Networked Sensing, Estimation and Control Systems

Vijay Gupta
University of Notre Dame

Richard M. Murray
California Institute of Technology

Ling Shi
Hong Kong University of Science and Technology

Bruno Sinopoli
Carnegie Mellon University

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Chapter 4
Markovian Jump Linear Systems

In this chapter, we present a short overview of Markovian jump linear systems. A more thorough and complete treatment is given in books such as [**?**]. As in other chapters, our focus will be on the Linear Quadratic Gaussian (LQG) control of such systems. As we shall see, even though such systems are non-linear, they can be analyzed using tools that are similar to those used in linear system analysis.

4.1 Introduction to Markovian Jump Linear Systems

A useful category of system models are those in which the system operates in multiple modes. Although each of the individual modes in linear, the switching between these modes introduces non-linearity into the overall system description. A general theory of such systems is developed in the hybrid systems community. However, much tighter results can be developed if a further assumptions holds, that the *mode switches are governed by a stochastic process that is statistically independent from the state values*. In the case when the stochastic process can be described by a Markov chain, the system is called a Markovian jump linear system. Although the individual modes of such systems may be continuous or discrete, we will concentrate on the latter case here.

More formally, consider a discrete time discrete state Markov process with state $r_k \in \{1, 2, \cdots, m\}$ at time k. Denote the transition probability $P(r_{k+1} = j | r_k = i)$ by q_{ij}, and the resultant transition probability matrix by Q. We will assume that the Markov chain is irreducible and recurrent. Also denote

$$P(r_k = j) = \pi_{j,k},$$

with $\pi_{j,0}$ as given. The evolution of a Markovian jump linear system (MJLS), denoted by $\mathcal{S}_1$ for future reference, can be described by the following equations

$$\begin{aligned} x_{k+1} &= A_{r_k} x_k + B_{r_k} u_k + F_{r_k} w_k \\ y_k &= C_{r_k} x_k + G_{r_k} v_k, \end{aligned} \tag{4.1}$$

where w_k is zero mean white Gaussian noise with covariance R_w, v_k is zero mean white Gaussian noise with covariance R_v and the notation X_{r_k} implies that the matrix $X \in \{X_1, X_2, \cdots, X_m\}$ with the matrix X_i being chosen when $r_k = i$. The initial state x_0 is assumed to be a zero mean Gaussian random variable with variance Π_0. For simplicity, we will consider $F_{r_k} = G_{r_k} \equiv I$ for all values of r_k in the sequel. We also assume that x_0, $\{w_k\}$, $\{v_k\}$ and $\{r_k\}$ are mutually independent. The particular case when $q_{ij} = q_j, \forall i, j$ (i.e., the random process governing the switching of the modes is a Bernoulli process) is sometimes referred to as a Bernoulli jump linear system.

Such systems have been studied for a long time in the fault isolation community, and have received new impetus with the advent of networked control systems. We now consider some examples of applicability of Markovian jump linear systems.

Example 4.1
Consider the following example of a failure prone production system, which is the discrete time equivalent of the model presented in [AK86]. Consider a manufacturing system producing a single commodity. There is a constant demand rate d for the commodity, and the goal of the manufacturing system is to try to meet this demand. The manufacturing system is, however, subject to occasional breakdowns and so at any time k, the system can be in one of two states: a functional ($r_k = 1$) state and a breakdown ($r_k = 2$) state. The transitions between these two states are usually modeled to occur as a Markov chain with given mean time between failures and mean repair time. When the manufacturing system is in the breakdown state it cannot produce the commodity, while if it is in the functional state it can produce at any rate u up to a maximum production rate $r > d > 0$. Let x_k be the inventory of the commodity at time k, i.e., x_k = (total production up to time k) - (total demand up to time k). Then the system is a Markovian jump linear system that evolves as

$$x_{k+1} = \begin{cases} x_k + u_k - d & r_k = 1 \\ x_k - d & r_k = 2, \end{cases}$$

where u_k is the controlled production rate. A negative x_k denotes backlog, and u_k satisfies a saturation constraint. ∇

