Feedback Systems

An Introduction for Scientists and Engineers

SECOND EDITION

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Chapter 7

State Feedback

Intuitively, the state may be regarded as a kind of information storage or memory or accumulation of past causes. We must, of course, demand that the set of internal states Σ be sufficiently rich to carry all information about the past history of Σ to predict the effect of the past upon the future. We do not insist, however, that the state is the least *such information although this is often a convenient assumption.*

R. E. Kalman, P. L. Falb, and M. A. Arbib, *Topics in Mathematical System Theory*, 1969 [KFA69].

This chapter describes how the feedback of a system's state can be used to shape the local behavior of a system. The concept of reachability is introduced and used to investigate how to design the dynamics of a system through assignment of its eigenvalues. In particular, we show that under certain conditions it is possible to assign the system eigenvalues arbitrarily by appropriate feedback of the system state.

7.1 Reachability

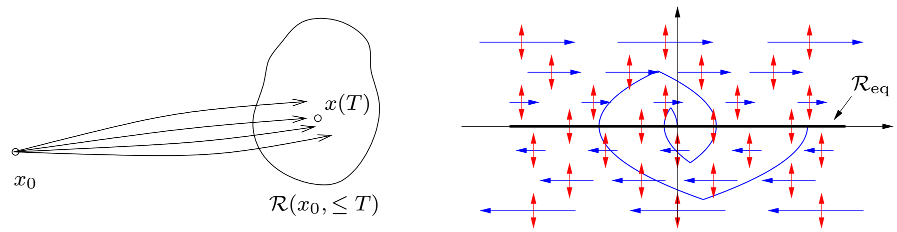
One of the fundamental properties of a control system is what set of points in the state space can be reached through the choice of a control input. It turns out that the property of reachability is also fundamental in understanding the extent to which feedback can be used to design the dynamics of a system.

Definition of Reachability

We begin by disregarding the output measurements of the system and focusing on the evolution of the state, given by

$$\frac{dx}{dt} = Ax + Bu, \tag{7.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, A is an $n \times n$ matrix, and B a column vector. A fundamental question is whether it is possible to find control signals so that any point in the



(a) Reachable set (b) Reachability through control

Figure 7.1: The reachable set for a control system. The set $\mathcal{R}(x_0, \leq T)$ shown in (a) is the set of points reachable from x_0 in time less than T. The phase portrait in (b) shows the dynamics for a double integrator, with the natural dynamics drawn as horizontal arrows and the control inputs drawn as vertical arrows. The set of achievable equilibrium points is the x axis. By setting the control inputs as a function of the state, it is possible to steer the system to the origin, as shown on the sample path.

state space can be reached through some choice of input. To study this, we define the *reachable set* $\mathcal{R}(x_0, \leq T)$ as the set of all points x_f such that there exists an input $u(t)$, $0 \leq t \leq T$ that steers the system from $x(0) = x_0$ to $x(T) = x_f$, as illustrated in Figure 7.1a.

Definition 7.1 (Reachability)**.** A linear system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u\colon [0, T] \to \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.

The definition of reachability addresses whether it is possible to reach all points in the state space in a *transient* fashion. In many applications, the set of points that we are most interested in reaching is the set of equilibrium points of the system (since we can remain at those points with constant input u). The set of all possible equilibrium points for constant controls is given by

$$\mathcal{R}_{\text{eq}} = \{x_e : Ax_e + Bu_e = 0 \text{ for some } u_e \in \mathbb{R}\}.$$

This means that possible equilibrium points lie in a one- (or possibly higher) dimensional subspace. If the matrix A is invertible, this subspace is one-dimensional and is spanned by $A^{-1}B$.

The following example provides some insight into the possibilities.

Example 7.1 Double integrator

Consider a linear system consisting of a double integrator whose dynamics are given by

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = u.$$

Figure 7.1b shows a phase portrait of the system. The open loop dynamics ($u = 0$) are shown as horizontal arrows pointed to the right for $x_2 > 0$ and to the left for $x_2 < 0$. The control input is represented by a double-headed arrow in the vertical direction, corresponding to our ability to set the value of $\dot{x}_2$. The set of equilibrium points $\mathcal{E}$ corresponds to the x_1 axis, with $u_e = 0$.

Suppose first that we wish to reach the origin from an initial condition $(a, 0)$. We can directly move the state up and down in the phase plane, but we must rely on the natural dynamics to control the motion to the left and right. If $a > 0$, we can move toward the origin by first setting $u < 0$, which will cause x_2 to become negative. Once $x_2 < 0$, the value of x_1 will begin to decrease and we will move to the left. After a while, we can set u to be positive, moving x_2 back toward zero and slowing the motion in the x_1 direction. If we bring x_2 to a positive value, we can move the system state in the opposite direction.

Figure 7.1b shows a sample trajectory bringing the system to the origin. Note that if we steer the system to an equilibrium point, it is possible to remain there indefinitely (since $\dot{x}_1 = 0$ when $x_2 = 0$), but if we go to a point in the state space with $x_2 \neq 0$, we can pass through the point only in a transient fashion. ∇

To find general conditions under which a linear system is reachable, we will first give a heuristic argument based on formal calculations with impulse functions. We note that if we can reach all points in the state space through some choice of input, then we can also reach all equilibrium points.

