Networked Sensing, Estimation and Control Systems

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Chapter 2
State Estimation and Sensor Fusion

In this chapter, we provide an overview of Kalman filter and linear quadratic Gaussian (LQG) control. We will first provide a quick summary of basic theories of probability and stochastic process, which will be used to derive the Kalman filter equations. Some properties of Kalman filter and its steady-state error covariance matrix will also be provided. After that, we will introduce LQG control and derive the optimal control law using a dynamic programming approach.

The material of this chapter will be the foundation of most subsequent chapters including Chapters 4, 6 and 8. In particular, Chapter 6 considers the effect of data packet drops on Kalman filtering and LQG control and Chapter 8 considers distributed Kalman filtering.

2.1 Review of Probability and Random Process

We assume the readers have some exposure to the theory of probability and random process. The material presented in this section only serves as a quick review of some basic concepts and tools from probability and random process that will be helpful to understand and derive some important results in subsequent sections and chapters. Good introductory books on probability and random process are [GS01] and [LG93].

Random Variables

Consider an experiment with many (possibly infinite) outcomes. All these outcomes form the *sample space* Ω. A subset $A \subset \Omega$ is called an *event*. Two events A_1, A_2 are called *mutually disjoint* if $A_1 \cap A_2 = \emptyset$. The *complement* of an event A is defined as $\bar{A} = \Omega \setminus A$. A *probability measure* $P(\cdot)$ is a mapping from Ω into the interval $[0, 1]$ such that the following axioms of probability are satisfied:

1. $P(A) \geq 0$ for all $A \subset \Omega$.
2. $P(\Omega) = 1$.
3. If $\{A_i, i = 1, 2, \ldots\}$ is a collection of disjoint members of $\mathcal{F}$, i.e., $A_i \cap A_j = \emptyset$ for all i, j, then $P(\cup A_i) = \sum_i P(A_i)$.

From the axioms of probability, it follows that

$$P(A) \leq 1, \quad P(\emptyset) = 0, \quad P(\bar{A}) = 1 - P(A), \quad P(\cup A_i) \leq \sum_i P(A_i).$$

The joint probability of two events A and B is $P(A \cap B)$ which is often written as $P(AB)$ for simplicity. The conditional probability of A given B i.e., the probability

that A occurs if B occurs in an experiment is

$$P(A|B) = \frac{P(AB)}{P(B)}, \quad \text{assuming } P(B) \neq 0.$$

A and B are *mutually independent* if

$$P(AB) = P(A)P(B).$$

If $P(B) \neq 0$, the conditional probability $P(A|B)$ can be calculated from *Bayes' Rule* as

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

If $A_i, i = 1, 2, \ldots$ are mutually disjoint and $\cup A_i = \Omega$, then

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

and

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_i P(B|A_i)P(A_i)}.$$

A *random variable* is a function $X : \Omega \to \mathbb{R}$. The *cumulative distribution function* of a random variable X is a function $F_X : R \to [0, 1]$ given by

$$F_X(x) = P(X \leq x).$$

The cumulative distribution function F has the following properties

1. $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.
2. If $x \leq y$, then $F_X(x) \leq F_X(y)$.
3. F_X is right-continuous.

When F_X is differentiable, we can define the associated *probability density function* $p_X(x)$ as

$$p_X(x) = \frac{dF_X(x)}{dx}.$$

The *joint cumulative distribution function* of two random variables X and Y, denoted as $F_{XY}(x, y)$, is given by

$$F_{XY}(x, y) = P(X \leq x) \cap P(Y \leq y).$$

If its derivative exists, the associated joint probability density function is given by

$$p_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

Given $F_{XY}(x, y)$, the *marginal distribution functions* of X and Y can be calculated as

$$F_X(x) = P(X \leq x) = F_{XY}(x, \infty), \; F_Y(y) = P(Y \leq y) = F_{XY}(\infty, y).$$

It follows that the *marginal density functions* of X and Y are

$$p_X(x) = \int_{-\infty}^{\infty} F_{XY}(x,y)dy,\ p_Y(y) = \int_{-\infty}^{\infty} F_{XY}(x,y)dx.$$

The *conditional density function* of X given Y is given by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)}.$$

The density function of X can also be calculated as

$$p_X(x) = \int_{-\infty}^{\infty} p_{X|Y}(x|y)p_Y(y)dy.$$

If X and Y are independent random variables, then the following statements holds and are equivalent to each other:

1. $F_{XY}(x,y) = F_X(x)F_Y(y)$.
2. $p_{XY}(x,y) = p_X(x)p_Y(y)$.
3. $p_{X|Y}(x|y) = p_X(x)$.

A random variable X is completely specified by its distribution function $F_X(x)$ or density function $p_X(x)$. In many situations, $F_X(x)$ or $p_X(x)$ are difficult to obtain. It turns out the *mean* μ_X and *variance* σ_X^2 may provide us enough (useful) information about X. The mean and variance of a random variable X are defined as follows:

$$\begin{aligned}\mu_X &= \mathbb{E}[X] = \int_{-\infty}^{\infty} xp_X(x)dx,\\ \sigma_X^2 &= \mathbb{E}\big[(X-\mathbb{E}[X])^2\big] = \int_{-\infty}^{\infty} (X-\mathbb{E}[X])^2 p_X(x)dx.\end{aligned}$$

We denote $\mathbb{E}[\cdot]$ as the *expectation operator*. Since $\mathbb{E}[\cdot]$ is a linear operator, σ_X^2 can also be calculated as

$$\sigma_X^2 = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2.$$

If X is a zero-mean random variable, i.e., $\mathbb{E}[X] = 0$, then $\sigma_X = \mathbb{E}[X^2]$. The kth *moment* of X is $m_k = \mathbb{E}[X^k]$ and the kth *central moment* is $\mu_k = \mathbb{E}\big[(X-\mathbb{E}[X])^k\big]$.