Example 4.2
Consider a linear process evolving as

$$x_{k+1} = Ax_k + Bu_k + w_k,$$

and being observed by a sensor of the form

$$y_k = Cx_k + v_k.$$

The measurements from the sensor are transmitted to an estimator across an analog erasure link. At any time k, the estimator receives measurement y_k with probability $1-p$, and with a probability p no measurement is received. As discussed in another chapter, this is a common model for a dynamic process being estimated across an analog erasure channel. This is a Bernoulli jump linear system with two modes $r_k \in \{0, 1\}$. For both the modes, the system matrices $A_0 = A_1 = A$ and $B_0 = B_1 = B$. Mode 0 corresponds to no measurement being received and for this case $C_0 = 0$. Mode 1 corresponds to measurement being received, and for this case $C_1 = C$. ∇

4.2 Stability of Markovian jump linear systems

In this section, we discuss the stability of autonomous Markovian jump linear systems. We will see that the necessary and sufficient condition for stability can be presented an algebraic condition in terms of the spectral radius of a suitable matrix.

Since an Markovian jump linear systems is a stochastically varying system, numerous notions of stability may be defined. We will primarily be interested in mean square stability. Thus, define the state covariance $C_k = \mathbb{E}[x_k x_k^T]$, where the expectation is taken with respect to the initial state, process and measurement noise, and the discrete modes till time k. The system is stable if the steady state covariance is bounded, i.e., if $\lim_{k\to\infty} C_k < C^\star$, where $C^\star$ is a constant matrix, and the inequality is understood in the positive definite sense.

The stability condition for Markovian jump linear systems is given by the following result.

Theorem 4.1. *Consider the system $\mathcal{S}_1$ with the control input $u_k = 0$. The system is stable if and only if the condition*

$$\rho\left((Q^T \otimes I)\, diag(A_i \otimes A_i)\right) < 1$$

holds, where $\rho(M)$ is the spectral radius of matrix M, Q is the transition probability matrix of the Markov chain governing the mode switches of the system, $\otimes$ denotes the Kronecker product, I is the identity matrix of suitable dimensions, and $diag(A_i \otimes A_i)$ denotes a block diagonal matrix formed by using the matrices $A_i \otimes A_i$ for various mode values i.

Proof. Consider the term

$$C_k^i = \mathbb{E}[x_k x_k^T | r_k = i]\pi_{i,k},$$

so that the covariance is given by

$$C_k = \sum_{i=1}^{m} C_k^i.$$

We will study the evolution of terms C_k^i. Conditioning on the state value at time $k-1$ yields

$$\begin{aligned}
C_k^i &= \sum_{j=1}^{m} P(r_{k-1} = j | r_k = i)\pi_{i,k}]\mathbb{E}[x_k x_k^T | r_k = i, r_{k-1} = j] \\
&= \sum_{j=1}^{m} P(r_k = i | r_{k-1} = j)\pi_{j,k-1}\mathbb{E}[x_k x_k^T | r_k = i, r_{k-1} = j] \\
&= \sum_{j=1}^{m} q_{ji}\pi_{j,k-1}\mathbb{E}[x_k x_k^T | r_{k-1} = j],
\end{aligned}$$

where in the second line we have used the Bayes law, and in the third line we have used the fact that given the Markov mode at time $k-1$, x_k is conditionally independent of the Markov mode at time k. Now given the Markov mode at time $k-1$, the covariance of the state at time k can be related to the covariance at time $k-1$. Thus, we obtain

$$\begin{aligned}
C_k^i &= \sum_{j=1}^{m} q_{ji}\pi_{j,k-1}\Big(A_j \mathbb{E}[x_{k-1}x_{k-1}^T | r_{k-1} = j]A_j^T + R_w\Big) \\
&= \sum_{j=1}^{m} q_{ji} A_j C_{k-1}^j A_j^T + \sum_{j=1}^{m} q_{ji}\pi_{j,k-1} R_w.
\end{aligned}$$

We can vectorize this equation and use the identity

$$\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$$

to obtain

$$\mathrm{vec}(C_k^i) = \sum_{j=1}^{m} q_{ji}(A_j \otimes A_j)\mathrm{vec}(C_{k-1}^j) + \pi_{i,k}\mathrm{vec}(R_w). \tag{4.2}$$

For values of $i = 1, \cdots, m$, these coupled linear equations define the stability of C_k. We can stack the vectors $\mathrm{vec}(C_k^i)$ for various values of i, and obtain that the dynamical system recursion is governed by the matrix $\left((Q^T \otimes I)\mathrm{diag}(A_i \otimes A_i)\right)$. Thus, we need to consider the spectral radius of this matrix. □

For a Bernoulli jump linear system, the condition reduces to the following simple form.

