie algebra.

4.2 Examples of nonholonomic systems, continued

In this subsection, we illustrate the classification of nonholonomic systems on the examples that were developed in Section 3.

Example 7.7. Hopping robot in flight
Recall from Section 3 that the configuration space for the hopping robot in flight is given by (ψ, l, θ): the leg angle, leg extension, and body angle of the robot. Since we control the leg angle and extension directly, we choose their velocities as our inputs and the control system associated with the hopper is given by

$$\begin{aligned}
\dot{\psi} &= u_1 \\
\dot{l} &= u_2 \\
\dot{\theta} &= -\frac{m(l+d)^2}{I + (l+d)^2} u_1.
\end{aligned}$$

The controllability Lie algebra is given by

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{I+(l+d)^2} \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ \frac{2Im(l+d)}{(I+m(l+d)^2)^2} \end{bmatrix}.$$

In a neighborhood of $l = 0$, span$\{g_1, g_2, g_3\}$ is full rank and hence the hopping robot has degree of nonholonomy 2 with growth vector $(2, 3)$ and relative growth vector $(2, 1)$.

Example 7.8. Planar space robot
From Example 7.3, we have that the angular momentum conservation constraint yields

$$a_{13}(\psi)\dot{\theta}_1 + a_{23}(\psi)\dot{\theta}_2 + a_{33}(\psi)\dot{\theta} = 0,$$

where the vector of configuration variables is $q = (\psi_1, \psi_2, \theta)$. Using the control equations derived in Example 7.3, we have

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ -\frac{ml^2 + mr\cos\psi_1}{I + 2ml^2 + 2mr^2 + 2mrl\cos\psi_1 + 2mrl\cos\psi_2} \end{bmatrix}$$

$$g_2 = \begin{bmatrix} 1 \\ 0 \\ -\frac{ml^2 + mr\cos\psi_2}{I + 2ml^2 + 2mr^2 + 2mrl\cos\psi_1 + 2mrl\cos\psi_2} \end{bmatrix}$$

and the Lie bracket is

$$g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ \frac{2m^2 l^2 r(-l\sin\psi_1 - r\sin(\psi_1 - \psi_2) + l\sin\psi_2)}{(I + 2ml^2 + 2mr^2 + 2mlr\cos\psi_1 + 2mlr\cos\psi_2)^2} \end{bmatrix}.$$

The vector field g_3 is zero when $\psi_1 = \psi_2$ and hence the filtration $\{\Delta_i\}$ is not regular. By computing higher order Lie brackets, however, it is possible to show that $\overline{\Delta}_q = T_q\mathbb{R}^3$ in a neighborhood of $q = 0$ and the system is controllable.

Example 7.9. Disk rolling on a plane
From Example 7.4, the control system which describes a disk rolling on a plane is described by the distribution spanned by

$$g_1 = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The control Lie algebra is constructed by computing the following vector fields:

$$g_3 = [g_1, g_2] = \begin{bmatrix} \rho\sin\theta \\ -\rho\cos\theta \\ 0 \\ 0 \end{bmatrix} \qquad g_4 = [g_2, g_3] = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 0 \end{bmatrix}.$$

For all q, $\operatorname{span}\{g_1, g_2, g_3, g_4\}$ is full rank and hence the rolling disk has degree of nonholonomy 3 with growth vector $(2, 3, 4)$. The relative growth vector for this system is $(2, 1, 1)$.

Example 7.10. Kinematic car
Recall that (x, y, θ, ϕ) denotes the configuration of the car, parameterized by the location of the rear wheel(s), the angle of the car body with respect to the horizontal, and the steering angle with respect to the car body. The constraints for the front and rear wheels to roll without slipping are given by the following equations:

$$\begin{aligned} \sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - l\cos\phi\,\dot{\theta} &= 0 \\ \sin\theta\,\dot{x} - \cos\theta\,\dot{y} &= 0. \end{aligned}$$

Converting this to a control system with the driving and steering velocity as inputs gives the control system of equation (7.14).