The *covariance* of two random variables X and Y is defined as $\mathbb{E}\big[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\big]$. X and Y are *uncorrelated* if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. If X and Y are uncorrelated, it is easy to verify that the covariance of X and Y is equal to zero. Clearly if X and Y are independent, then they are uncorrelated. However the converse does not hold in general.

The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} xp_{X|Y}(x|y)dx$$

which is a number that depends on the value of y. Similarly, the conditional expectation of X given Y is

$$\mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x p_{X|Y}(x|Y)dx$$

which is also a random variable that depends on Y, i.e., it is a *function of the random variable* Y. The following property is very important and has great practical value in evaluating $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \mathbb{E}_Y\big[\mathbb{E}_X[X|Y]\big],$$

i.e., we first find the conditional expectation of X (conditioned on Y), and then remove the condition by taking the expectation with respect to Y. From this property, one can easily verify that if X and Y are independent, then

$$\mathbb{E}[X|Y] = \mathbb{E}[X].$$

Furthermore if X and Y are jointly independently of Z, then

$$\mathbb{E}[XY|Z] = \mathbb{E}[X|Z]\mathbb{E}[Y|Z].$$

Random Processes

A random process $X(t)$ is a generalization of a random variable. For a random variable, each experiment leads to a number (or a vector), while for a random process, each experiment leads to a function. For a fixed outcome $\omega \in \Omega$, one obtains the function $X(t, \omega)$, which is also called the *sample path* or *sample function* of the process. For a fixed t, $X(t, \omega)$ is a random variable with the underlying probability space Ω. The *mean process* of $X(t)$ is the time function $\mathbb{E}[X(t)]$. The autocorrelation of $X(t)$ is $\mathbb{E}[X(t_1)X(t_2)^T]$ and the autocovariance of $X(t)$ is $\mathbb{E}\big[\big(X(t_1) - m(t_1)\big)\big(X(t_2) - m(t_2)\big)^T\big]$.

A random process $X(t)$ is called a *Gaussian random process* if for any finite set $\{t_1, t_2, \ldots, t_N\}$, the random variables $\{X(t_1), X(t_2), \ldots, X(t_N)\}$ have a joint Gaussian distribution, i.e., their joint probability density function is given by

$$p_X(x) = \frac{1}{(2\pi)^{N/2}\sqrt{\det[\mathcal{C}_X]}} \exp\left[-\frac{1}{2}(x - m_X)^T \mathcal{C}_X^{-1}(x - m_X)\right] \tag{2.1}$$

where $m_X = [m_X(t_1)\, m_X(t_2)\, \ldots\, m_X(t_N)]^T$ is the mean vector and $\mathcal{C}_X = \big[\mathrm{cov}\big(X(t_i), X(t_j)\big)\big]$ is the covariance matrix. Gaussian processes have the following properties.

Theorem 2.1. *Let $X(t)$ be a Gaussian process. Then $X(t)$ is completely determined by m_X and $\mathcal{C}_X$.*

Theorem 2.2. *Let X and Y have a joint Gaussian distribution with mean and covariance given by*

$$\mu = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_y \end{bmatrix}.$$

Then X conditioned on $Y = y$ is Gaussian with mean and covariance given by

$$\mu_{X|Y=y} = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y}) \text{ and } \Sigma_{X|Y=y}\Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx}.$$

In other words,

$$\mathbb{E}[X|Y=y] = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y-\bar{y}). \tag{2.2}$$

The proof can be found in Anderson and Moore [AM90].

Stability of stochastic systems

Consider the following system dynamics:

$$x_{k+1} = f(x_k, w_k), \tag{2.3}$$

where x_0 and w_k are random vectors. System (2.3) is said to be

1. *second moment stable* if
$$\lim_{k\to\infty} \mathbb{E}[||x_k||^2] = 0,$$
2. *almost sure stable* if
$$P(\lim_{k\to\infty} ||x_k|| = 0]) = 1,$$

where the expectation is taken with respect to x_0 and $w_i, i = 0, \ldots, k$.

For a convex function f, $x_1, \ldots, x_n$ in its domain, and positive weights α_i, Jensen's inequality can be stated as:

$$f\left(\frac{\sum \alpha_i x_i}{\sum \alpha_i}\right) \leq \frac{\sum \alpha_i f(x_i)}{\sum \alpha_i}. \tag{2.4}$$

Jensen's inequality can also be stated in probabilistic form. Let X be a random variable and f be a convex function. Then

$$f\left(\mathbb{E}[X]\right) \leq \mathbb{E}\left(f(X)\right). \tag{2.5}$$

The above two inequalities are reversed if f is concave.

Markov Chains

2.2 Optimal Estimation

Minimum mean square error estimator

Suppose we wish to know some quantity X, and we are not able to make a direct and accurate measurement of X. However we can make some indirect measurement Y that is related to X. Our task is to get an "optimal" estimate of X from Y.

One question that immediately arises before we attempt to solve the estimation problem is: what is a good estimate and when an estimate is "optimal"?