Testing for Reachability

When the initial state is zero, the response of the system (7.1) to an input $u(t)$ is given by

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau. \tag{7.2}$$

If we choose the input to be a impulse function $\delta(t)$ as defined in Section 6.3, the state becomes

$$x_\delta(t) = \int_0^t e^{A(t-\tau)} B \delta(\tau)\, d\tau = e^{At} B.$$

(Note that the state changes instantaneously in response to the impulse.) We can find the response to the derivative of an impulse function by taking the derivative of the impulse response (Exercise 6.1):

$$x_{\dot{\delta}}(t) = \frac{dx_\delta}{dt} = A e^{At} B.$$

Continuing this process and using the linearity of the system, the input

$$u(t) = \alpha_1 \delta(t) + \alpha_2 \dot{\delta}(t) + \alpha_3 \ddot{\delta}(t) + \cdots + \alpha_n \delta^{(n-1)}(t)$$

gives the state

$$x(t) = \alpha_1 e^{At} B + \alpha_2 A e^{At} B + \alpha_3 A^2 e^{At} B + \cdots + \alpha_n A^{n-1} e^{At} B.$$

Taking the limit as t goes to zero through positive values, we get

$$\lim_{t \to 0+} x(t) = \alpha_1 B + \alpha_2 A B + \alpha_3 A^2 B + \cdots + \alpha_n A^{n-1} B.$$

On the right is a linear combination of the columns of the matrix

$$W_r = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}. \tag{7.3}$$

To reach an arbitrary point in the state space, we thus require that W_r has n independent columns (full rank). The matrix W_r is called the *reachability matrix* and it is full rank if and only if its determinant is nonzero.

Although we have only considered the scalar input case, it turns out that this same test works in the multi-input case, where we require that W_r be full column rank (have n linearly independent columns). In addition, it can be shown that only the terms up to $A^{n-1}B$ must be computed; additional terms add no new directions to W_r (see Exercise 7.3).

An input consisting of a sum of impulse functions and their derivatives is a very violent signal. To see that an arbitrary point can be reached with smoother signals we can make use of the convolution equation. Assuming that the initial condition is zero, the state of a linear system is given by

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \int_0^t e^{A\tau} B u(t-\tau) d\tau.$$

It follows from the theory of matrix functions, specifically the Cayley–Hamilton theorem (Exercise 7.3), that

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau),$$

where $\alpha_i(\tau)$ are scalar functions, and we find that

$$\begin{aligned} x(t) = B \int_0^t \alpha_0(\tau) u(t-\tau)\, d\tau + AB \int_0^t \alpha_1(\tau) u(t-\tau)\, d\tau \\ + \cdots + A^{n-1}B \int_0^t \alpha_{n-1}(\tau) u(t-\tau)\, d\tau. \end{aligned}$$

Again we observe that the right-hand side is a linear combination of the columns of the reachability matrix W_r given by equation (7.3). This basic approach leads to the following theorem.

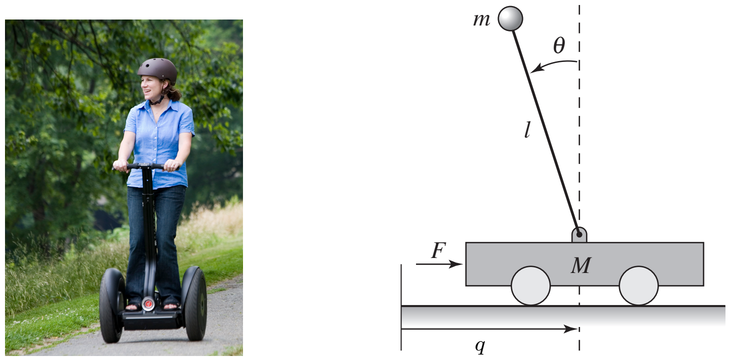
Theorem 7.1 (Reachability rank condition)**.** *A linear system of the form* (7.1) *is reachable if and only if the reachability matrix W_r is invertible (full column rank).*

The formal proof of this theorem is beyond the scope of this text but follows along the lines of the previous sketch and can be found in most books on linear control theory, such as Callier and Desoer [CD91] or Lewis [Lew03]. It is also interesting to note that Theorem 7.1 makes no mention of the time T that was in our definition of reachability. For a linear system, it turns out that we can find an input taking x_0 to x_f for *any* $T > 0$, though the size of the input required can be very large when T is very small.

We illustrate the concept of reachability with the following example.

Example 7.2 Balance system

Consider the balance system introduced in Example 3.2 and shown in Figure 7.2.



(a) Segway (b) Cart–pendulum system

Figure 7.2: Balance system. The Segway® Personal Transporter shown in (a) is an example of a balance system that uses torque applied to the wheels to keep the rider upright. A simplified diagram for a balance system is shown in (b). The system consists of a mass m on a rod of length l connected by a pivot to a cart with mass M.

Recall that this system is a model for a class of examples in which the center of mass is balanced above a pivot point. One example is the Segway® Personal Transporter shown in Figure 7.2a, about which a natural question to ask is whether we can move from one stationary point to another by appropriate application of forces through the wheels.

The nonlinear equations of motion for the system are given in equation (3.9) and repeated here:

$$\begin{aligned}(M+m)\ddot{q} - ml\cos\theta\,\ddot{\theta} &= -c\dot{q} - ml\sin\theta\,\dot{\theta}^2 + F,\\ (J+ml^2)\ddot{\theta} - ml\cos\theta\,\ddot{q} &= -\gamma\dot{\theta} + mgl\sin\theta.\end{aligned} \tag{7.4}$$

For simplicity, we take $c = \gamma = 0$. Linearizing around the equilibrium point $x_e = (0,0,0,0)$, the dynamics matrix and the control matrix are

$$A = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & m^2l^2g/\mu & 0 & 0\\ 0 & M_t mgl/\mu & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0\\ 0\\ J_t/\mu\\ lm/\mu \end{pmatrix},$$

where $\mu = M_t J_t - m^2 l^2$, $M_t = M + m$, and $J_t = J + ml^2$. The reachability matrix is

$$W_r = \begin{pmatrix} 0 & J_t/\mu & 0 & gl^3m^3/\mu^2\\ 0 & lm/\mu & 0 & gl^2m^2M_t/\mu^2\\ J_t/\mu & 0 & gl^3m^3/\mu^2 & 0\\ lm/\mu & 0 & gl^2m^2M_t/\mu^2 & 0 \end{pmatrix}. \tag{7.5}$$