To calculate the growth vector, we build the filtration

$$g_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{l\cos^2\phi} \\ 0 \end{bmatrix} \qquad g_4 = [g_1, g_3] = \begin{bmatrix} -\frac{\sin\theta}{l\cos^2\phi} \\ \frac{\cos\theta}{\cos^2\phi} \\ 0 \\ 0 \end{bmatrix}.$$

The vector fields $\{g_1, g_2, g_3, g_4\}$ are linearly independent when $\phi \neq \pm\pi$. Thus the system has degree of nonholonomy 3 with growth vector $r = (2, 3, 4)$ and relative growth vector $\sigma = (2, 1, 1)$. The system is regular away from $\phi = \pm\pi/2$, at which point g_1 is undefined.

Example 7.11. Spherical finger rolling on a plane
Let the inputs be the two components of rolling velocities, i.e., $u_1 = \omega_x$ and $u_2 = \omega_y$. The associated control system is derived in (7.19), which in vector field form reads

$$g_1 = \begin{bmatrix} 0 \\ \sec q_1 \\ -\sin q_5 \\ -\cos q_5 \\ -\tan q_1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} -1 \\ 0 \\ -\cos q_5 \\ \sin q_5 \\ 0 \end{bmatrix}.$$

Constructing the filtration, we have

$$g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ \tan q_1 \sec q_1 \\ -\tan q_1 \sin q_5 \\ -\tan q_1 \cos q_5 \\ -\sec^2 q_1 \end{bmatrix} \qquad g_4 = [g_1, g_3] = \begin{bmatrix} 0 \\ 0 \\ -\cos q_5 \\ \sin q_5 \\ 0 \end{bmatrix}$$

$$g_5 = [g_2, g_3] = \begin{bmatrix} 0 \\ -(1 + \sin^2 q_1) \sec^3 q_1 \\ 2 \sin q_5 \sec^2 q_1 \\ 2 \cos q_5 \sec^2 q_1 \\ 2 \tan q_1 \sec^2 q_1 \end{bmatrix}.$$

In a neighborhood of $q = 0$ (more specifically in a neighborhood not containing $q_1 = \frac{\pi}{2}$) the vector fields $\{g_1, g_2, g_3, g_4, g_5\}$ are linearly independent, thus establishing that the degree of nonholonomy is 3 and that the growth vector is $(2, 3, 5)$. The relative growth vector is $(2, 1, 2)$.

4.3 Philip Hall basis

Let $\mathcal{L}(\}_\infty, \ldots, \}_\Updownarrow)$ be the Lie algebra generated by a set of vector fields $g_1, \ldots, g_m$. One approach to equipping $\mathcal{L}(\}_\infty, \ldots, \}_\Updownarrow)$ with a basis is to list all the generators and all of their Lie products. The problem is that not all Lie products are linearly independent because of skew-symmetry and the Jacobi identity. The Philip Hall basis is a particular way to select a basis which takes into account skew-symmetry and the Jacobi identity.

Given a set of generators $\{g_1, \cdots, g_m\}$, we define the length of a Lie product recursively as

$$\begin{aligned} l(g_i) &= 1 & i &= 1, \cdots, m \\ l([A, B]) &= l(A) + l(B), \end{aligned}$$

where A and B are themselves Lie products. Alternatively, $l(A)$ is the number of generators in the expansion for A. A Lie algebra is *nilpotent* if there exists an integer k such that all Lie products of length greater than k are zero. The integer k is called the order of nilpotency.

A *Philip Hall basis* is an ordered set of Lie products $H = \{B_i\}$ satisfying:

1. $g_i \in H, i = 1, \ldots, m$
2. If $l(B_i) < l(B_j)$ then $B_i < B_j$
3. $[B_i, B_j] \in H$ if and only if
   (a) $B_i, B_j \in H$ and $B_i < B_j$ and

(b) either $B_j = g_k$ for some k or $B_j = [B_l, B_r]$ with $B_l, B_r \in H$ and $B_l \leq B_i$.