Theorem 4.2. *Consider the system $\mathcal{S}_1$ with the control input $u_k = 0$ and the additional assumption that the Markov transition probability matrix is such that for all states i and j, $q_{ij} = q_i$. The system is stable if and only if the condition*

$$\rho\left(\mathbb{E}[A_i \otimes A_i]\right) < 1$$

holds, where the expectation is taken over the probabilities $\{q_i\}$.

Proof. In this case, we have $q_{ij} = q_j, \forall i$. Moreover, r_k and x_k are independent, so that $C_k^i = C_k\pi_{i,k} = C_k q_{i,k}$. Thus, (4.2) yields

$$\begin{aligned}\mathrm{vec}(C_k) &= \sum_{j=1}^{m}(A_j \otimes A_j)\mathrm{vec}(C_{k-1})q_{j,k} + \mathrm{vec}(R_w)\\ &= \mathbb{E}[A_i \otimes A_i]\mathrm{vec}(C_{k-1}) + \mathrm{vec}(R_w),\end{aligned}$$

which yields the desired stability condition. □

4.3 LQG control

We will develop the LQG controller of Markovian jump linear systems in three steps. We will begin by considering the optimal linear quadratic regulator. We will then consider the optimal estimation problem for Markovian jump linear systems in the minimum mean squared error (MMSE). Finally, we will present a separation principle that will allow us to solve the LQG problem as a combination of the above filters.

Optimal Linear Quadratic Regulator

The Linear Quadratic Regulator (LQR) problem for the system $\mathcal{S}_1$ is posed by assuming that the noises w_k and v_k are not present. Moreover, the matrix $C_{r_k} \equiv I$ for all choices of the mode r_k. The problem aims at designing the control input u_k to minimize the finite horizon cost function

$$J_{LQR}(K) = \sum_{k=1}^{K}\left(\mathbb{E}_{\{r_j\}_{j=k+1}^{K}}\left[x_k^T Q x_k + u_k^T R u_k\right]\right) + x_{K+1}^T P_{K+1} x_{K+1},$$

where the expectation at time k is taken with respect to the future values of the Markov state realization, and P_{K+1}, Q and R are all assumed to be positive definite. The controller at time k has access to control inputs $\{u_j\}_{j=0}^{k-1}$, state values $\{x_j\}_{j=0}^{k}$ and the Markov state values $\{r_j\}_{j=0}^{k}$. Finally, the system is said to be stabilizable if the infinite horizon cost function $J_\infty \stackrel{def}{=} \lim_{K\to\infty} \frac{J_{LQR}}{K}$ is finite.

The solution to this problem can readily be obtained through dynamic programming arguments. The optimal control is given by the following result.

Theorem 4.3. *Consider the LQR problem posed above for the system $\mathcal{S}_1$.*

1. *At time k, if $r_k = i$, then the optimal control input is given by*

$$u_k = -\left(R + B_i^T P_{i,k+1} B_i\right)^{-1} B_i^T P_{i,k+1} A_i x_k,$$

where for $j = 1, 2, \cdots, m$,

$$P_{j,k} = \sum_{t=1}^{m} q_{tj} \Big(Q + A_t^T P_{t,k+1} A_t - A_t^T P_{t,k+1} B_t \left(R + B_t^T P_{t,k+1} B_t\right)^{-1} B_t^T P_{t,k+1} A_t \Big), \quad (4.3)$$

and $P_{j,K+1} = P_{K+1}, \forall j = 1, 2, \cdots, m$.

2. *Assume that the Markov states reach a stationary probability distribution. A necessary and sufficient condition for stabilizability of the system is that there exist m positive definite matrices X_1, X_2, $\cdots$, X_m and m^2 matrices $K_{1,1}$, $K_{1,2}$, $\cdots$, $K_{1,m}$, $K_{2,1}$, $\cdots$, $K_{m,m}$ such that for all $j = 1, 2, \cdots, m$,*

$$X_j > \sum_{i=1}^{m} q_{ij} \left((A_i^T + K_{i,j} B_i^T) X_i (A_i^T + K_{i,j} B_i^T)^T + Q + K_{ij} R K_{ij}^T \right).$$

3. *A necessary condition for stabilizability is that*

$$q_{i,i}\rho(A_i)^2 < 1, \qquad \forall i = 1, 2, \cdots, m,$$

where $\rho(A_i)$ is the spectral radius of the matrix A_i that governs the dynamics of unstabilizable modes of the process in the i-th mode.

