Feedback Systems:
Notes on Linear Systems Theory

Richard M. Murray
Control and Dynamical Systems
California Institute of Technology

DRAFT – Fall 2020

October 30, 2020

These notes are a supplement for the second edition of *Feedback Systems* by Åström and Murray (referred to as FBS2e), focused on providing some additional mathematical background and theory for the study of linear systems.

Chapter 5

State Estimation

Preliminary reading The material in this chapter extends the material Chapter 8 of FBS2e. Readers should be familiar with the material in Sections 8.1–8.3 in preparation for the more advanced topics discussed here.

5.1 Concepts and Definitions

Let $\mathcal{D} = (\mathcal{U}, \Sigma, \mathcal{Y}, s, r)$ be an input/output dynamical system on time range $\mathcal{T}$ with input/output map

$$\rho(t, t_0, x_0, u(\,\cdot\,)) = r(t, s(t, t_0, x_0, u(\,\cdot\,)), u(t)).$$

Let $x_0, z_0 \in \Sigma$ be different initial conditions and let $t_1, t_2 \in \mathcal{T}$ be two different times with $t_1 < t_2$. We write $\mathcal{U}_{[t_1,t_2]}$ to represent input signals that are restricted to the time range $[t_1, t_2] \in \mathcal{T}$.

Definition 5.1 (Distinguishability)**.** A control $u \in \mathcal{U}_{[t_1,t_2]}$ *distinguishes between* $x(t_0) = x_0$ *and* $x(t_0) = z_0$ if

$$\rho(t_2, t_0, x_0, u(\,\cdot\,)) \neq \rho(t_2, t_0, z_0, u(\,\cdot\,)).$$

The initial states (x_0, t_0) and (z_0, t_0) are *distinguishable on* $[t_1, t_2]$ if there exists $t \in [t_1, t_2]$ and input $u \in \mathcal{U}_{[t_1,t]}$ that distinguishes between x_0 and z_0 on $[t_0, t]$.

If a system is distinguishable for every pair of initial states $x_0, z_0 \in \Sigma$ then in principle it should be possible to find an input $u(\,\cdot\,)$ such that measurement of the input $u(\,\cdot\,)$ and output $y(\,\cdot\,)$ allows the initial state to be determined.

Definition 5.2 (Observability)**.** The system $\mathcal{D}$ is *observable on* $[t_1, t_2]$ if every initial state x_0 is distinguishable on $[t_1, t_2]$ from all other initial states $z_0 \neq x_0$. The system is *observable* if for every $t_1 < t_2$, $\mathcal{D}$ is observable on $[t_1, t_2]$.

If a system is observable then for any initial condition x_0 there exists an input $u(\,\cdot\,)$ that can be used to uniquely determine the initial condition by measurement of the output over an (arbitrary) interval of time. A process that measures u and y and returns x_0 is called an *observer.*. For many systems (including linear systems), the initial state can be determined for *any* input that is applied to the system. The definition of observability in FBS2e is specialized to the linear case and hence does not make use of the ability to choose the input $u(t)$.

For an observable system, it is possible to create a *state estimator* (also sometimes called an observer) that measures the input $u(t)$ and output $y(t)$ of the system and provides an estimate $\hat{x}(t)$ of the current state. To see how such an estimator might be constructed, consider a nonlinear system of the form

$$\frac{dx}{dt} = f(x, u), \qquad y = h(x).$$

A common form for an estimator is to construct a copy of the system dynamics and update the estimated state based on the error between the predicted output and the measured output:

$$\frac{d\hat{x}}{dt} = f(\hat{x}, u) + \alpha(y - h(\hat{x})).$$

We see that the estimator requires the current input u in order to update the estimated state to match the dynamics of the underlying model.

As in the case of reachability, it may not always be possible to observe the entire initial state, but only the initial state restricted to some subspace. However, if the unobservable portion of the state is asymptotically stable, then an estimator that converges to the current state can still be constructed since the unobservable portion of the state is converging to zero. This leads to the concept of *detectability* of a system, which is roughly the equivalent of stabilizability.