The proof that a Philip Hall basis is indeed a basis for the Lie algebra generated by $\{g_1, \ldots, g_m\}$ is beyond the scope of this book and may be found in [38] and [104]. A Philip Hall basis which is nilpotent of order k can be constructed from a set of generators using this definition. The simplest approach is to construct all possible Lie products with length less than k and use the definition to eliminate elements which fail to satisfy one of the properties. In practice, the basis can be built in such a way that only condition 3 need be checked.

Example 7.12. Philip Hall basis of order 3
A basis for the nilpotent Lie algebra of order 3 generated by g_1, g_2, g_3 is

$$
\begin{array}{llll}
g_1 & g_2 & g_3 & \\
[g_1, g_2] & [g_2, g_3] & [g_3, g_1] & \\
[g_1, [g_1, g_2]] & [g_1, [g_1, g_3]] & [g_2, [g_1, g_2]] & [g_2, [g_1, g_3]] \\
[g_2, [g_2, g_3]] & [g_3, [g_1, g_2]] & [g_3, [g_1, g_3]] & [g_3, [g_2, g_3]]
\end{array}
$$

Note that $[g_1, [g_2, g_3]]$ does not appear since

$$[g_1, [g_2, g_3]] + [g_2, [g_3, g_1]] + [g_3, [g_1, g_2]] = 0$$

by the Jacobi identity and the second two terms in the formula are already present.

5 Summary

The following are the key concepts covered in this chapter:

1. *Nonholonomic constraints* are linear velocity constraints of the form

$$\omega_i(q)\dot{q} = 0 \quad i = 1, \ldots, k$$

which cannot be integrated to give constraints on the configuration variables q alone. By choosing $g_j(q), j = 1, \ldots, n-k =: m$ to be a basis for the null space of the linear velocity constraints, we get the associated control system

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m.$$

The problem of *nonholonomic motion planning* consists of finding a trajectory $q(\cdot) : [0, T] \to \mathbb{R}^n$, given $q(0) = q_0$ and $q(T) = q_f$.

2. The *Lie bracket* between two vector fields f and g on $\mathbb{R}^n$ is a new vector field $[f, g]$ defined by

$$[f, g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q).$$

3. A *distribution* Δ is a smooth assignment of a subspace of the tangent space to each point $q \in \mathbb{R}^n$. One important way of generating it is as the span of a number of vector fields:

$$\Delta = \text{span}\{g_1, \ldots, g_m\}.$$

The distribution Δ is said to be *regular* if the dimension of Δ_q does not vary with q. The distribution Δ is said to be *involutive* if it is closed under the Lie bracket, that is if for all $f, g \in \Delta$, we have $[f, g] \in \Delta$.

4. A distribution Δ of dimension k is said to be *integrable* if there exist $n-k$ independent functions whose differentials annihilate the distribution. *Frobenius' theorem* asserts that a regular distribution is integrable if and only if it is involutive. A Pfaffian system or codistribution Ω

$$\Omega = \text{span}\{\omega_1, \ldots, \omega_k\}$$

is *completely nonholonomic* if the involutive closure of the distribution $\Delta = \Omega^\perp$ spans $\mathbb{R}^n$ for all q.

5. Consider the system

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m.$$

The *controllability Lie algebra* is the Lie algebra generated by the vector fields $g_1, \ldots, g_m$. It is the smallest Lie algebra containing $g_1, \ldots, g_m$. *Chow's theorem* asserts that if the controllability Lie algebra is full rank, we can steer this system from any initial to any final point.