Proof. The proof follows by standard dynamic programming arguments. We begin by rewriting the cost function J_{LQR} to identify terms in the cost that depend on x_k and u_k:

$$J_{LQR}(K) = \sum_{k=1}^{K-1} \Big(\mathbb{E}_{\{r_j\}_{j=k+1}^{K}} \left[x_k^T Q x_k + u_k^T R u_k \right] \Big) + T_K$$
$$T_K = \mathbb{E}_{r_K} \left[x^T[K] Q x_K + u_K^T R u_K \right] + x_{K+1}^T P_{K+1} x_{K+1}.$$

We rewrite T_K by explicitly conditioning it on the value of r_K.

$$T_K = \sum_{i=1}^{m} \pi_{i,K} \left(x^T[K] Q x_K + u_K^T R u_K + x_{K+1}^T P_{i,K+1} x_{K+1} | r_K = i \right),$$

where $P_{i,K+1} = P_{K+1}, \forall i$. At the time of calculation of u_K, the mode r_K is known. To choose the control input for any value of the mode, we complete the square of each of the terms in the summation. For the i-th term we obtain

$$
\begin{aligned}
&\left(x^T[K]Qx_K + u_K^T Ru_K + x_{K+1}^T P_{i,K+1} x_{K+1} | r_K = i\right) \\
&\quad = x^T[K]Qx_K + u_K^T Ru_K + (A_i x_K + B_i u_K)^T P_{i,K+1}(A_i x_K + B_i u_K) \\
&\quad = x^T[K]M_{i,K}x_K + (u_K + S_{i,K}^{-1} B_i^T P_{i,K+1} A_i x_K)^T S_{i,K}(U_K + S_{i,K}^{-1} B_i^T P_{i,K+1} A_i x_K),
\end{aligned}
$$

where

$$
\begin{aligned}
S_{i,K} &= R + B_i^T P_{i,K+1} B_i \\
M_{i,K} &= Q + A_i^T P_{i,K+1} A_i - A_i^T P_{i,K+1} B_i S_{i,K}^{-1} B_i^T P_{i,K+1} A_i.
\end{aligned}
$$

Thus, the optimal choice of u_K for the case $r_K = i$ is

$$u_K = -S_{i,K}^{-1} B_i^T P_{i,K+1} x_K.$$

With the optimal choice of u_K for all values of $i = 1, \cdots, m$, the term T_K reduces to

$$
\begin{aligned}
T_K &= \sum_{i=1}^{m} \pi_{i,K} \left(x_K^T M_{i,K} x_K | r_K = i\right) \\
&= \sum_{i=1}^{m} \pi_{i,K} \sum_{j=1}^{m} q_{ji} \left(x_K^T M_{i,K} x_K | r_K = i, r_{K-1} = j\right) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{m} \pi_{i,K} q_{ji} \left(x_K^T M_{i,K} x_K | r_{K-1} = j\right) \\
&= \sum_{j=1}^{m} \left(x_K^T (\sum_{i=1}^{m} \pi_{i,K} q_{ji} M_{i,K}) x_K | r_{K-1} = j\right) \\
&= \sum_{j=1}^{m} \left(x_K^T \pi_{j,K-1} P_{j,K} x_K | r_{K-1} = j\right) \\
&= \mathbb{E}_{r_{K-1}} \left[x_K^T P_{j,K} x_K\right],
\end{aligned}
$$

where

$$\pi_{j,K-1} P_{j,K} = \sum_{i=1}^{m} \pi_{i,K} q_{ji} M_{i,K}.$$

Thus, the cost function J_{LQR} can be rewritten as

$$
\begin{aligned}
J_{LQR}(K) &= \sum_{k=1}^{K-2} \left(\mathbb{E}_{\{r_j\}_{j=k+1}^{K-1}} \left[x_k^T Q x_k + u_k^T R u_k\right]\right) + T_{K-1} \\
T_{K-1} &= \mathbb{E}_{r_{K-1}} \left[x_{K-1}^T Q x_{K-1} + u_{K-1}^T R U_{K-1} + x_K^T P_{i,K} x_K\right].
\end{aligned}
$$