Intuitively a "good" estimate should make the estimation error $\hat{X} - X$ "small" since we wish to reconstruct X as perfectly as possible. An "optimal" estimate should make $\hat{X} - X$ the "smallest" among all other estimates. Many metrics can be used to define the size of the error $\hat{X} - X$ (hence we are able to say if it is "small" or not). Since $\hat{X} - X$ is typically a random variable, the metric that we shall use throughout the book is the following *mean squared error* (MSE)

$$\mathbb{E}[(\hat{X} - X)^T(\hat{X} - X)].$$

Therefore given $Y = y$ (i.e., the measurement that we take), our task is to construct the optimal estimate $\hat{X}$ that minimizes

$$\mathbb{E}[(\hat{X} - X)^T(\hat{X} - X)|Y = y].$$

It turns out that the optimal $\hat{X}$ has a very simple form, given in the following theorem.

Theorem 2.3. *The optimal estimate $\hat{X}^*$ that minimizes*

$$\mathbb{E}[(\hat{X} - X)^T(\hat{X} - X)|Y = y]$$

is given by the following conditional expectation of X

$$\hat{X}^* = \mathbb{E}[X|Y = y].$$

Proof. We can rewrite $\mathbb{E}[(\hat{X} - X)^T(\hat{X} - X)|Y = y]$ as follows

$$\begin{aligned} & \mathbb{E}[(\hat{X} - X)^T(\hat{X} - X)|Y = y] \\ = \;& \mathbb{E}[X^T X|Y = y] - 2\hat{X}^T\mathbb{E}[X|Y = y] + \hat{X}^T\hat{X} \\ = \;& \big(\hat{X} - \mathbb{E}[X|Y = y]\big)^T\big(\hat{X} - \mathbb{E}[X|Y = y]\big) + \mathbb{E}\big[X^T X - \mathbb{E}[X]^T\mathbb{E}[X^T]|Y = y\big]. \end{aligned}$$

Since $\mathbb{E}\big[X^T X - \mathbb{E}[X]^T\mathbb{E}[X^T]|Y = y\big]$ is independent of $\hat{X}$, we conclude that

$$\hat{X}^* = \mathbb{E}[X|Y = y].$$

□

$\hat{X}^* = \mathbb{E}[X|Y = y]$ is also called the *minimum mean squared error* (MMSE) estimate of X.

Example 2.1 Estimate a Gaussian random variable
Consider the following equation

$$Y = X + N \tag{2.6}$$

where X and N are both scalar zero-mean Gaussian random variables with covariances σ_x and σ_n respectively. Further assume X and N are uncorrelated. Suppose we make a measurement of X and get y. The MMSE estimate of X is then given by

$$\hat{X} \;\; = \;\; \mathbb{E}[X|Y = y] = \frac{\sigma_x}{\sigma_x + \sigma_n} y.$$

∇

Sampling of a continuous-time system

A wide variety of physical systems are modeled in the continuous-time domain. In this book, we focus on continuous-time systems with dynamics of the form

$$\frac{dx}{dt} = A_c x + B_c u + w, \quad y = C_c + v, \tag{2.7}$$

where $x(t) \in \mathbb{R}^n$ is the state vector with unknown initial value $x(0)$, $u(t) \in \mathbb{R}^p$ is the input vector, $y(t) \in \mathbb{R}^m$ is the observation vector, and $w(t)$ and $v(t)$ are process disturbance and measurement noise. We assume $w(t)$ and $v(t)$ are mutually uncorrelated zero-mean Gaussian processes with autocovariances

$$\mathbb{E}[w(s)w(t)^T] = \delta_{st}\Sigma_{wc}, \quad \mathbb{E}[v(s)v(t)^T] = \delta_{st}\Sigma_{vc},$$

where $\delta_{st} = 1$ if $s = t$ and $\delta_{st} = 0$ otherwise.

As more and more controllers are implemented digitally, we need a procedure to convert the continuous-time system (2.7) into an equivalent discrete-time system. This procedure is called *sampling* or *discretization*. A frequently seen approach to implement the control law on a digital computer is to use a digital to analogue converter that holds the analog signal until the next time step, called *zero-order-hold* control.

Consider the following periodic sampling scheme: we sample the system (2.7) at time instances $t = k\tau,\ k = 0, 1, \ldots$, where $\tau > 0$ is the *sampling period*. It can be shown (see Astrom-Wittenmark) that the equivalent discrete-time system of (2.7) is given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad y_k = Cx_k + v_k, \tag{2.8}$$

where x_k and y_k correspond to $x(t)$ and $y(t)$ at time $t = k\tau$, and A, B and C are given by

$$A = e^{A_c\tau}, \quad B = \int_0^\tau e^{A_c t}dtB_c, \quad C = C_c. \tag{2.9}$$

In the discrete-time setting, the process and measurement noises are also uncorrelated zero-mean Gaussian random processes with covariance

$$\mathbb{E}[w_s w_k] = \delta_{sk}\Sigma_w, \quad \mathbb{E}[v_s v_k^T] = \delta_{sk}\Sigma_v,$$

where

$$\Sigma_w = \int_0^\tau e^{A_c t}\Sigma_{wc}e^{A_c^T t}dt, \quad \Sigma_v = \Sigma_{vc}.$$

Computing Σ_w directly from the above formula is sometimes difficult due to the integral of matrix exponentials. An easier approach to compute it is given as follows. Define M and N as

$$M = \begin{bmatrix} -A_c & \Sigma_{wc} \\ 0 & A_c^T \end{bmatrix}\tau, \quad N = e^M.$$

Then can be shown that

$$N = \begin{bmatrix} * & X^{-1}\Sigma_w \\ 0 & X^T \end{bmatrix}.$$

Therefore Σ_w can be computed from

$$\Sigma_w = (X^T)^T X^{-1}\Sigma_w,$$

i.e., Σ_w is obtained by multiplying the transpose of the lower-right submatrix of N with the upper-right submatrix of N.