To compute the determinant we permute the first and the last columns of the matrix W_r and use the fact that such a permutation changes the determinant by a

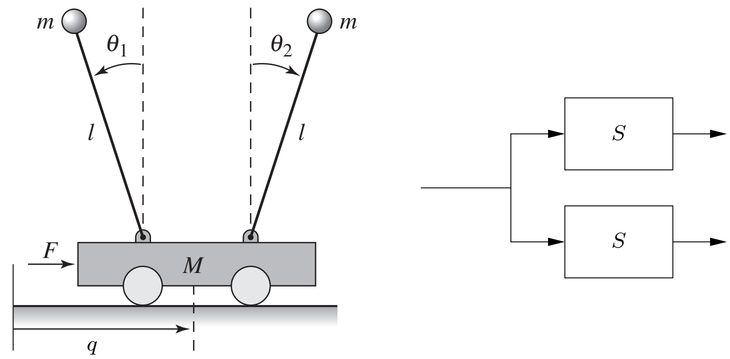


Figure 7.3: An unreachable system. The cart–pendulum system shown on the left has a single input that affects two pendula of equal length and mass. Since the forces affecting the two pendula are the same and their dynamics are identical, it is not possible to arbitrarily control the state of the system. The figure on the right is a block diagram representation of this situation.

factor of -1. This gives a block diagonal matrix with two identical blocks and the determinant becomes

$$\det(W_\mathrm{r}) = -\Big(\frac{gl^4m^4}{\mu^3} - \frac{gl^2m^2J_\mathrm{t}M_\mathrm{t}}{\mu^3}\Big)^2 = -\frac{g^2l^4m^4}{\mu^6}(MJ + mJ + Mml^2)^2,$$

and we can conclude that the system is reachable. This implies that we can move the system from any initial state to any final state and, in particular, that we can always find an input to bring the system from an initial state to an equilibrium point. ∇

It is useful to have an intuitive understanding of the mechanisms that make a system unreachable. An example of such a system is given in Figure 7.3. The system consists of two identical systems with the same input. We cannot separately cause the first and the second systems to do something different since they have the same input. Hence we cannot reach arbitrary states, and so the system is not reachable (Exercise 7.4).

More subtle mechanisms for nonreachability can also occur. For example, if there is a linear combination of states that always remains constant, then the system is not reachable. To see this, suppose that there exists a row vector H such that

$$0 = \frac{d}{dt}Hx = H(Ax + Bu), \quad \text{for all } x \text{ and } u.$$

Then H is in the left null space of both A and B and it follows that

$$HW_\mathrm{r} = H\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = 0.$$

Hence the reachability matrix is not full rank. In this case, if we have an initial condition x_0 and we wish to reach a state x_f for which $Hx_0 \neq Hx_\mathrm{f}$, then since $Hx(t)$ is constant, no input u can move the state from x_0 to x_f.

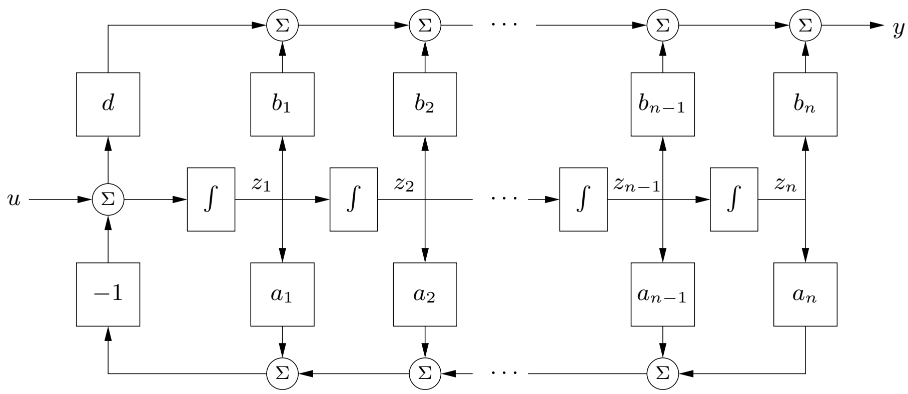


Figure 7.4: Block diagram for a system in reachable canonical form. The individual states of the system are represented by a chain of integrators whose input depends on the weighted values of the states. The output is given by an appropriate combination of the system input and other states.

Reachable Canonical Form

As we have already seen in previous chapters, it is often convenient to change coordinates and write the dynamics of the system in the transformed coordinates $z = Tx$. One application of a change of coordinates is to convert a system into a canonical form in which it is easy to perform certain types of analysis.

A linear state space system is in *reachable canonical form* if its dynamics are given by

$$\begin{aligned} \frac{dz}{dt} &= \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & 0 & & & \\ & 1 & 0 & & 0 \\ & 0 & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix} z + du. \end{aligned} \tag{7.6}$$

A block diagram for a system in reachable canonical form is shown in Figure 7.4. We see that the coefficients that appear in the A and B matrices show up directly in the block diagram. Furthermore, the output of the system is a simple linear combination of the outputs of the integration blocks.

The characteristic polynomial for a system in reachable canonical form is given by

$$\lambda(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n. \tag{7.7}$$

The reachability matrix also has a relatively simple structure:

$$\tilde{W}_{\mathrm{r}} = \begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix} = \begin{pmatrix} 1 & -a_1 & a_1^2 - a_2 & & \\ 0 & 1 & -a_1 & & * \\ & & \ddots & \ddots & \\ & 0 & & 1 & -a_1 \\ & & & & 1 \end{pmatrix},$$

where $*$ indicates a possibly nonzero term and we use a tilde to remind us that A and B are in a special form. The matrix W_{r} is full rank since no column can be written as a linear combination of the others because of the triangular structure of the matrix.

