Optimization-Based Control

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Chapter 5
Kalman Filtering

In this chapter we derive the Kalman filter in continuous time (also referred to as the Kalman-Bucy filter).

Prerequisites. Readers should have basic familiarity with continuous-time stochastic systems at the level presented in Chapter 4.

5.1 Linear Quadratic Estimators

Consider a stochastic system

$$
\begin{aligned}
\dot{X} &= AX + Bu + FV \\
Y &= CX + W,
\end{aligned}
$$

where X represents that state, u is the (deterministic) input, V represents disturbances that affect the dynamics of the system and W represents measurement noise. Assume that the disturbance V and noise W are zero-mean, Gaussian white noise (but not necessarily stationary):

$$
\begin{aligned}
p(v) &= \frac{1}{\sqrt[n]{2\pi}\sqrt{\det R_V}} e^{-\frac{1}{2} v^T R_V^{-1} v} \qquad && E\{V(s)V^T(t)\} = R_V(t)\delta(t-s) \\
p(w) &= \frac{1}{\sqrt[n]{2\pi}\sqrt{\det R_W}} e^{-\frac{1}{2} v^T R_W^{-1} v} \qquad && E\{W(s)W^T(t)\} = R_W(t)\delta(t-s)
\end{aligned}
$$

We also assume that the cross correlation between V and W is zero, so that the distrubances are not correlated with the noise. Note that we use multivariable Gaussians here, with (incremental) covariance matrix $R_V \in \mathbb{R}^{m\times m}$ and $R_W \in \mathbb{R}^{p\times p}$. In the scalar case, $R_V = \sigma_V^2$ and $R_W = \sigma_W^2$.

We formulate the optimal estimation problem as finding the estimate $\hat{X}(t)$ that minimizes the mean square error $E\{(X(t)-\hat{X}(t))(X(t)-\hat{X}(t))^T\}$ given $\{Y(\tau) : 0 \leq \tau \leq t\}$. It can be shown that this is equivalent to finding the expected value of X subject to the "constraint" given by all of the previous measurements, so that $\hat{X}(t) = E\{X(t)|Y(\tau), \tau \leq t\}$. This was the way that Kalman originally formulated the problem and it can be viewed as solving a *least squares* problem: given all previous $Y(t)$, find the estimate $\hat{X}$ that satisfies the dynamics and minimizes the square error with the measured data. We omit the proof since we will work directly with the error formulation.

Theorem 5.1 (Kalman-Bucy, 1961)**.** *The optimal estimator has the form of a linear observer*

$$\dot{\hat{X}} = A\hat{X} + BU + L(Y - C\hat{X})$$

where $L(t) = P(t)C^T R_W^{-1}$ *and* $P(t) = E\{(X(t) - \hat{X}(t))(X(t) - \hat{X}(t))^T\}$ *and satisfies*

$$\begin{aligned} \dot{P} &= AP + PA^T - PC^T R_W^{-1}(t)CP + FR_V(t)F^T \\ P(0) &= E\{X(0)X^T(0)\} \end{aligned}$$

Sketch of proof. The error dynamics are given by

$$\dot{E} = (A - LC)E + \xi, \qquad \xi = FV - LW, \quad R_\xi = FR_V F^T + LR_W L^T$$

The covariance matrix $P_E = P$ for this process satisfies

$$\begin{aligned} \dot{P} &= (A - LC)P + P(A - LC)^T + FR_V F^T + LR_W L^T \\ &= AP + PA^T + FR_V F^T - LCP - PC^T L^T + LR_W L^T \\ &= AP + PA^T + FR_V F^T + (LR_W - PC^T)R_W^{-1}(LR_W + PC^T)^T \\ &\qquad - PC^T R_W CP, \end{aligned}$$

where the last line follows by completing the square. We need to find L such that $P(t)$ is as small as possible, which can be done by choosing L so that $\dot{P}$ decreases by the maximimum amount possible at each instant in time. This is accomplished by setting

$$LR_W = PC^T \qquad \Longrightarrow \qquad L = PC^T R_W^{-1}.$$

□

Note that the Kalman filter has the form of a *recursive* filter: given $P(t) = E\{E(t)E^T(t)\}$ at time t, can compute how the estimate and covariance *change*. Thus we don't need to keep track of old values of the output. Furthermore, the Kalman filter gives the estimate $\hat{X}(t)$ *and* the covariance $P_E(t)$, so you can see how well the error is converging.