Most of the results developed in this book also extend to cases where the sensor measurement y_k involves a direct input term, i.e.,

$$y_k = Cx_k + Du_k + v_k. \tag{2.10}$$

For simplicity, we shall use the system model as described by (2.8) for the remainder of the book unless otherwise explicitly stated.

Kalman filtering

Consider the following system as described by equation(2.8):

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad y_k = Cx_k + v_k, \tag{2.11}$$

where $x_k \in \mathbb{R}^n$ is the state vector with unknown initial value x_0, $u_k \in \mathbb{R}^p$ is the input vector, $y_k \in \mathbb{R}^m$ is the observation vector, and w_k and v_k are process and measurement noises (or disturbances).

Clearly nothing can be said on any estimator without defining a structure on w_k and v_k. In this book, we are particularly interested in w_k and v_k that have the following properties:

- w_k and v_k are zero-mean Gaussian random vectors;
- $\mathbb{E}[w_k w_j^T] = \delta_{kj}\Sigma_w$ with $\Sigma_w \geq 0$;
- $\mathbb{E}[v_k v_j^T] = \delta_{kj}\Sigma_v$ with $\Sigma_v > 0$;
- $\mathbb{E}[w_k v_j^T] = 0 \; \forall j, k$,

where $\delta_{kj} = 0$ if $k \neq j$ and $\delta_{kj} = 1$ otherwise. We also assume the initial value x_0 of system (2.11) is a zero-mean Gaussian random vector that is uncorrelated with w_k and v_k for all $k \geq 0$. The covariance of x_0 is given by $\Pi_0 \geq 0$. Furthermore we assume $(A, \sqrt{Q})$ is stabilizable.

Let $Y_k = \{y_0, y_1, \ldots, y_k\}$ be the measurements available at time k and $U_k = \{u_0, u_1, \ldots, u_k\}$ be the input applied to the system up to time k. We are interested in looking for the MMSE $\hat{x}_k$ of x_k at each time $k \geq 0$ given Y_k and U_{k-1}. From Theorem 2.3, we know that $\hat{x}_k$ is given by

$$\hat{x}_k = \mathbb{E}[x_k | Y_k, U_{k-1}], \tag{2.12}$$

and the corresponding error covariance P_k is given by

$$P_k = \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T | Y_k, U_{k-1}]. \tag{2.13}$$

Calculating $\hat{x}_k$ and P_k according to equation (2.12) and (2.13) is not trivial and is computationally intensive as k increases. The celebrated Kalman filter provides a simple and elegant way to compute $\hat{x}_k$ and P_k recursively.

The Kalman filter [Kal60] is a well-established methodology for model-based fusion of sensor data [GA93, Gus00, May79, KSH00, AM90] that has played a central role in systems theory and has found wide applications in many fields such as control, signal processing, and communications.

Assume that $\hat{x}_{k-1}$ and P_{k-1} defined as in equation (2.12) and (2.13) are available. Consider the one-step state prediction $\hat{x}_{k|k-1}$ (also called the *a priori state estimate*) given by

$$\hat{x}_{k|k-1} \quad = \quad \mathbb{E}[x_k|Y_{k-1}, U_{k-1}]$$

and the associated estimation error covariance (also called the *a priori error covariance*) $P_{k|k-1}$ given by

$$P_{k|k-1} \quad = \quad \mathbb{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T|Y_{k-1}, U_{k-1}].$$

From (2.11), we have

$$\begin{aligned} \hat{x}_{k|k-1} &= \mathbb{E}[x_k|Y_{k-1}, U_{k-1}] \\ &= \mathbb{E}[Ax_{k-1} + Bu_{k-1} + w_{k-1}|Y_{k-1}, U_{k-1}] \\ &= A\hat{x}_{k-1} + Bu_{k-1}, \end{aligned} \tag{2.14}$$

where we use the fact that w_{k-1} is independent of any y_t ($t \leq k-1$) and the expectation operator is linear. Consequently,

$$P_{k|k-1} = AP_{k-1}A^T + \Sigma_w. \tag{2.15}$$

Now consider y_k conditioned on Y_{k-1} and U_{k-1} which has mean

$$\mathbb{E}[y_k|Y_{k-1}, U_{k-1}] = \mathbb{E}[Cx_k + v_k|Y_{k-1}, U_{k-1}] = C\hat{x}_{k|k-1}$$

and covariance

$$\mathbb{E}\big[\big(y_k - \mathbb{E}[y_k]\big)\big(y_k - \mathbb{E}[y_k]\big)^T|Y_{k-1}, U_{k-1}\big] = CP_{k|k-1}C^T + \Sigma_v,$$

where we have used the fact that v_k is independent of Y_{k-1}. The cross covariance of x_k and y_k conditioned on Y_{k-1} and U_{k-1} is given by

$$\mathbb{E}\big[\big(x_k - \mathbb{E}[x_k]\big)\big(y_k - \mathbb{E}[y_k]\big)^T|Y_{k-1}, U_{k-1}\big] = P_{k|k-1}C^T.$$

From the above analysis, we see that the random vector $[x_k' \quad y_k']'$ conditioned on Y_{k-1} and U_{k-1} is Gaussian with mean and covariance

$$\begin{bmatrix} \hat{x}_{k|k-1} \\ C\hat{x}_{k|k-1} \end{bmatrix} \text{ and } \begin{bmatrix} P_{k|k-1} & P_{k|k-1}C^T \\ CP_{k|k-1} & CP_{k|k-1}C^T + \Sigma_v \end{bmatrix}.$$

Therefore from Theorem 2.2, x_k conditioned on y_k (and on Y_{k-1} and U_{k-1}, i.e., conditioned on Y_k and U_{k-1}) has mean

$$\mathbb{E}[x_k|Y_k, U_{k-1}] = \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1})$$

and covariance

$$(I - K_kC)P_{k|k-1}$$

where $K_k = P_{k|k-1}C^T[CP_{k|k-1}C^T + \Sigma_v]^{-1}$ is the so-called *Kalman gain*.