We now consider the problem of finding a change of coordinates such that the dynamics of a system can be written in reachable canonical form. Let A, B represent the dynamics of a given system and $\tilde{A}, \tilde{B}$ be the dynamics in reachable canonical form. Suppose that we wish to transform the original system into reachable canonical form using a coordinate transformation $z = Tx$. As shown in the previous chapter, the dynamics matrix and the control matrix for the transformed system are

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB.$$

The reachability matrix for the transformed system then becomes

$$\tilde{W}_{\mathrm{r}} = \begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix}.$$

Transforming each element individually, we have

$$\begin{aligned}
\tilde{A}\tilde{B} &= TAT^{-1}TB = TAB,\\
\tilde{A}^2\tilde{B} &= (TAT^{-1})^2TB = TAT^{-1}TAT^{-1}TB = TA^2B,\\
&\vdots\\
\tilde{A}^n\tilde{B} &= TA^nB,
\end{aligned}$$

and hence the reachability matrix for the transformed system is

$$\tilde{W}_{\mathrm{r}} = T\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = TW_{\mathrm{r}}. \tag{7.8}$$

If W_{r} is invertible, we can thus solve for the transformation T that takes the system into reachable canonical form:

$$T = \tilde{W}_{\mathrm{r}}W_{\mathrm{r}}^{-1}.$$

The following example illustrates the approach.

Example 7.3 Transformation to reachable form
Consider a simple two-dimensional system of the form

$$\frac{dx}{dt} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

We wish to find the transformation that converts the system into reachable canonical form:

$$\tilde{A} = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The coefficients a_1 and a_2 can be determined from the characteristic polynomial for the original system:

$$\lambda(s) = \det(sI - A) = s^2 - 2\alpha s + (\alpha^2 + \omega^2) \quad \Longrightarrow \quad \begin{aligned} a_1 &= -2\alpha,\\ a_2 &= \alpha^2 + \omega^2. \end{aligned}$$

The reachability matrix for each system is

$$W_r = \begin{pmatrix} 0 & \omega \\ 1 & \alpha \end{pmatrix}, \qquad \tilde{W}_r = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix}.$$

The transformation T becomes

$$T = \tilde{W}_r W_r^{-1} = \begin{pmatrix} -(a_1+\alpha)/\omega & 1 \\ 1/\omega & 0 \end{pmatrix} = \begin{pmatrix} \alpha/\omega & 1 \\ 1/\omega & 0 \end{pmatrix},$$

and hence the coordinates

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tx = \begin{pmatrix} \alpha x_1/\omega + x_2 \\ x_1/\omega \end{pmatrix}$$

put the system in reachable canonical form. ∇

We summarize the results of this section in the following theorem.

Theorem 7.2 (Reachable canonical form)**.** *Let A and B be the dynamics and control matrices for a reachable system and suppose that the characteristic polynomial for A is given by*

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n.$$

Then there exists a transformation $z = Tx$ such that in the transformed coordinates the dynamics and control matrices are in reachable canonical form (7.6)*.*

One important implication of this theorem is that for any reachable system, we can assume without loss of generality that the coordinates are chosen such that the system is in reachable canonical form. This is particularly useful for proofs, as we shall see later in this chapter. However, for high-order systems, small changes in the coefficients a_i can give large changes in the eigenvalues. Hence, the reachable canonical form is not always well conditioned and must be used with some care.

7.2 Stabilization by State Feedback

The state of a dynamical system is a collection of variables that permits prediction of the future evolution of a system given its future inputs. We now explore the idea of designing the dynamics of a system through feedback of the state. We will assume that the system to be controlled is described by a linear state model and has a single input (for simplicity). The feedback control law will be developed step by step using a single idea: the positioning of closed loop eigenvalues in desired locations.

State Space Controller Structure

Figure 7.5 is a diagram of a typical control system using state feedback. The full system consists of the process dynamics, which we take to be linear, the controller elements K and k_f, the reference input (or command signal) r, and process disturbances v. The goal of the feedback controller is to regulate the output of the

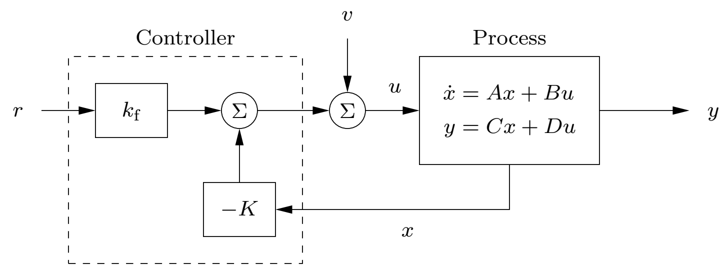


Figure 7.5: A feedback control system with state feedback. The controller uses the system state x and the reference input r to command the process through its input u. We model disturbances via the additive input v.

system y such that it tracks the reference input in the presence of disturbances and also uncertainty in the process dynamics.

An important element of the control design is the performance specification. The simplest performance specification is that of stability: given a constant reference r and in the absence of any disturbances, we would like the equilibrium point of the system to be asymptotically stable. More sophisticated performance specifications typically involve giving desired properties of the step or frequency response of the system, such as specifying the desired rise time, overshoot, and settling time of the step response. Finally, we are often concerned with the disturbance attenuation properties of the system: to what extent can we experience disturbance inputs v and still hold the output y near the desired value?

Consider a system described by the linear differential equation

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{7.9}$$

where we have ignored the disturbance signal v for now. Our goal is to drive the output y to a given reference value r and hold it there.