If the noise is stationary (R_V, R_W constant) and *if* $\dot{P}$ is stable, then the observer gain converges to a constant and satisfies the *algebraic Riccati equation*:

$$L = PC^T R_W^{-1} \qquad AP + PA^T - PC^T R_W^{-1} CP + FR_V F^T.$$

This is the most commonly used form of the controller since it gives an explicit formula for the estimator gains that minimize the error covariance. The gain matrix for this case can solved use the `lqe` command in MATLAB.

Another property of the Kalman filter is that it extracts the maximum possible information about output data. To see this, consider the *residual random prossess*

$$R = Y - C\hat{X}$$

(this process is also called the *innovations process*). It can be shown for the Kalman filter that the correlation matrix of R is given by

$$R_R(t, s) = W(t)\delta(t - s).$$

This implies that the residuals are a white noise process and so the output error has *no* remaining dynamic information content.

5.2 Extensions of the Kalman Filter

Correlated disturbances and noise

The derivation of the Kalman filter assumes that the disturbances and noise are independent and white. Removing the assumption of independence is straightforward and simply results in a cross term ($E\{V(t)W(s)\} = R_{VW}\delta(s - t)$) being carried through all calculations.

To remove the assumption of white noise sources, we construct a filter that takes white noise as an input and produces a noise source with the appropriate correlation function (or equivalently, spectral power density function). The intuition behind this approach is that we must have an internal model of the noise and/or disturbances in order to capture the correlation between different times.

Extended Kalman filters

Consider a *nonlinear* system

$$\begin{aligned} \dot{X} &= f(X, U, V) && X \in \mathbb{R}^n, u \in \mathbb{R}^m \\ Y &= CX + W && V, W \text{ Gaussian white noise processes with} \\ & && \quad \text{covariance matrices } R_V \text{ and } R_W. \end{aligned}$$

Nonlinear observer:

$$\dot{\hat{X}} = f(\hat{X}, U, 0) + L(Y - C\hat{X})$$

Error dynamics: $E = X - \hat{X}$

$$\begin{aligned} \dot{E} &= f(X, U, V) - f(\hat{X}, U, 0) - LC(X - \hat{X}) \\ &= F(E, \hat{X}, U, V) - LCe \qquad F(E, \hat{X}, U, V) = f(E + \hat{X}, U, V) - f(\hat{X}, U, 0) \end{aligned}$$

Now linearize around *current* estimate $\hat{X}$

$$\begin{aligned} \hat{E} &= \frac{\partial F}{\partial E}E + \underbrace{F(0, \hat{X}, U, 0)}_{=0} + \underbrace{\frac{\partial F}{\partial V}V}_{\text{noise}} - \underbrace{LCe}_{\text{observer gain}} + \text{h.o.t} \\ &= \tilde{A}E + \tilde{F}V - LCE \end{aligned}$$

where

$$\left.\begin{aligned} \tilde{A} &= \left.\frac{\partial F}{\partial e}\right|_{(0,\hat{X},U,0)} &= \left.\frac{\partial f}{\partial X}\right|_{(\hat{X},U,0)} \\ \tilde{F} &= \left.\frac{\partial F}{\partial V}\right|_{(0,\hat{X},U,0)} &= \left.\frac{\partial f}{\partial V}\right|_{(\hat{X},U,0)} \end{aligned}\right\} \text{Depend on current estimate } \hat{X}$$

Idea: design observer for the linearized system around *current* estimate

$$\dot{\hat{X}} = f(\hat{X}, U, 0) + L(Y - C\hat{X}) \qquad L = PC^T R_W^{-1}$$
$$\dot{P} = (\tilde{A} - LC)P + P(\tilde{A} - LC)^T + \tilde{F}R_V\tilde{F}^T + LR_W L^T \qquad P(t_0) = E\{X(t_0)X^T(t_0)\}$$

This is called the (Schmidt) *extended Kalman filter* (EKF)

Remarks:

1. Can't prove very much about EKF due to nonlinear terms
2. In applications, works *very* well. One of the most used forms of the Kalman filter
3. Unscented Kalman filters
4. Information form of the Kalman filter (next week?)