5.2 Observability for Linear State Space Systems

For linear systems we have that

$$\rho(t, t_0, x_0, u(\,\cdot\,)) = Ce^{At}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau.$$

It follows that whether two initial states are distinguishable is independent of the input $u(\,\cdot\,)$ since the effect of the input is the same for any initial condition. Hence observability depends only on the pair (A, C), and we say that the system is observable if the pair (A, C) is observable.

In FBS2e, a simple characterization of observability is given by considering the output $y(t) = Cx(t)$ and its derivatives, leading to the observability rank condition. A more insightful analysis is obtained by considering the linear operator $\mathcal{M}_T : \Sigma \to \mathcal{Y}$ given by

$$\big(\mathcal{M}_T(x_0)\big)(t) = Ce^{At}x_0.$$

The question of observability is equivalent to whether the map $\mathcal{M}_T$ is an injection (one-to-one) so that given any output $y(t)$ in the range of $\mathcal{M}_T$ there exists a unique x_0 such that $y(t) = (\mathcal{M}_T x_0)(t)$.

To characterize the injectivity of $\mathcal{M}_T$ we compute the adjoint operator $\mathcal{M}_T^* : \mathcal{Y}^* \to \Sigma^*$, which can be shown to be

$$\mathcal{M}_T^*(\sigma(\,\cdot\,)) = \int_{t_0}^{T} e^{A^\mathsf{T}\tau}C^\mathsf{T}\sigma(\tau)\,d\tau,$$

where $\sigma(\,\cdot\,) \in \mathcal{Y}^*$. The map $\mathcal{M}_T$ is injective if its rank is equal to the dimension of Σ, and the rank of $\mathcal{M}_T$ is equal to the rank of the operator $\mathcal{M}_T^*\mathcal{M}_T$, given by

$$W_\mathrm{o}(T) = \int_{t_0}^{T} e^{A^\mathsf{T}\tau}C^\mathsf{T}Ce^{A\tau}\,d\tau.$$

W_o is an $n \times n$ square, symmetric matrix. In the case that $W_o(T)$ is not full rank, the (right) null space of $W_o(T)$ gives the subspace of initial conditions whose values cannot be distinguished through measurement of the output. As in the case of reachability, it can be shown that the rank of $W_o(T)$ is independent of T.

To see how this definition of $\mathcal{M}_T^*$ arises, recall that for a linear space V we can define the dual space V^* of linear functions on V. If $V = \mathbb{R}^n$ then we can associate linear functionals $\alpha \in V^*$ with vectors and we write

$$\langle \alpha, x \rangle = \alpha^\mathsf{T} x,$$

where we think of x and α as a column vectors. Similarly, in the signal space $\mathcal{Y} : \mathcal{T} \to U \subset \mathbb{R}^m$, a dual element $\sigma \in \mathcal{Y}^*$ acts on a signal $y \in \mathcal{Y}$ according to the formula

$$\langle \sigma, y \rangle = \int_0^T \sigma^\mathsf{T}(\tau) y(\tau)\, d\tau.$$

Given a linear mapping $\mathcal{M}_T : \Sigma \to \mathcal{Y}$ defined as $(\mathcal{M}_T x_0)(t) = Ce^{At}x_0$, its dual $\mathcal{M}_T^* : \mathcal{Y}^* \to \Sigma^*$ must satisfy

$$\langle \mathcal{M}_T^* \sigma, x_0 \rangle = \langle \sigma, \mathcal{M}_T x_0 \rangle = \int_0^T \sigma^\mathsf{T}(\tau) C e^{A\tau} x_0\, d\tau,$$

and hence

$$\mathcal{M}_T^* \sigma = \int_{t_0}^T e^{A^\mathsf{T}\tau} C^\mathsf{T} \sigma(\tau)\, d\tau.$$

△

Given the input $u(\,\cdot\,)$ and the output $y(\,\cdot\,)$ on an interval $[t_0, T]$ the values of the initial state can be computed using W_o. Assume first that $u(\,\cdot\,) = 0$ so that the system dynamics are given by $y(t) = Ce^{At}x_0$. The value for x_0 is given by

$$x_0 = (\mathcal{M}_T^* \mathcal{M}_T)^{-1} \mathcal{M}_T^* y(\,\cdot\,) = \big(W_0(T)\big)^{-1} \int_{t_0}^T e^{A^\mathsf{T}\tau} C^\mathsf{T} y(\tau)\, d\tau.$$