Let us summarize what we have said so far. Given the system (2.11), the MMSE estimate $\hat{x}_k$ of x_k is given by $\hat{x}_k = \mathbb{E}[x_k|Y_k, U_{k-1}]$, which can be computed recursively as follows

1. *time update:*

$$\begin{aligned} \hat{x}_{k|k-1} &= A\hat{x}_{k-1} + Bu_{k-1}, \\ P_{k|k-1} &= AP_{k-1}A^T + \Sigma_w. \end{aligned}$$

2. *measurement update:*

$$\begin{aligned} K_k &= P_{k|k-1}C^T[CP_{k|k-1}C^T + \Sigma_v]^{-1}, \\ \hat{x}_k &= \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}), \\ P_k &= (I - K_kC)P_{k|k-1}. \end{aligned}$$

The initial values of the recursion are set as $\hat{x}_0 = 0$ and $P_0 = \Pi_0$. The Kalman filter essentially consists of the above two update steps.

Lemma 2.1. *The Kalman gain K_k and the error covariance P_k satisfy*

$$K_k = P_kC^T\Sigma_v^{-1}. \tag{2.16}$$

Proof. Since $P_k = (I - K_kC)P_{k|k-1}$, it suffices to show

$$(I - K_kC)P_{k|k-1}C^T\Sigma_v^{-1} = K_k$$

which is equivalent to

$$\begin{aligned} & P_{k|k-1}C^T\Sigma_v^{-1} = K_k(I + CP_{k|k-1}C^T\Sigma_v^{-1}) \\ \iff\ & P_{k|k-1}C^T\Sigma_v^{-1} = P_{k|k-1}C^T[CP_{k|k-1}C^T + \Sigma_v]^{-1}(I + CP_{k|k-1}C^T\Sigma_v^{-1}) \\ \impliedby\ & \Sigma_v = (I + CP_{k|k-1}C^T\Sigma_v^{-1})^{-1}(CP_{k|k-1}C^T + \Sigma_v) \\ \iff\ & \Sigma_v = \Sigma_v(\Sigma_v + CP_{k|k-1}C^T)^{-1}(CP_{k|k-1}C^T + \Sigma_v) \end{aligned}$$

where the last equation holds trivially. □

Let $\mathbb{S}_+^n$ be the set of n by n positive semi-definite matrices. To simplify the analysis, define the function $h : \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$h(X) \triangleq AXA^T + \Sigma_w, \tag{2.17}$$

and $\tilde{g} : \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$\tilde{g}(X) \triangleq X - XC^T[CXC^T + \Sigma_v]^{-1}CX. \tag{2.18}$$

Further define $g : \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$g(X) \triangleq h \circ \tilde{g} = AXA^T + \Sigma_w - AXC^T[CXC^T + \Sigma_v]^{-1}CXA. \tag{2.19}$$

For functions $f, f_1, f_2 : \mathbb{S}_+^n \to \mathbb{S}_+^n$, $f_1 \circ f_2$ is defined as

$$f_1 \circ f_2(X) \triangleq f_1\big(f_2(X)\big), \tag{2.20}$$

and f^t is defined as

$$f^t(X) \triangleq \underbrace{f \circ f \circ \cdots \circ f}_{t \text{ times}}(X). \tag{2.21}$$

With these definitions, it can be verified that in the Kalman filter time update and measurement update equations, $P_{k+1|k}$ and P_{k+1} satisfy

$$\begin{aligned} P_{k+1|k} &= h(P_k), \\ P_{k+1|k} &= g(P_{k|k-1}), \\ P_{k+1} &= \tilde{g}(P_{k+1|k}), \\ P_{k+1} &= \tilde{g} \circ h(P_k). \end{aligned}$$

The equation $g(X) = X$ or

$$AXA^T + \Sigma_w - AXC^T[CXC^T + \Sigma_v]^{-1}CXA = X \tag{2.22}$$

is called the *discrete-time algebraic Riccati equation* (DARE). When $(A, \sqrt{\Sigma_w})$ is stabilizable and (A, C) is detectable, P_k converges to a unique positive semi-definite matrix $\overline{P}$ which satisfies $\overline{P} = \tilde{g} \circ h(\overline{P})$. $\overline{P}$ is called the steady-state error covariance, and it reflects how well the estimate $\hat{x}_k$ approximates x_k in the steady state.

Properties of the Kalman filter

We first introduce a few well-known lemmas without proofs.

Lemma 2.2 (Matrix Inversion Lemma)**.** *Let $X > 0$. If $X = B^{-1} + CD^{-1}C'$, then the inverse of X can be written as*

$$X^{-1} = B - BC(D + C'BC)^{-1}C'B.$$

The second lemma is the Schur Complement lemma. It provides a set of equivalent relationships for a positive definite matrix M.

Lemma 2.3 (Schur Complement)**.** *Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then the following three conditions are equivalent to each other.

1. $M > 0$.
2. $A > 0$ *and* $S_A \triangleq D - CA^{-1}B > 0$.
3. $D > 0$ *and* $S_D \triangleq A - BD^{-1}C > 0$.

The last one is the Block Matrix Inversion lemma, which, as its name suggests, inverts a block matrix using the Schur complement of the matrix.