6. Given a distribution Δ, the *filtration* associated with Δ is defined by $\Delta_1 = \Delta$ and

$$\Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}],$$

where

$$[\Delta_1, \Delta_{i-1}] = \text{span}\{[g, h] : g \in \Delta_1, h \in \Delta_{i-1}\}.$$

The filtration is said to be *regular* if each of the Δ_i are regular. For a regular filtration, the smallest integer κ at which rank Δ_κ is equal to that of $\Delta_{\kappa+1}, \Delta_{\kappa+2}, \ldots$ is called the *degree of nonholonomy* of the distribution. The *growth vector* $r \in \mathbb{Z}^\kappa$ for a regular filtration is defined as $r_i :=$ rank Δ_i. The *relative growth vector* $\sigma \in \mathbb{Z}^\kappa$ is defined as $\sigma_i = r_i - r_{i-1}$ with $r_0 = 0$.

7. Given $\Delta = \text{span}\{g_1, \ldots, g_m\}$, a *Lie product* is any nested set of Lie brackets of the generators g_i. A Lie algebra generated by Δ is said to be *nilpotent* if there exists an integer k such that all Lie products of length greater than k are zero. A *Philip Hall basis* is an ordered set of Lie products chosen by a set of rules so as to keep track of the restrictions imposed by the properties of the Lie bracket, namely skew-symmetry and the Jacobi identity.

6 Bibliography

The topic of holonomy and nonholonomy of Pfaffian constraints has captured the attention of many of the earliest writers on classical mechanics. A nice description of the mechanics point of view is given in [81]. Chapter 1 of Rosenberg [99] makes mention of the different kinds of constraints: holonomic, rheonomic, scleronomic. The examples in this chapter are drawn from our interest in fingers rolling on the surface of an object [60, 76], mobile robots and parking problems [78, 112], and space robots [119, 32]. A recent collection of papers on nonholonomic motion planning is [61].

Work on nonlinear controllability has a long history as well, with recognition of the connections between Chow's theorem and controllability in Hermann and Krener [40]. Good textbook presentations of the work on nonlinear control are available in [43], and [83]. The theory of nonholonomic distributions presented here was originally developed by Vershik and Gershkovic [117]. The notation we follow is theirs and is presented in [78].

A somewhat less obvious application of the methods of this chapter is in the analysis of control algorithms for redundant manipulators. In this application, one looks for an algorithm such that closed trajectories of the end-effector generate closed paths in the joint space of the manipulator. This is closely related to the integrability of a set of constraints. A good description of this is in the work of Shamir and Yomdin [105], Baillieul and Martin [5], Chiacchio and Siciliano [17], and De Luca and Oriolo [23].

7 Exercises

1. Show that the controllability rank condition is also a necessary condition for local controllability under the usual smoothness and regularity assumptions.

2. Show that the differential constraint in $\mathbb{R}^5$ given by

$$\begin{bmatrix} 0 & 1 & \rho \sin q_5 & \rho \cos q_3 & \cos q_5 \end{bmatrix} \dot{q} = 0$$

is nonholonomic.

3. Use the definition of the Lie bracket to prove the properties listed in Proposition 7.1.

4. Consider the system Σ,

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m.$$

Let $u : [0, T] \to \mathbb{R}^m$ be input which steers Σ from q_0 to q_f in T units of time.

   (a) Show that the input $\tilde{u} : [0, 1] \to \mathbb{R}^m$ defined by

   $$\tilde{u}(t) = u(t/T)$$

   steers σ from q_0 to q_f in 1 unit of time.

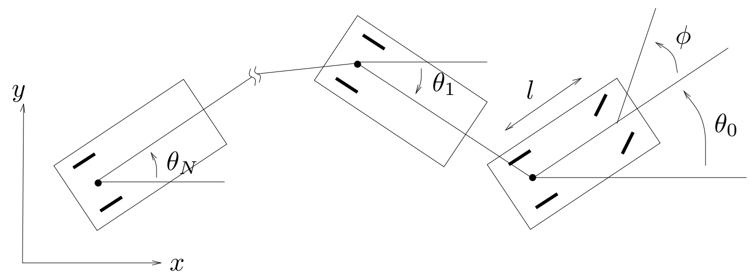
   (b) Show that the input $\bar{u} : [0, 1] \longrightarrow \mathbb{R}^m$ defined by

   $$\bar{u}(t) = -\tilde{u}(1 - t)$$

   steers σ from q_f to q_0 in 1 unit of time.