In the case that the model is not correct and so the output trajectory is not in the range of $\mathcal{M}_T$, this estimate represents the best estimate (in a least squares sense) of the initial state. If the system has a nonzero input, then this estimate should be applied to the function

$$\tilde{y}(t) = y(t) - \int_{t_0}^t Ce^{A(t-\tau)} B u(\tau)\, d\tau$$

which removes the contribution of the input from the measured state $y(t)$.

If a linear system is stable, we can compute the *observability Gramian* $W_o \in \mathbb{R}^{n \times n}$:

$$W_o = \lim_{T \to \infty} W_o(T) = \int_{t_0}^\infty e^{A^\mathsf{T}\tau} C^\mathsf{T} C e^{A\tau}\, d\tau.$$

A (stable) linear system is observable if and only if the observabilty Gramian has rank n. The observability Gramian can be computed using linear algebra:

Theorem 5.1. *$A^\mathsf{T} W_o + W_o A = -C^\mathsf{T} C$.*

As in the case of reachability, there are a number of equivalent conditions for observability of a linear system.

Theorem 5.2 (Observability conditions)**.** *A linear system with dynamics matrix A and output matrix C is observable if an only if the following equivalent conditions hold:*

1. *$W_o(T)$ has rank n for any $T > 0$.*

2. *W_o has rank n (requires A stable).*

3. *Popov-Bellman-Hautus (PBH) test:*

$$\text{rank} \begin{bmatrix} C \\ sI - A \end{bmatrix} = n$$

for all $s \in \mathbb{C}$ (suffices to check for eigenvalue of A).

4. *Observability rank test:*

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

5.3 Combining Estimation and Control

As discussed in FBS2e, a state estimator can be combined with a state feedback controller to design the dynamics of a reachable and observable linear system. In the case that a system is not reachable and/or not observable, the ability to stabilize the system to an equilibrium point will depend on whether the unreachable modes have unstable eigenvalues and the ability to detect the system state around an equilibrium will depend on whether the unobservable modes have stable eigenvalues. The *Kalman decomposition*, described briefly in FBS2e, Section 8.3, can be used to understand the structure of the system. We expand on this description here by describing some of the properties of the subspaces in the Kalman decomposition.

Consider a linear system in standard form

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

Assume that the system is neither reachable nor observable. As we saw in Section 3.2, if a system is not reachable then there exists a subspace E_{r} of reachable states. It can be shown that this subspace is A invariant (easy proof: look at the structure of A in Theorem 3.6) and $E_{\mathrm{r}} = \operatorname{range} W_{\mathrm{r}}$. Similarly, we see from the analysis above that there exists a subspace $E_{\overline{\mathrm{o}}}$ of states that are not observable, consisting of states in the null space of $W_{\mathrm{o}}(T)$. This subspace is also A invariant (Exercise 5.4).

Another characterization of E_{r} and $E_{\overline{\mathrm{o}}}$ is that E_{r} is the smallest A-invariant subspace containing B, which is equivalent to $\operatorname{range} W_{\mathrm{c}}$. Similarly, $E_{\overline{\mathrm{o}}}$ is the largest A-invariant subspace that annihilates C. We can now decompose the state space as $\mathbb{R}^n = E_{\mathrm{ro}} \oplus E_{\mathrm{r}\overline{\mathrm{o}}} \oplus E_{\overline{\mathrm{r}}\mathrm{o}} \oplus E_{\overline{\mathrm{r}}\overline{\mathrm{o}}}$, where $A \oplus B = \{x + y : x \in A, y \in B\}$. Define $E_{\mathrm{r}\overline{\mathrm{o}}} = E_{\mathrm{r}} \cap E_{\overline{\mathrm{o}}}$. This space is uniquely defined and is A invariant.

Using these subspaces, it is possible to construct a transformation that extends the decomposition in Theorem 3.6 to account for both reachability and observability.