If we rewrite T_K by explicitly conditioning it on the value of r_{K-1},

$$
\begin{aligned}
T_{K-1} = \sum_{i=1}^{m} \pi_{i,K-1} \Big(& x_{K-1}^T Q x_{K-1} + u_{K-1}^T R U_{K-1} \\
& + x_K^T P_{i,K} x_K | r_{K-1} = i \Big),
\end{aligned}
$$

we see that the problem of choosing U_{K-1} is formally identical to the problem that we solved above for choosing u_K. Thus, the same argument can be repeated at any time step recursively. At a general time k, the control input u_k given $r_k = i$ is given by

$$u_k = -S_{i,k}^{-1} B_i^T P_{i,k+1} x_k,$$

where

$$\begin{aligned} S_{i,k} &= R + B_i^T P_{i,k+1} B_i \\ \pi_{j,k-1} P_{j,k} &= \sum_{i=1}^{m} \pi_{i,k} q_{ji} M_{i,k} \\ M_{i,k} &= Q + A_i^T P_{i,k+1} A_i - A_i^T P_{i,k+1} B_i S_{i,k}^{-1} B_i^T P_{i,k+1} A_i, \end{aligned}$$

with boundary value $P_{i,K+1} = P_{K+1} \forall i$. This proves the first part of the theorem.

To prove the second and third parts, we need to study the stability of the terms $P_{i,0}$ as the horizon $K \to \infty$. □

The above conditions reduce to simpler form for Bernoulli jump linear systems. For this case, the LQR and stabilizability problems can be solved to yield the following result.

Theorem 4.4. *Consider system $\mathcal{S}_1$ with the additional assumption that the Markov transition probability matrix is such that for all states i and j, $q_{ij} = q_i$ (in other words, the states are chosen independently and identically distributed from one time step to the next). Consider the LQR problem posed above for the system $\mathcal{S}_1$.*

1. *At time k, if $r_k = i$, then the optimal control input is given by*

$$u_k = -\left(R + B_i^T P_{k+1} B_i\right)^{-1} B_i^T P_{k+1} A_i x_k,$$

where

$$P_k = \sum_{t=1}^{m} q_t \Big(Q + A_t^T P_{k+1} A_t - A_t^T P_{k+1} B_t \left(R + B_t^T P_{k+1} B_t\right)^{-1} B_t^T P_{k+1} A_t \Big).$$

2. *Assume that the Markov states reach a stationary probability distribution. A sufficient condition for stabilizability of the system is that there exists a positive definite matrix X, and m matrices $K_1, K_2, \cdots, K_m$ such that*

$$X > \sum_{i=1}^{m} q_i \left((A_i^T + K_i B_i^T) X (A_i^T + K_i B_i^T)^T + Q + K_i R K_i^T \right).$$

3. *A necessary condition for stabilizability is that*

$$q_i \rho(A_i)^2 < 1, \qquad \forall i = 1, 2, \cdots, m,$$

where $\rho(A_i)$ is the spectral radius of the matrix A_i that governs the dynamics of unstabilizable modes of the process in the i-th mode.

Proof. The result follows readily from the LQR solution of Markovian jump linear systems. Specifically, if we substitute $q_{tj} = q_j \forall t$ in (4.3), we see that all matrices $P_{j,k}$ are identical for $j = 1, \cdots, m$. If we denote this value by P_k, we obtain the desired form of the LQR control law. Similarly the stability conditions in the theorem also follow from those for Markovian jump linear systems in Theorem 4.3. □

Optimal Minimum Mean Squared Error Estimator

The minimum mean squared error estimate problem for the system $\mathcal{S}_1$ is posed by assuming that the control u_{r_k} is identically zero. The objective is to identify at every time step k, an estimate $\hat{x}_{k+1}$ of the state x_{k+1} that minimizes the mean squared error covariance

$$\Pi_{k+1} = \mathbb{E}_{\{w(j)\},\{v(j)\},x_0}\left[(x_{k+1} - \hat{x}_{k+1})(x_{k+1} - \hat{x}_{k+1})^T\right],$$

where the expectation is taken with respect to the process and measurement noises, and the initial state value (but not the Markov state realization). The estimator at time k has access to observations $\{y(j)\}_{j=0}^{k}$ and the Markov state values $\{r_j\}_{j=0}^{k}$. Moreover, the error covariance is said to be stable if the expected steady state error covariance $\lim_{k\to\infty}\mathbb{E}_{\{r_j\}_{j=0}^{k-1}}[\Pi_k]$ is bounded, where the expectation is taken with respect to the Markov process.