5. *Spheres rolling on spheres*
Derive the control equation for a unit sphere in rolling contact with another sphere of radius ρ with the same inputs as in Example 7.6. Show that the system is controllable if and only if $\rho \neq 1$.

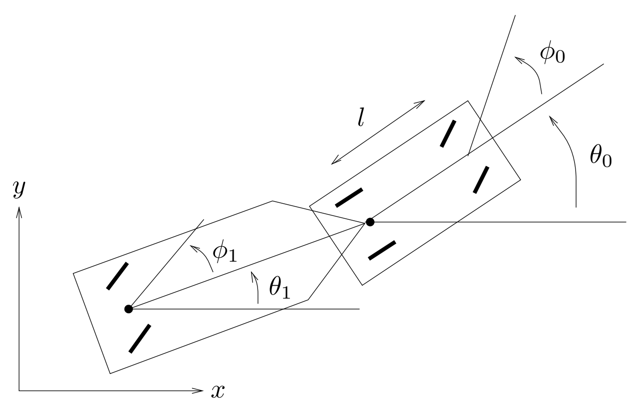
6. *Car with N trailers*
The figure below shows a car with N trailers attached. We attach the hitch of each trailer to the center of the rear axle of the previous trailer. The wheels of the individual trailers are aligned with the body of the trailer. The constraints are again based on allowing the wheels only to roll and spin, but not slip. The dimension of the state space is $N + 4$ with 2 controls.



Parameterize the configuration by the states of the automobile plus the angle of each of the trailers with respect to the horizontal. Show that the control equation for the system has the form

$$
\begin{aligned}
\dot{x} &= \cos\theta_0\, u_1 \\
\dot{y} &= \sin\theta_0\, u_1 \\
\dot{\phi} &= u_2 \\
\dot{\theta}_0 &= \frac{1}{l}\tan\phi\, u_1 \\
\dot{\theta}_i &= \frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_{j-1}-\theta_j)\right)\sin(\theta_{i-1}-\theta_i)u_1.
\end{aligned}
$$

7. *Firetruck*
A firetruck can be modeled as a car with one trailer, with the difference that the trailer is steerable, as shown in the figure below.



The constraints on the system are similar to that of the car in Section 3, with the difference that back wheels are steerable. Derive the nonlinear control system for a firetruck corresponding to the control inputs for driving the cab and steering both the cab and the trailer, and show that it represents a controllable system.

8. Prove that a 1-dimensional distribution $\Delta_q = \text{span}\{f(q)\}$ is involutive. More specifically, show that for any two smooth functions α and β
$$[\alpha f, \beta f] \in \Delta.$$

9. Prove that the two definitions of Lie bracket given in this chapter, namely,
$$[f, g] = \frac{\partial g}{\partial q} f - \frac{\partial f}{\partial q} g,$$
and
$$L_{[f,g]}\alpha = L_f(L_g\alpha) - L_g(L_f\alpha) \qquad \forall \alpha : \mathbb{R}^n \to \mathbb{R},$$
are equivalent.

10. Use induction and Jacobi's identity to prove that
$$[\Delta_i, \Delta_j] \subset [\Delta_1, \Delta_{i+j-1}] \subset \Delta_{i+j},$$
where $\Delta = \Delta_1 \subset \Delta_2 \subset \cdots$ is a filtration associated with a distribution $\triangle$.

11. Let $\Delta_i, i = 1, \ldots, \kappa$ be a regular filtration associated with a distribution. Show that if $\text{rank}(\Delta_{i+1}) = \text{rank}(\Delta_i)$ then Δ_i is involutive. (Hint: use Exercise 10).

12. *Satellite with 2 rotors*
Figure 7.9 shows a model of a satellite body with two symmetrically attached rotors, where the rotors' axes of rotation intersect at a point. The constraint on the system is conservation of angular momentum.