Theorem 5.3 (Kalman decomposition)**.** *Let (A, B, C) represent an input/output linear system with n states and define $r = \mathrm{rank}\, W_c \leq n$ and $q = \mathrm{nullity}\, W_o \leq n$. Then there exists a transformation $T \in \mathbb{R}^{n\times n}$ such that the dynamics can be transformed into the block matrix form*

$$\begin{aligned}\frac{dx}{dt} &= \begin{bmatrix} A_{ro} & 0 & * & 0 \\ * & A_{r\overline{o}} & * & * \\ 0 & 0 & A_{\overline{r}o} & 0 \\ 0 & 0 & * & A_{\overline{r}\overline{o}} \end{bmatrix} x + \begin{bmatrix} B_{ro} \\ B_{r\overline{o}} \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_{ro} & 0 & C_{\overline{r}o} & 0 \end{bmatrix} x,\end{aligned}$$

where $*$ represents non-zero elements. Furthermore, in this basis:

$$E_{ro} = \mathrm{span} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad E_{r\overline{o}} = \mathrm{span} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}, \qquad E_{\overline{r}o} = \mathrm{span} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}, \qquad E_{\overline{r}\overline{o}} = \mathrm{span} \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$

Proof. (Sketch) Choose E_{ro} such that $E_{\mathrm{r}} = E_{\mathrm{ro}} \oplus E_{\mathrm{r}\overline{\mathrm{o}}}$. This space is not uniquely defined and is not A invariant, but can be constructed by completing the basis for E_{r}. Choose $E_{\overline{\mathrm{r}}\overline{\mathrm{o}}}$ such that $E_{\overline{\mathrm{o}}} = E_{\mathrm{r}\overline{\mathrm{o}}} \oplus E_{\overline{\mathrm{r}}\overline{\mathrm{o}}}$. This space is also not uniquely defined nor is it A invariant, but can be constructed by completing the basis for $E_{\overline{\mathrm{o}}}$. Finally, choose $E_{\overline{\mathrm{r}}\mathrm{o}}$ such that $\mathbb{R}^n = E_{\mathrm{ro}} \oplus E_{\mathrm{r}\overline{\mathrm{o}}} \oplus E_{\overline{\mathrm{r}}\mathrm{o}} \oplus E_{\overline{\mathrm{r}}\overline{\mathrm{o}}}$. $E_{\overline{\mathrm{r}}\mathrm{o}}$ is not uniquely defined and is not A invariant, but can be constructed by completing the basis for $\mathbb{R}^n$. The transformation T is constructed by using the basis elements for each subspace and rewriting the dynamics in terms of this basis. $\square$

While the Kalman decomposition allows us to identify subspaces corresponding to reachable and observable states, it does not give us any indications of which states are difficult to reach and/or difficult to reach. For control systems we are particularly interested in those states that are difficult to control and also do not significantly influence the output (so they are hard to observe) since those states can be discounted during the control design process.

Consider a system (A, B, C) with controllability Gramian W_{c} and observability Gramian W_{o}. We define the *Hankel singular values* as the eigenvalues of the matrix $W_{\mathrm{co}} := W_{\mathrm{c}} W_{\mathrm{o}}$. Intuitively, a system (or subsystem) with small Hankel singular values correspond to a situation in which it is difficult to observe a state and/or difficult to control that state. It turns out that it is possible to find a state transformation in which this intuition can be made more precise.

Theorem 5.4. *Suppose that W_c and W_o are such that they have independent eigenspaces (either no shared eigenvalues or shared eigenvalues have no mutual (generalized) eigenvalues. Then there exists a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that in the transformed set of coordinates $z = Tx$,*

$$\tilde{W}_c = \tilde{W}_o = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{bmatrix},$$

where $\{\sigma_i\}$ are the Hankel singular values for the system.