We begin by assuming that all components of the state vector are measured. Since the state at time t contains all the information necessary to predict the future behavior of the system, the most general time-invariant control law is a function of the state and the reference input:

$$u = \alpha(x, r).$$

If the control law is restricted to be linear, it can be written as

$$u = -Kx + k_f r, \tag{7.10}$$

where r is the reference value, assumed for now to be a constant.

This control law corresponds to the structure shown in Figure 7.5. The negative sign is a convention to indicate that negative feedback is the normal situation. The term $k_f r$ represents a feedforward signal from the reference to the control. The closed loop system obtained when the feedback (7.10) is applied to the system (7.9) is given by

$$\frac{dx}{dt} = (A - BK)x + Bk_f r. \tag{7.11}$$

We attempt to determine the feedback gain K so that the closed loop system has the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1} s + p_n. \tag{7.12}$$

This control problem is called the *eigenvalue assignment problem* or *pole placement problem* (we will define poles more formally in Chapter 9).

Note that k_f does not affect the stability of the system (which is determined by the eigenvalues of $A - BK$) but does affect the steady-state solution. In particular, the equilibrium point and steady-state output for the closed loop system are given by

$$x_\mathrm{e} = -(A - BK)^{-1} B k_\mathrm{f} r, \qquad y_\mathrm{e} = C x_\mathrm{e} + D u_\mathrm{e},$$

and hence k_f should be chosen such that $y_\mathrm{e} = r$ (the desired output value). Since k_f is a scalar, we can easily solve to show that if $D = 0$ (the most common case),

$$k_\mathrm{f} = -1/\big(C(A - BK)^{-1}B\big). \tag{7.13}$$

Notice that k_f is exactly the inverse of the zero frequency gain of the closed loop system. The solution for $D \neq 0$ is left as an exercise.

Using the gains K and k_f, we are thus able to design the dynamics of the closed loop system to satisfy our goal. To illustrate how to construct such a state feedback control law, we begin with a few examples that provide some basic intuition and insights.

Example 7.4 Vehicle steering

In Example 6.13 we derived a normalized linear model for vehicle steering. The dynamics describing the lateral deviation were given by the normalized dynamics

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \gamma \\ 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = 0,$$

where $\gamma = a/b$ is the ratio of the distance between the center of mass and the rear wheel, a, and the wheelbase b. We want to design a controller that stabilizes the dynamics and tracks a given reference value r of the lateral position of the vehicle. To do this we introduce the feedback

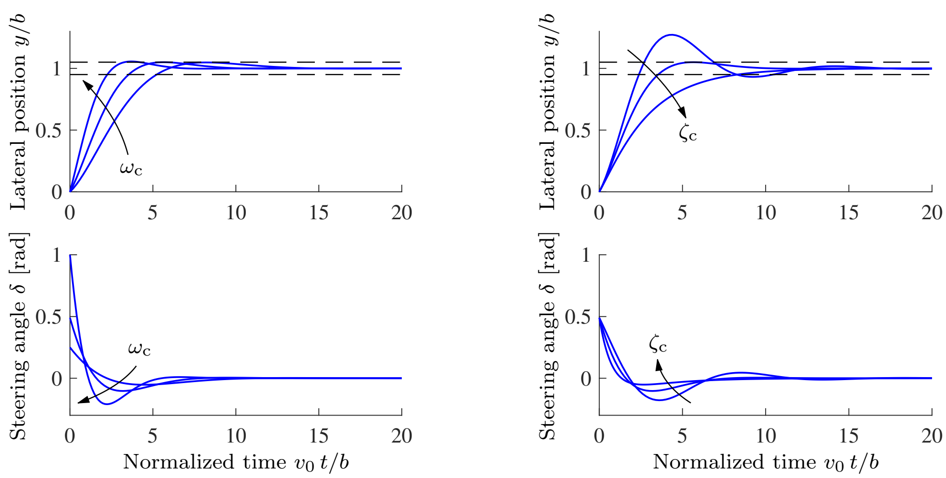
$$u = -Kx + k_\mathrm{f} r = -k_1 x_1 - k_2 x_2 + k_\mathrm{f} r,$$

and the closed loop system becomes

$$\begin{aligned} \frac{dx}{dt} &= (A - BK)x + B k_\mathrm{f} r = \begin{pmatrix} -\gamma k_1 & 1 - \gamma k_2 \\ -k_1 & -k_2 \end{pmatrix} x + \begin{pmatrix} \gamma k_\mathrm{f} \\ k_\mathrm{f} \end{pmatrix} r, \\ y &= Cx + Du = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \end{aligned} \tag{7.14}$$

The closed loop system has the characteristic polynomial

$$\det(sI - A + BK) = \det \begin{pmatrix} s + \gamma k_1 & \gamma k_2 - 1 \\ k_1 & s + k_2 \end{pmatrix} = s^2 + (\gamma k_1 + k_2)s + k_1.$$



(a) Unit step response for varying ω_c

(b) Unit step response for varying ζ_c

Figure 7.6: State feedback control of a steering system. Unit step responses (from zero initial condition) obtained with controllers designed with $\zeta_c = 0.7$ and $\omega_c = 0.5$, 0.7, and 1 [rad/s] are shown in (a). The dashed lines indicate $\pm 5\%$ deviations from the setpoint. Notice that response speed increases with increasing ω_c, but that large ω_c also give large initial control actions. Unit step responses obtained with a controller designed with $\omega_c = 0.7$ and $\zeta_c = 0.5$, 0.7, and 1 are shown in (b).