Lemma 2.4 (Block Matrix Inversion)**.** *Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} > 0.$$

Then M^{-1} can be computed as

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS_A^{-1}CA^{-1} & -A^{-1}BS_A^{-1} \\ -S_A^{-1}CA^{-1} & S_A^{-1} \end{bmatrix},$$

or it can be computed as

$$M^{-1} = \begin{bmatrix} S_D^{-1} & -S_D^{-1}BD^{-1} \\ -D^{-1}CS_D^{-1} & D^{-1} + D^{-1}CS_D^{-1}BD^{-1} \end{bmatrix}.$$

Many useful properties of the functions h, $\tilde{g}$ and g are presented below.

Lemma 2.5. *For any $X, Y \in \mathbb{S}_+^n$, and $X \leq Y$,*

1. $h(X) \leq h(Y)$.
2. $g(X) \leq g(Y)$.
3. $\tilde{g}(X) \leq \tilde{g}(Y)$.
4. $\tilde{g}(X) \leq X$.
5. $g(X) \leq h(X)$.

When the measurement matrix C is invertible, the function g exhibits a very nice property. When we apply g to any $X \geq 0$, we have a bounded value. The following lemma gives this bound.

Lemma 2.6. *Assume C^{-1} exists and let $\overline{M} = C^{-1}RC^{-1'}$. Then for any $X \geq 0, \tilde{g}(X) \leq \overline{M}$.*

Proof. For any $t > 0$, we have $\tilde{g}(t\overline{M}) = \frac{t}{t+1}\overline{M} \leq \overline{M}$. For all $X \geq 0$, since $\overline{M} > 0$, it is clear that there exists $t_1 > 0$ such that $t_1\overline{M} > X$. Therefore $\tilde{g}(X) \leq \tilde{g}(t_1\overline{M}) \leq \overline{M}$. □

The steady-state error covariance $\overline{P}$ has the following property.

Lemma 2.7. $\overline{P} \leq h(\overline{P})$.

Proof. Let P^* satisfy $P^* = g(P^*)$. Then one can verify that $\overline{P} = \tilde{g}(P^*)$. Since $g = h \circ \tilde{g}$, we have

$$\overline{P} = \tilde{g}(P^*) \leq P^* = g(P^*) = h \circ \tilde{g}(P^*) = h(\overline{P}).$$

□

Let $0 \leq \lambda \leq 1$. Consider the following modified DARE.

$$g_\lambda(X) \triangleq AXA^T + \Sigma_w - \lambda AXC^T[CXC^T + \Sigma_v]^{-1}CXA = X. \tag{2.23}$$

The modified DARE will be studied in detail in Chapter 6 and the parameter λ will represent data packet arrival rate. Some preliminary results on the modified DARE are stated in the following lemmas. The proofs are omitted and can be found in the appendix of [SSS+03].

Lemma 2.8. *Let the operator*

$$\phi(K, X) = (1 - \lambda)(AXA' + \Sigma_w) + \lambda(FXF' + V) \tag{2.24}$$

where $F = A + KC, \quad V = \Sigma_w + K\Sigma_v K'$. Assume $X \in \mathbb{S}_+^n$, $\Sigma_v > 0$, $\Sigma_w \geq 0$, and $(A, \Sigma_w^{\frac{1}{2}})$ is controllable. Then the following facts are true:

1. *With* $K_X = -AXC'\left(CXC' + \Sigma_v\right)^{-1}$, $g_\lambda(X) = \phi(K_X, X)$

2. $g_\lambda(X) = \min_K \phi(K, X) \leq \phi(K, X)$, $\forall K$

3. *If* $X \leq Y$, *then* $g_\lambda(X) \leq g_\lambda(Y)$

4. *If* $\lambda_1 \leq \lambda_2$ *then* $g_{\lambda_1}(X) \geq g_{\lambda_2}(X)$

5. *If* $\alpha \in [0, 1]$, *then*
   $g_\lambda(\alpha X + (1-\alpha)Y) \geq \alpha g_\lambda(X) + (1-\alpha) g_\lambda(Y)$

6. $g_\lambda(X) \geq (1-\lambda)AXA' + \Sigma_w$

7. *If* $\bar{X} \geq g_\lambda(\bar{X})$, *then* $\bar{X} > 0$

8. *If* X *is a random variable, then*
   $(1-\lambda)A\mathbb{E}[X]A' + \Sigma_w \leq \mathbb{E}[g_\lambda(X)] \leq g_\lambda(\mathbb{E}[X])$

The next two Lemmas show that when the modified DARE has a solution $\hat{P}$, this solution is also stable, i.e., every sequence based on the difference Riccati equation $P_{t+1} = g_\lambda(P_t)$ converges to $\hat{P}$ for all initial positive semidefinite conditions $P_0 \geq 0$.

Lemma 2.9. *Define the linear operator*

$$\mathcal{L}(Y) = (1-\lambda)(AYA') + \lambda(FYF')$$

Suppose there exists $\overline{Y} > 0$ *such that* $\overline{Y} > \mathcal{L}(\overline{Y})$.