Proof. (sketch) It can be shown that in transformed coordinates the controllability and observability Gramians are given by

$$\tilde{W}_{\mathrm{c}} = T^{-1} W_{\mathrm{c}} (T^{\mathsf{T}})^{-1}, \qquad \tilde{W}_{\mathrm{o}} = T^{\mathsf{T}} W_{\mathrm{c}} T.$$

We seek to find a transformation such that

$$\tilde{W}_\mathrm{c}\tilde{W}_\mathrm{o} = T^{-1}W_\mathrm{c}W_\mathrm{o}T = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n^2 \end{bmatrix} =: \Sigma^2. \tag{S5.1}$$

Since W_o is symmetric and positive definite, it follows that it can be written as $W_\mathrm{o} = R^\mathsf{T} R$ where $R \in \mathbb{R}^{n\times n}$ is invertible. We can now write equation (S5.1) as

$$(RT)^{-1}RW_\mathrm{c}R^\mathsf{T}(RT) = \Omega^2, \tag{S5.2}$$

from which it follows that RW_c is diagonalizable and furthermore there exists an orthogonal transformation U with $U^\mathsf{T}U = I$ such that

$$RW_\mathrm{c}R^\mathsf{T} = U\Omega^2U^\mathsf{T}.$$

Finally, it can be shown that

$$T = R^{-1}U\Sigma^{\frac{1}{2}}, \qquad T^{-1} = \Sigma^{-\frac{1}{2}}U^\mathsf{T}R.$$

$\square$

In this representation we can define a set of subspaces Σ_i corresponding to each singular value $\sigma_i > 0$. If we assume the σ_i's are ordered from largest to smallest, then these subspaces correspond to the most controllable/observable to the least controllable/observable.

5.4 Exercises

5.1 (FBS 8.1) Consider the multi-input, multi-output system given by

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$. Show that the states can be determined from the input u and the output y and their derivatives if the observability matrix W_o given by equation (8.4) has n independent rows.

5.2 (FBS 8.2) Consider a system under a coordinate transformation $z = Tx$, where $T \in \mathbb{R}^{n\times n}$ is an invertible matrix. Show that the observability matrix for the transformed system is given by $\widetilde{W}_\mathrm{o} = W_\mathrm{o}T^{-1}$ and hence observability is independent of the choice of coordinates.

5.3 (FBS 8.3) Show that the system depicted in Figure 8.2 is not observable.

5.4 Show that the set of unobservable states for a linear system with dynamics matrix A and output matrix C is an A-invariant subspace and that it is equal to the largest A-invariant subspace annihilated by C.

5.5 (FBS 8.4) Show that if a system is observable, then there exists a change of coordinates $z = Tx$ that puts the transformed system into observable canonical form.

5.6 (FBS 8.9) Consider the linear system (8.2), and assume that the observability matrix W_o is invertible. Show that

$$\hat{x} = W_o^{-1} \begin{bmatrix} y & \dot{y} & \ddot{y} & \cdots & y^{(n-1)} \end{bmatrix}^T$$

is an observer. Show that it has the advantage of giving the state instantaneously but that it also has some severe practical drawbacks.

5.7 (FBS 8.15) Consider a linear system characterized by the matrices

$$A = \begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -3 & 0 & 2 \\ 1 & 1 & -4 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \quad D = 0.$$

Construct a Kalman decomposition for the system. (Hint: Try to diagonalize.)

5.8 Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix} x + \begin{bmatrix} 3 \\ 7 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & -1 \end{bmatrix} x.$$

Transform the system to observable canonical form.

5.9 Consider a control system having state space dynamics

$$\frac{dx}{dt} = \begin{bmatrix} -\alpha - \beta & 1 \\ -\alpha\beta & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(a) Construct an observer for the system and find expressions for the observer gain $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$ such that the observer has natural frequency ω_0 and damping ratio ζ.

(b) Suppose that we choose a different output

$$\tilde{y} = \begin{bmatrix} 1 & \gamma \end{bmatrix} x.$$

Are there any values of γ for which the system is *not* observable? If so, provide an example of an initial condition and output where it is not possible to uniquely determine the state of the system by observing its inputs and outputs.

5.10 Show that the design of an observer by eigenvalue assignment is unique for single-output systems. Construct examples that show that the problem is not necessarily unique for systems with many outputs.

5.11 Consider the normalized double integrator described by

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Cx$$

Determine an observer and find the observer gain that gives dynamics characterized by the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.