Suppose that we would like to use feedback to design the dynamics of the system to have the characteristic polynomial

$$p(s) = s^2 + 2\zeta_c\omega_c s + \omega_c^2.$$

Comparing this polynomial with the characteristic polynomial of the closed loop system, we see that the feedback gains should be chosen as

$$k_1 = \omega_c^2, \qquad k_2 = 2\zeta_c\omega_c - \gamma\omega_c^2.$$

Equation (7.13) gives $k_f = k_1 = \omega_c^2$, and the control law can be written as

$$u = k_1(r - x_1) - k_2 x_2 = \omega_c^2(r - x_1) - (2\zeta_c\omega_c - \gamma\omega_c^2)x_2.$$

To find reasonable values of ω_c we have to balance the speed of response with the available control authority. The unit step responses for the closed loop system for different values of the design parameters are shown in Figure 7.6. The effect of ω_c is shown in Figure 7.6a, which shows that the response speed increases with increasing ω_c. All responses have overshoot less than 5%, as indicated by the dashed lines, which corresponds to 15 cm assuming a wheelbase $b = 3$ m. The settling times range from 3 to 6 normalized time units, which corresponds to about 2–4 s at $v_0 = 15$ m/s. The effect of ζ_c is shown in Figure 7.6b. The response speed and the overshoot increase with decreasing damping.

To select the specific gains to use, we can evaluate how the choice of parameters affects vehicle handling characteristics. For example, a lateral error of 20% of the wheelbase is relatively large and we might choose ω_c to exert a relatively large steering angle to correct for such an error. For $\omega_c = 0.7$ and a step input of size 0.2 (in normalized units), Figure 7.6a indicates that the initial steering angle will be 0.1 rad, which is aggressive but not unreasonable at moderate speeds. The value for ζ_c can be also be chosen as 0.7, which gives a fast response with approximately 5% overshoot. ∇

The example of the vehicle steering system illustrates how state feedback can be used to set the eigenvalues of a closed loop system to arbitrary values. We see that for this example we can set the eigenvalues to any location. We now show that this is a general property for reachable systems.

State Feedback for Systems in Reachable Canonical Form

The reachable canonical form has the property that the parameters of the system are the coefficients of the characteristic polynomial. It is therefore natural to consider systems in this form when solving the eigenvalue assignment problem.

Consider a system in reachable canonical form, i.e.,

$$
\begin{aligned}
\frac{dz}{dt} &= \tilde{A}z + \tilde{B}u = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & 0 & & & \\ & 1 & 0 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \\
y &= \tilde{C}z = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} z.
\end{aligned}
\tag{7.15}
$$

It follows from equation (7.7) that the open loop system has the characteristic polynomial

$$
\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n.
$$

Before making a formal analysis we can gain some insight by investigating the block diagram of the system shown in Figure 7.4. The characteristic polynomial is given by the parameters a_k in the figure. Notice that the parameter a_k can be changed by feedback from state z_k to the input u. It is thus straightforward to change the coefficients of the characteristic polynomial by state feedback.

Returning to equations, introducing the control law

$$
u = -\tilde{K}z + k_f r = -\tilde{k}_1 z_1 - \tilde{k}_2 z_2 - \cdots - \tilde{k}_n z_n + k_f r, \tag{7.16}
$$

the closed loop system becomes

$$
\begin{aligned}
\frac{dz}{dt} &= \begin{pmatrix} -a_1 - \tilde{k}_1 & -a_2 - \tilde{k}_2 & -a_3 - \tilde{k}_3 & \dots & -a_n - \tilde{k}_n \\ 1 & 0 & & & \\ & 1 & 0 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} k_f \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} r, \\
y &= \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} z.
\end{aligned}
\tag{7.17}
$$

The feedback changes the elements of the first row of the A matrix, which corresponds to the parameters of the characteristic polynomial. The closed loop system thus has the characteristic polynomial

$$s^n + (a_1 + \tilde{k}_1)s^{n-1} + (a_2 + \tilde{k}_2)s^{n-2} + \cdots + (a_{n-1} + \tilde{k}_{n-1})s + a_n + \tilde{k}_n.$$

Requiring this polynomial to be equal to the desired closed loop polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n,$$

we find that the controller gains should be chosen as

$$\tilde{k}_1 = p_1 - a_1, \qquad \tilde{k}_2 = p_2 - a_2, \qquad \cdots \qquad \tilde{k}_n = p_n - a_n.$$

This feedback simply replaces the parameters a_i in the system (7.15) by p_i. The feedback gain for a system in reachable canonical form is thus

$$\tilde{K} = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix}. \tag{7.18}$$

To have zero frequency gain equal to unity, we compute the equilibrium point z_e by setting the right hand side of equation (7.17) to zero and then compute the corresponding output. It can be seen that $z_{\mathrm{e},1}, \dots, z_{\mathrm{e},n-1}$ must all be zero and we are left with

$$(-a_n - \tilde{k}_n)z_{\mathrm{e},n} + k_\mathrm{f} r = 0 \quad \text{and} \quad y_\mathrm{e} = b_n z_{\mathrm{e},n}.$$

It follows that in order for y_e to be equal to r then the parameter k_f should be chosen as

$$k_\mathrm{f} = \frac{a_n + \tilde{k}_n}{b_n} = \frac{p_n}{b_n}. \tag{7.19}$$

Notice that it is essential to know the precise values of parameters a_n and b_n in order to obtain the correct zero frequency gain. The zero frequency gain is thus obtained by precise calibration. This is very different from obtaining the correct steady-state value by integral action, which we shall see in later sections.

Eigenvalue Assignment

We have seen through the examples how feedback can be used to design the dynamics of a system through assignment of its eigenvalues. To solve the problem in the general case, we simply change coordinates so that the system is in reachable canonical form. Consider the system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du. \tag{7.20}$$

We can change the coordinates by a linear transformation $z = Tx$ so that the transformed system is in reachable canonical form (7.15). For such a system the feedback is given by equation (7.16), where the coefficients are given by equation (7.18). Transforming back to the original coordinates gives the control law

$$u = -\tilde{K}z + k_\mathrm{f} r = -\tilde{K}Tx + k_\mathrm{f} r.$$

The form of the controller is a feedback term $-Kx$ and a feedforward term $k_\mathrm{f} r$.