1. *For all* $W \geq 0$,
$$\lim_{k \to \infty} \mathcal{L}^k(W) = 0$$

2. *Let* $U \geq 0$ *and consider the linear system*
$$Y_{k+1} = \mathcal{L}(Y_k) + U \quad \textit{initialized at} \quad Y_0.$$
   Then, the sequence Y_k *is bounded.*

Lemma 2.10. *Consider the operator* $\phi(K, X)$ *defined in equation (6.28). Suppose there exists a matrix* $\overline{K}$ *and a positive definite matrix* Z *such that*

$$Z > 0 \quad \textit{and} \quad Z > \phi(\overline{K}, Z).$$

Then, for any P_0, *the sequence* $P_t = g_\lambda^t(P_0)$ *is bounded, i.e., there exists* $M_{P_0} \geq 0$ *dependent of* P_0 *such that*

$$P_t \leq M \quad \textit{for all} \quad t.$$

2.3 Linear Quadratic Optimal Control

The optimal linear quadratic regulator problem is posed as follows. Consider the process

$$x_{k+1} = Ax_k + Bu_k$$

with the initial condition x_0, where $x_k \in \mathbf{R}^n$ is the state and $u_k \in \mathbf{R}^m$ is the control input that needs to be designed to minimize the cost

$$J_K = \sum_{k=0}^{K} \left(x_k^T Q x_k + u_k^T R u_k \right) + x_{k+1}^T P_{k+1} x_{k+1},$$

with $Q > 0$ and $R \geq 0$. If the parameter K is finite, the problem is termed the finite horizon LQR problem. The case when $K \to \infty$ is termed the infinite horizon LQR problem. We shall assume that the par (A, B) is controllable. In general, the results given below extend to the case when the matrices A, B, Q and R are time varying.

The solution to the finite horizon problem can be obtained through standard dynamic programming arguments. The following theorem summarizes the results.

Theorem 2.4. *Consider the finite horizon LQR problem posed above. The optimal control law is a linear function of the state*

$$u_k = -\left(B^T P_{k+1} B + R \right)^{-1} B^T P_{k+1} A x_k,$$

where the matrix P_k evolves according to the backward Riccati recursion

$$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B \left(B^T P_{k+1} B + R \right)^{-1} B^T P_{k+1} A$$

with the initial condition P_{k+1}. Moreover, the achieved cost is given by $x_0^T P_0 x_0$.

Proof. We begin by rewriting the cost function J_K to identify terms in the cost that depend on x_k and u_k:

$$\begin{aligned} J_K &= \sum_{k=1}^{K-1} \left(x_k^T Q x_k + u_k^T R u_k \right) + T_k \\ T_k &= x_k^T Q x_k + u_k^T R u_k + x_{k+1}^T P_{k+1} x_{k+1}. \end{aligned}$$

The only term in the cost that can be affected by the choice of u_k is T_k. To choose u_k, we minimize T_k by a completion of squares argument. We obtain

$$\begin{aligned} T_k &= x_k^T Q x_k + u_k^T R u_k + (Ax_k + Bu_k)^T P_{k+1} (Ax_k + Bu_k) \\ &= x_k^T P_k x_k + (u_k + S_k^{-1} B^T P_{k+1} A x_k)^T S_k (u_k + S_k^{-1} B^T P_{k+1} A x_k), \end{aligned}$$

where

$$\begin{aligned} S_k &= B^T P_{k+1} B + R \\ P_k &= Q + A^T P_{k+1} A - A^T P_{k+1} B S_k^{-1} B^T P_{k+1} A. \end{aligned}$$

Thus, the optimal choice of u_k is

$$u_k = -S_k^{-1} B^T P_{k+1} x_k.$$

With the optimal choice of u_k the term T_k reduces to $x_k^T P_k x_k$. Thus, the cost function J_K can be rewritten as

$$\begin{aligned} J_K &= \sum_{k=1}^{K-2} \left(x_k^T Q x_k + u_k^T R u_k \right) + T_{K-1} \\ T_{K-1} &= x_{K-1}^T Q x_{K-1} + u_{K-1}^T R u_{K-1} + x_k^T P_k x_k. \end{aligned}$$

Thus, the problem of choosing u_{K-1} is formally identical to the problem that we solved above for choosing u_k, and the same argument can be repeated at any time step recursively. At a general time k, the control input u_k given $r_k = i$ is given by

$$u_k = -\left(B^T P_{k+1} B + R \right)^{-1} B^T P_{k+1} A x_k,$$

where the matrix P_k evolves according to the backward Riccati recursion

$$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B \left(B^T P_{k+1} B + R \right)^{-1} B^T P_{k+1} A$$

with the final condition P_{k+1}. Moreover, the cost J_0 obtained through this procedure equals $x_0^T P_0 x_0$. □

For the infinite horizon case, we provide the solution without proof below.

Theorem 2.5. *Consider the infinite horizon LQR problem posed above. The optimal control law is a linear function of the state*

$$u_k = -\left(B^T P B + R \right)^{-1} B^T P A x_k,$$

where the matrix P is the unique positive semi-definite solution of the Riccati equation

$$P = A^T P A + Q - A^T P B \left(B^T P B + R \right)^{-1} B^T P A.$$

Moreover, the achieved cost is given by $x_0^T P x_0$.

2.4 LQG Problem

The finite horizon Linear Quadratic Guassian optimal control problem is posed as follows. Consider the process

$$x_{k+1} = A x_k + B u_k + w_k$$

with the initial condition x_0 as zero mean Gaussian, where $x_k \in \mathbf{R}^n$ is the state, $u_k \in \mathbf{R}^m$ is the control input that needs to be designed, and w_k is the process noise modeled Gaussian and white with mean zero and covariance $\Sigma_w > 0$. The process is observed using a sensor that generates measurements of the form

$$y_k = C x_k + v_k,$$

where the sensor noise v_k is modeled Gaussian and white with mean zero and covariance $\Sigma_v > 0$. The noise sequences $\{w(j)\}$, $\{v(j)\}$ and the initial condition x_0 are assumed to be independent. The cost function that needs to be minimized is

$$J_K = \sum_{k=0}^{K} \mathbb{E}[\left(x_k^T Q x_k + u_k^T R u_k\right)] + \mathbb{E}[x_{k+1}^T P_{k+1} x_{k+1}],$$

with $Q > 0$ and $R \geq 0$. The expectation is taken with respect to all the random parameters in the system. The controller at time k is allowed access to the measurements until time k and control inputs until time $k-1$. If the parameter K is finite, the problem is termed the finite horizon LQG problem. As $K \to \infty$, the cost would necessarily diverge. Thus, the infinite horizon LQG problem considers the cost

$$J_\infty = \lim_{K \to \infty} \frac{1}{K} J_K.$$

We shall assume that the par (A, B) is controllable and (A, C) is observable. In general, the results given below extend to the case when the matrices A, B, C, Q and R are time varying.