Since the estimator has access to the Markov state values till time k, the optimal estimate can be calculated through a time-varying Kalman filter. Thus, if at time k, $r_k = i$, the estimate evolves as

$$\hat{x}_{k+1} = A_i\hat{x}_k + K_k\left(y_k - C_i\hat{x}_k\right),$$

where

$$\begin{aligned} K_k &= A_i\Pi_k C_i^T\left(C_i\Pi_k C_i^T + R_v\right)^{-1} \\ \Pi_{k+1} &= A_i\Pi_k A_i^T + R_w - A_i\Pi_k C_i^T\left(C_i\Pi_k C_i^T + R_v\right)^{-1} C_i\Pi_k A_i^T. \end{aligned}$$

The error covariance Π_k is available through the above calculations. However, calculating $\mathbb{E}_{\{r_j\}_{j=0}^{k-1}}[\Pi_k]$ seems to be intractable. Instead, the normal approach is to consider an upper bound to this quantity[1] that will also help in obtaining sufficient conditions for the error covariance to be stable.

The intuition behind obtaining the upper bound is simple. The optimal estimator presented above optimally utilizes the information about the Markov states till time k. Consider an alternate estimator that at every time step k, averages over the values of the Markov states $r_0, \cdots, r_{k-1}$. Such an estimator is sub-optimal and the error covariance for this estimator forms an upper bound for $\mathbb{E}_{\{r_j\}_{j=0}^{k-1}}[\Pi_k]$. A more formal derivation for the upper bound is presented below.

Theorem 4.5. *The term $\mathbb{E}_{\{r_j\}_{j=0}^{k-1}}[\Pi_k]$ obtained from the optimal estimator is upper bounded by $M_k = \sum_{j=1}^{m} M_{j,k}$ where*

$$\begin{aligned} M_{j,k} = \sum_{t=1}^{m} q_{tj}\Big(&R_w + A_t M_{t,k-1}A_t^T \\ &- A_t M_{t,k-1}C_t^T\left(R_v + C_t M_{t,k-1}C_t^T\right)^{-1} C_t M_{t,k-1}A_t^T\Big), \end{aligned}$$

with $M_{j,0} = \Pi_0$ $\forall j = 1, 2, \cdots, m$. Moreover, assume that the Markov states reach a stationary probability distribution. A sufficient condition for stabilizability of the

[1] We say that A is upperbounded by B if $B - A$ is positive semi-definite.

system is that there exist m positive definite matrices X_1, X_2, $\cdots$, X_m and m^2 matrices $K_{1,1}$, $K_{1,2}$, $\cdots$, $K_{1,m}$, $K_{2,1}$, $\cdots$, $K_{m,m}$ such that for all $j = 1, 2, \cdots, m$,

$$X_j > \sum_{i=1}^{m} q_{ij} \left((A_i + K_{i,j} C_i) X_i (A_i + K_{i,j} C_i)^T + R_w + K_{ij} R_v K_{ij}^T \right).$$

Finally, a necessary condition for stabilizability is that

$$q_{i,i} \rho(A_i)^2 < 1, \qquad \forall i = 1, 2, \cdots, m,$$

where $\rho(A_i)$ is the spectral radius of the matrix A_i that governs the dynamics of unobservable modes of the process in the i-th mode.

Proof. We begin by defining

$$M_{j,k} = \pi_{j,k-1} \mathbb{E} \left[\Pi_k | r_{k-1} = j \right],$$

so that

$$\mathbb{E} \left[\Pi_k \right] = \sum_{i=1}^{m} M_{j,k}.$$

Now we can bound each term $M_{j,k}$ as follows.

$$\begin{aligned} M_{j,k+1} &= \pi_{j,k} \sum_{i=1}^{m} \mathbb{E} \left[\Pi_{k+1} | r_k = j, r_{k-1} = i \right] P(r_{k-1} = i | r_k = j) \\ &= \sum_{i=1}^{m} \mathbb{E} \left[A_j \Pi_k A_j^T + R_w - A_j \Pi_k C_j^T (C_j \Pi_k C_j^T + R_v)^{-1} C_j \Pi_k A_j^T | r_{k-1} = i \right] q_{ij} \pi_{i,k-1}, \end{aligned}$$

since given r_{k-1}, Π_k and r_k are independent. Further, note that the Riccati operator

$$f_j(M) = A_j M A_j^T + R_w - A_j M C_j^T (C_j M C_j^T + R_v)^{-1} C_j M A_j^T$$

is both concave and increasing. Since it is concave, Jensen's inequality yields

$$\begin{aligned} M_{j,k+1} \leq \sum_{i=1}^{m} \big(& A_j \mathbb{E}[\Pi_k | r_{k-1} = i] A_j^T + R_w \\ & - A_j \mathbb{E}[\Pi_k | r_{k-1} = i] C_j^T (C_j \mathbb{E}[\Pi_k | r_{k-1} = i] C_j^T + R_v)^{-1} \cdot \\ & C_j \mathbb{E}[\Pi_k | r_{k-1} = i] A_j^T \big) q_{ij} \pi_{i,k-1}. \end{aligned}$$