The results obtained can be summarized as follows.

Theorem 7.3 (Eigenvalue assignment by state feedback)**.** *Consider the system given by equation (7.20), with one input and one output. Let $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ be the characteristic polynomial of A. If the system is reachable, then there exists a control law*

$$u = -Kx + k_\mathrm{f} r$$

that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n$$

and unity zero frequency gain between r and y. The feedback gain is given by

$$K = \tilde{K}T = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} \tilde{W}_\mathrm{r} W_\mathrm{r}^{-1}, \tag{7.21}$$

where a_i are the coefficients of the characteristic polynomial of the matrix A and the matrices W_r and $\tilde{W}_\mathrm{r}$ are given by

$$W_\mathrm{r} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}, \qquad \tilde{W}_\mathrm{r} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ & 1 & a_1 & \cdots & a_{n-2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & a_1 \\ & & & & 1 \end{pmatrix}^{-1}.$$

The feedforward gain is given by

$$k_\mathrm{f} = -1/\big(C(A-BK)^{-1}B\big).$$

For simple problems, the eigenvalue assignment problem can be solved by introducing the elements k_i of K as unknown variables. We then compute the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)$$

and equate coefficients of equal powers of s to the coefficients of the desired characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n.$$

This gives a system of linear equations to determine k_i. The equations can always be solved if the system is reachable, exactly as we did in Example 7.4.

Equation (7.21), which is called Ackermann's formula [Ack72, Ack85], can be used for numeric computations. It is implemented in the MATLAB function `acker`. The MATLAB function `place` is preferable for systems of high order because it is better conditioned numerically.

Example 7.5 Predator–prey

Consider the problem of regulating the population of an ecosystem by modulating the food supply. We use the predator–prey model introduced in Example 5.16 and described in more detail in Section 4.7. The dynamics for the system are given by

$$\begin{aligned} \frac{dH}{dt} &= (r+u)H\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H}, && H \geq 0, \\ \frac{dL}{dt} &= b\frac{aHL}{c+H} - dL, && L \geq 0. \end{aligned}$$

We choose the following nominal parameters for the system, which correspond to the values used in previous simulations:

$$\begin{aligned} a &= 3.2, & b &= 0.6, & c &= 50, \\ d &= 0.56, & k &= 125 & r &= 1.6. \end{aligned}$$

We take the parameter r, corresponding to the growth rate for hares, as the input to the system, which we might modulate by controlling a food source for the hares. This is reflected in our model by the term $(r + u)$ in the first equation, where here r represents a constant parameter (not the reference signal) and u represents the controlled modulation. We choose the number of lynxes L as the output of our system.

To control this system, we first linearize the system around the equilibrium point of the system (H_e, L_e), which can be determined numerically to be $x_e \approx (20.6, 29.5)$. This yields a linear dynamical system

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.13 & -0.93 \\ 0.57 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 17.2 \\ 0 \end{pmatrix} v, \qquad w = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where $z_1 = H - H_e$, $z_2 = L - L_e$, and $v = u$. It is easy to check that the system is reachable around the equilibrium point $(z, v) = (0, 0)$, and hence we can assign the eigenvalues of the system using state feedback.

Selecting the eigenvalues of the closed loop system requires balancing the ability to modulate the input against the natural dynamics of the system. This can be done by the process of trial and error or by using some of the more systematic techniques discussed in the remainder of the text. For now, we simply choose the desired closed loop eigenvalues to be at $\lambda = \{-0.1, -0.2\}$. We can then solve for the feedback gains using the techniques described earlier, which results in

$$K = \begin{pmatrix} 0.025 & -0.052 \end{pmatrix}.$$

Finally, we solve for the feedforward gain k_f, using equation (7.13) to obtain $k_f = 0.002$.

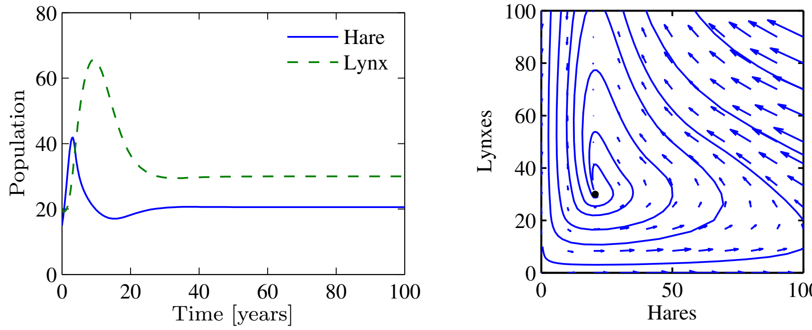
Putting these steps together, our control law becomes

$$v = -Kz + k_f L_d,$$

where L_d is the desired number of lynxes. In order to implement the control law, we must rewrite it using the original coordinates for the system, yielding

$$\begin{aligned} u &= u_e - K(x - x_e) + k_f(L_d - y_e) \\ &= -\begin{pmatrix} 0.025 & -0.052 \end{pmatrix}\begin{pmatrix} H - 20.6 \\ L - 29.5 \end{pmatrix} + 0.002\,(L_d - 29.5). \end{aligned}$$

This rule tells us how much we should modulate u as a function of the current number of lynxes and hares in the ecosystem. Figure 7.7a shows a simulation of the resulting closed loop system using the parameters defined above and starting with an initial population of 15 hares and 20 lynxes. Note that the system stabilizes the population of lynxes at the reference value ($L_d = 30$). A phase portrait of the system is given in Figure 7.7b, showing how other initial conditions converge to the stabilized equilibrium population. Notice that the dynamics are very different from the natural dynamics (shown in Figure 4.20). ∇



(a) Initial condition response

(b) Phase portrait

Figure 7.7: Simulation results for the controlled predator–prey system. The population of lynxes and hares as a function of time is shown in (a), and a phase portrait for the controlled system is shown in (b). Feedback is used to make the population stable at $H_e = 20.6$ and $L_e = 30$.