    (a) Assuming that the initial angular momentum of the system is zero, show that the (body) angular velocity, ω_1, of the satellite body is related to the rotor velocities (u_1, u_2) by
    $$\omega_1 = b_1 u_1 + b_2 u_2 \tag{7.21}$$
    where $b_1, b_2 \in \mathbb{R}^3$ are constant vectors.
    Equation (7.21) gives rise to a differential equation in the rotation group $SO(3)$ for the satellite body
    $$\dot{R}(t) = R(t)(\widehat{b}_1 u_1 + \widehat{b}_2 u_2). \tag{7.22}$$

    (b) Obtain a local coordinate description of (7.22) using the Euler parameters of $SO(3)$ (from Chapter 2) and show that the resulting system is controllable.

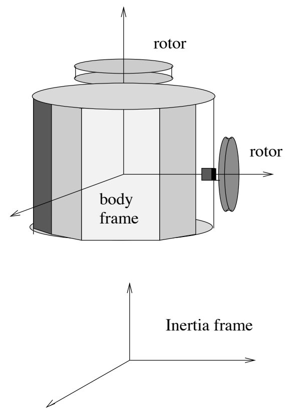
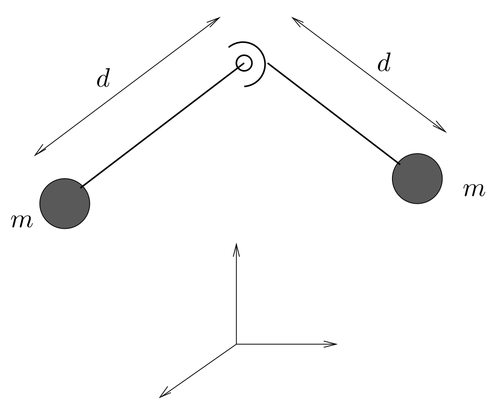


Figure 7.9: A model of a satellite body with two rotors. The satellite can be repositioned by controlling the rotor velocities. (Figure courtesy of Greg Walsh)

13. The figure below shows a simplified model of a *falling cat*. It consists of two pendulums coupled by a spherical joint. The configuration space of the system is $Q = \mathbb{S}^2 \times \mathbb{S}^2$, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$.



    (a) Derive the Pfaffian constraints arising from conservation of angular momentum and dualize the results to obtain the control system for nonholonomic motion planning.
    (b) Is the system in part (a) controllable?

14. Write a computer program to write a Philip Hall basis of given order for a set of m generators $g_1, \dots, g_m$. Use your program to generate a Philip Hall basis of order 5 for a system with 2 generators.

15. Consider the system of Exercise 6. Write a computer program (or use *Mathematica* or any other symbolic manipulation software packages) to compute the filtration associated with the system. Show that the system is controllable, with degree of nonholonomy $N+2$ and relative growth vector $(2, 1, \ldots, 1)$.

16. In this chapter, we restricted ourselves to constraints of the form

$$\omega_i(q)\dot{q} = 0 \qquad i = 1, \ldots, k.$$

Consider what would happen if the constraints were of the form

$$\omega_i(q)\dot{q} = c_i \qquad i = 1, \ldots, k$$

for constants c_i. Dualize these constraints to get an associated control system of the form

$$\dot{q} = f(q) + \sum_{i=1}^{n-k} g_i(q) u_i.$$

What is the formula for $f(q)$? Apply this method to the space robot with nonzero angular momentum. What difficulties would one encounter in path planning for these examples? These systems are called systems *with drift*.

17. (Hard) Show that for a regular system the growth vector is bounded above by

$$\overline{\sigma}_i = \frac{1}{i}\left((\overline{\sigma}_1)^i - \sum_{j|i,\, j<i} j\overline{\sigma}_j \right) \qquad i > 1,$$

where $\overline{\sigma}_i$ is the maximum relative growth at the ith stage and $j|i$ means all integers j such that j divides i. If $\sigma_i = \overline{\sigma}_i$ for all i, we say that the system has *maximum growth*.