The solution to the finite horizon problem is provided by the separation principle. The principle essentially states that the optimal control input is calculated as the input in the LQR problem, but with the state x_k replaced by the minimum mean squared error (MMSE) estimate of the state x_k based on the measurements until time k and control inputs until time $k-1$. Note that the estimate can be calculated recursively through the Kalman filter. The following theorem summarizes the results.

Theorem 2.6. *Consider the finite horizon LQG problem posed above. The optimal control law is a linear function of the state*

$$u_k = -\left(B^T P_{k+1} B + R\right)^{-1} B^T P_{k+1} A \hat{x}_k,$$

where the matrix P_k evolves according to the backward Riccati recursion

$$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B \left(B^T P_{k+1} B + R\right)^{-1} B^T P_{k+1} A$$

with the final condition P_{k+1}, and $\hat{x}_k$ is the MMSE estimate of the state x_k based on the measurements until time k and control inputs until time $k-1$, calculated, e.g., using a Kalman filter.

Proof. The proof again follows dynamic programming arguments. We begin by rewriting the cost function J_K to identify terms in the cost that depend on x_k and u_k:

$$J_K = \mathbb{E}[\sum_{k=1}^{K-1} \left(x_k^T Q x_k + u_k^T R u_k\right)] + T_k$$
$$T_k = \mathbb{E}[x_k^T Q x_k + u_k^T R u_k + x_{k+1}^T P_{k+1} x_{k+1}].$$

The only term in the cost that can be affected by the choice of u_k is T_k. To choose u_k, we minimize T_k by a completion of squares argument. We obtain

$$\begin{aligned} T_k &= \mathbb{E}[x_k^T Q x_k + u_k^T R u_k + (A x_k + B u_k + w_k)^T P_{k+1} (A x_k + B u_k + w_k)] \\ &= \mathbb{E}[x_k^T P_k x_k + w_k^T P_{k+1} w_k + (u_k + S_k^{-1} B^T P_{k+1} A x_k)^T S_k (u_k + S_k^{-1} B^T P_{k+1} A x_k)], \end{aligned}$$

where we have used the fact that the process noise is white (hence w_k is independent of both x_k and u_k) and zero mean, and have defined

$$S_k = B^T P_{k+1} B + R$$
$$P_k = Q + A^T P_{k+1} A - A^T P_{k+1} B S_k^{-1} B^T P_{k+1} A.$$

Note that the controller does not have access to x_k and hence the quadratic term cannot be minimized to zero. Instead, the controller estimates (based on the measurements until time k and control inputs until time $k-1$) the term $S_k^{-1} B^T P_{k+1} A x_k$ in the MMSE sense. Thus, the optimal choice of u_k is

$$u_k = -S_k^{-1} B^T P_{k+1} \hat{x}_k.$$

Denote by $\Lambda_{e,k}$ the error covariance thus obtained. Since the controller utilizes all control inputs until time $K-1$ while calculating u_k, $\Lambda_{e,k}$ does not depend on the choice of control inputs $u_0, \cdots, u_{K-1}$. With the optimal choice of u_k, the term T_k reduces to $T_k = \mathbb{E}[x_k^T P_k x_k + w_k^T P_{k+1} w_k + \Lambda_{e,k}]$. Thus, the cost function J_K can be rewritten as

$$J_K = \mathbb{E}[\sum_{k=1}^{K-2} \left(x_k^T Q x_k + u_k^T R u_k\right)] + T_{K-1} + w_k^T P_{k+1} w_k + \Lambda_{e,k}$$
$$T_{K-1} = \mathbb{E}[x_{K-1}^T Q x_{K-1} + u_{K-1}^T R u_{K-1} + x_k^T P_k x_k].$$

Now note that the terms $w_k^T P_{k+1} w_k$ and $\Lambda_{e,k}$ are not impacted by the choice of $u[K-1]$ and hence may be dropped from the minimization problem. Thus, the problem of choosing $u[K-1]$ is formally identical to the problem that we solved above for choosing u_k, and the same argument can be repeated at any time step recursively. At a general time k, the control input u_k given $r[k] = i$ is given by

$$u_k = -\left(B^T P_{k+1} B + R\right)^{-1} B^T P_{k+1} A \hat{x}_k,$$

where the matrix P_k evolves according to the backward Riccati recursion

$$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B \left(B^T P_{k+1} B + R\right)^{-1} B^T P_{k+1} A$$

with the final condition P_{k+1}. □

The separation principle also holds for the infinite horizon case. In particular, under the assumptions above, both the backward Riccati recursion in the control calculation and the forward Riccati recursion in the Kalman filter are replaced by the corresponding Riccati equations.

2.5 Further Reading

Exercises

2.1 Show $\mathbb{E}[X|Y=y] = \frac{\sigma_x}{\sigma_x+\sigma_n} y$ in Example 2.1.