Application: parameter ID

Consider a linear system with unknown parameters ξ

$$\begin{aligned} \dot{X} &= A(\xi)X + B(\xi)U + FV \qquad \xi \in \mathbb{R}^p \\ Y &= C(\xi)X + W \end{aligned}$$

Parameter ID problem: given $U(t)$ and $Y(t)$, estimate the value of the parameters ξ.

One approach: treat ξ as unkown *state*

$$\left.\begin{aligned} \dot{X} &= A(\xi)X + B(\xi)U + FV \\ \dot{\xi} &= 0 \end{aligned}\right\} \quad \rightarrow \quad \begin{aligned} \frac{d}{dt}\begin{bmatrix} X \\ \xi \end{bmatrix} &= \overbrace{\begin{bmatrix} A(\xi) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X \\ \xi \end{bmatrix} + \begin{bmatrix} B(\xi) \\ 0 \end{bmatrix} U}^{f(\left[\begin{smallmatrix} X \\ \xi \end{smallmatrix}\right],U,V)} + \begin{bmatrix} F \\ 0 \end{bmatrix} V \\ Y &= \underbrace{C(\xi)X + W}_{h(\left[\begin{smallmatrix} X \\ \xi \end{smallmatrix}\right],W)} \end{aligned}$$

Now use EKF to *estimate* X and $\xi \implies$ determine unknown parameters $\xi \in \mathbb{R}^p$.

Remark: need various observability conditions on augmented system in order for this to work.

Unscented Kalman filter

5.3 LQG Control

Return to the original “H_2” control problem

Figure	$\dot{X} = AX + BU + FV$ $Y = CX + W$	V, W Gaussian white noise with covariance R_V, R_W

Stochastic control problem: find $C(s)$ to minimize

$$J = E\left\{ \int_0^\infty \left[(Y - r)^T R_V (Y - r)^T + U^T R_W U \right] \, dt \right\}$$

Assume for simplicity that the reference input $r = 0$ (otherwise, translate the state accordingly).

Theorem 5.2 (Separation principle)**.** *The optimal controller has the form*

$$\begin{aligned} \dot{\hat{X}} &= A\hat{X} + BU + L(Y - C\hat{X}) \\ U &= K(\hat{X} - X_d) \end{aligned}$$

where L is the optimal observer gain ignoring the controller and K is the optimal controller gain ignoring the noise.

This is called the *separation principle* (for H_2 control).

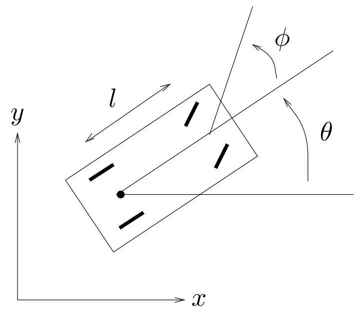
5.4 Application to the Caltech Ducted Fan

To illustrate the use of the Kalman filter, we consider the problem of estimating the state for the Caltech ducted fan, described already in Section **??**.

5.5 Further Reading

Exercises

5.1 Consider the problem of estimating the position of an autonomous mobile vehicle using a GPS receiver and an IMU (inertial measurement unit). The dynamics of the vehicle are given by



$$\dot{x} = \cos\theta\, v$$
$$\dot{y} = \sin\theta\, v$$
$$\dot{\theta} = \frac{1}{\ell}\tan\phi\, v,$$

We assume that the vehicle is disturbance free, but that we have noisy measurements from the GPS receiver and IMU and an initial condition error.

In this problem we will utilize the full form of the Kalman filter (including the $\dot{P}$ equation).

(a) Suppose first that we only have the GPS measurements for the xy position of the vehicle. These measurements give the position of the vehicle with approximately 1 meter accuracy. Model the GPS error as Gaussian white noise with $\sigma = 1.2$ meter in each direction and design an optimal estimator for the system. Plot the estimated states and the covariances for each state starting with an initial condition of 5 degree heading error at 10 meters/sec forward speed (i.e., choose $x(0) = (0, 0, 5\pi/180)$ and $\hat{x} = (0, 0, 0)$).

(b) An IMU can be used to measure angular rates and linear acceleration. Assume that we use a Northrop Grumman LN200 to measure the angular rate $\dot{\theta}$. Use the datasheet on the course web page to determine a model for the noise process and design a Kalman filter that fuses the GPS and IMU to determine the position of the vehicle. Plot the estimated states and the covariances for each state starting with an initial condition of 5 degree heading error at 10 meters/sec forward speed.

Note: be careful with units on this problem!