Now from the definition of $M_{i,k-1}$ and the fact that $f_j(.)$ is an increasing operator, we obtain the required bound.

□

The special case of a Bernoulli jump linear systems can be obtained from the above result by substituting $q_{ij} = q_j \forall i$. We state the result below.

Theorem 4.6. *Consider the estimation problem posed above for the system $\mathcal{S}_1$ with the additional assumption that the Markov transition probability matrix is such that for all states i and j, $q_{ij} = q_i$ (in other words, the states are chosen independently* *and identically distributed from one time step to the next). The term* $\mathbb{E}_{\{r_j\}_{j=0}^{k-1}}[\Pi_k]$ *obtained from the optimal estimator is upper bounded by* M_k *where*

$$M_k = \sum_{t=1}^{m} q_t \Big(R_w + A_t M_{k-1} A_t^T - A_t M_{k-1} C_t^T \left(R_v + C_t M_{k-1} C_t^T \right)^{-1} C_t M_{k-1} A_t^T \Big),$$

with $M_0 = \Pi_0$*. Further, a sufficient condition for stabilizability of the system is that there exists a positive definite matrix* X*, and* m *matrices* $K_1, K_2, \cdots, K_m$ *such that*

$$X > \sum_{i=1}^{m} q_i \left((A_i + K_i C_i) X (A_i + K_i C_i)^T + R_w + K_i R_v K_i^T \right).$$

Finally, a necessary condition for stabilizability is that

$$q_i \rho(A_i)^2 < 1, \qquad \forall i = 1, 2, \cdots, m,$$

where $\rho(A_i)$ *is the spectral radius of the matrix* A_i *that governs the dynamics of unobservable modes of the process in the* i*-th mode.*

Linear Quadratic Gaussian Control

Given the optimal linear quadratic regulator and minimum mean squared error estimator, the solution of the linear quadratic Gaussian control problem can be solved by utilizing a separation principle. The Linear Quadratic Gaussian (LQG) problem for the system $\mathcal{S}_1$ aims at designing the control input u_k to minimize the finite horizon cost function

$$J_{LQG} = \mathbb{E}\left[\sum_{k=1}^{K} \left(x_k^T Q x_k + u_k^T R u_k \right) + x_{K+1}^T P_{K+1} x_{k+1} \right],$$

where the expectation at time k is taken with respect to the future values of the Markov state realization, the measurement and process noises, and the initial state. Further, the matrices P_{K+1}, Q and R are all assumed to be positive definite. The controller at time k has access to control inputs $\{u_j\}_{j=0}^{k-1}$, measurements $\{y(j)\}_{j=0}^{k}$ and the Markov state values $\{r_j\}_{j=0}^{k}$. The system is said to be stabilizable if the infinite horizon cost function $J_\infty \overset{def}{=} \lim_{K\to\infty} \frac{J_{LQG}}{K}$ is finite.

The solution to this problem is provided by Theorems 4.3 and 4.5 because of the following separation principle.

Theorem 4.7. *Consider the LQG problem for the system* $\mathcal{S}_1$*. At time* k*, if* $r_k = i$*, then the optimal control input is given by*

$$u_k = -\left(R + B_i^T P_{i,k+1} B_i \right)^{-1} B_i^T P_{i,k+1} A_i \hat{x}_k,$$

where for $P_{i,k}$ *is calculated as in Theorem 4.3 and* $\hat{x}_k$ *is calculated using a time-varying Kalman filter.*

Given this separation principle, the stabilizability conditions provided in Theorems 4.3 and 4.5 can then be combined to yield the stabilizability conditions for the LQG case as well. Finally, we note that a similar separation principle also holds for Bernoulli jump linear systems. Thus, the LQG problem can be solved for this case as well.

4.4 Further Resources