The results of this section show that we can use state feedback to design the dynamics of a reachable system, under the strong assumption that we can measure all of the states. We shall address the availability of the states in the next chapter, when we consider output feedback and state estimation. In addition, Theorem 7.3, which states that the eigenvalues can be assigned to arbitrary locations, is also highly idealized and assumes that the dynamics of the process are known to high precision. The robustness of state feedback combined with state estimators is considered in Chapter 13 after we have developed the requisite tools.

7.3 Design Considerations

The location of the eigenvalues determines the behavior of the closed loop dynamics, and hence where we place the eigenvalues is the main design decision to be made. As with all other feedback design problems, there are trade-offs among the magnitude of the control inputs, the robustness of the system to perturbations, and the closed loop performance of the system. In this section we examine some of these trade-offs starting with the special case of second-order systems.

Second-Order Systems

One class of systems that occurs frequently in the analysis and design of feedback systems is second-order linear differential equations. Because of their ubiquitous nature, it is useful to apply the concepts of this chapter to that specific class of systems and build more intuition about the relationship between stability and performance.

A canonical second-order system is a differential equation of the form

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u, \qquad y = q. \tag{7.22}$$

In state space form, this system can be represented as

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x, \tag{7.23}$$

where $x = (q, \dot{q}/\omega_0)$ represents a normalized choice of states. The eigenvalues of this system are given by

$$\lambda = -\zeta\omega_0 \pm \omega_0\sqrt{(\zeta^2 - 1)},$$

and we see that the system is stable if $\omega_0 > 0$ and $\zeta > 0$. Note that the eigenvalues are complex if $\zeta < 1$ and real otherwise. Equations (7.22) and (7.23) can be used to describe many second-order systems, including damped oscillators, active filters, and flexible structures, as shown in the examples below.

The form of the solution depends on the value of ζ, which is referred to as the *damping ratio* for the system. If $\zeta > 1$, we say that the system is *overdamped*, and the natural response ($u = 0$) of the system is given by

$$y(t) = \frac{\beta x_{10} + x_{20}}{\beta - \alpha} e^{-\alpha t} - \frac{\alpha x_{10} + x_{20}}{\beta - \alpha} e^{-\beta t},$$

where $\alpha = \omega_0(\zeta + \sqrt{\zeta^2 - 1})$ and $\beta = \omega_0(\zeta - \sqrt{\zeta^2 - 1})$. We see that the response consists of the sum of two exponentially decaying signals. If $\zeta = 1$, then the system is *critically damped* and solution becomes

$$y(t) = e^{-\zeta\omega_0 t}\big(x_{10} + (x_{20} + \zeta\omega_0 x_{10})t\big).$$

Note that this is still asymptotically stable as long as $\omega_0 > 0$, although the second term within the outer parentheses is increasing with time (but more slowly than the decaying exponential that is multiplying it).

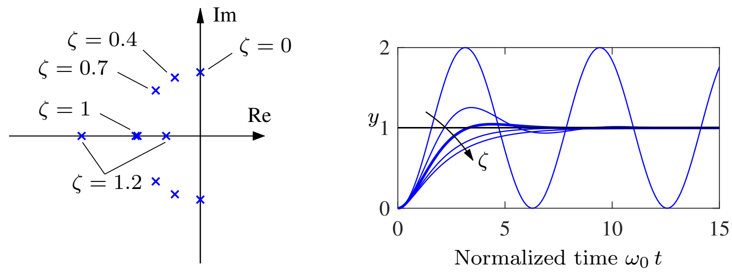
Finally, if $0 < \zeta < 1$, then the solution is oscillatory and equation (7.22) is said to be *underdamped*. The natural response of the system is given by

$$y(t) = e^{-\zeta\omega_0 t}\left(x_{10}\cos\omega_{\mathrm{d}} t + \Big(\frac{\zeta\omega_0}{\omega_{\mathrm{d}}} x_{10} + \frac{1}{\omega_{\mathrm{d}}} x_{20}\Big)\sin\omega_{\mathrm{d}} t\right),$$

where $\omega_{\mathrm{d}} = \omega_0\sqrt{1 - \zeta^2}$ is called the *damped frequency.* For $\zeta \ll 1$, $\omega_{\mathrm{d}} \approx \omega_0$ defines the oscillation frequency of the solution and ζ gives the damping rate relative to ω_0. The parameter ω_0 is referred to as the *natural frequency* of the system, stemming from the fact that for $\zeta = 0$ the oscillation frequency is given by ω_0.

Because of the simple form of a second-order system, it is possible to solve for the step and frequency responses in analytical form. The solution for the step response depends on the magnitude of ζ:

$$y(t) = \begin{cases} k\left(1 - e^{-\zeta\omega_0 t}\cos\omega_{\mathrm{d}} t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t}\sin\omega_{\mathrm{d}} t\right), & \text{if } \zeta < 1; \\ k\left(1 - e^{-\omega_0 t}(1 + \omega_0 t)\right), & \text{if } \zeta = 1; \\ k\left(1 - \frac{1}{2}\Big(\frac{\zeta}{\sqrt{\zeta^2-1}} + 1\Big)e^{-\omega_0 t(\zeta - \sqrt{\zeta^2-1})} \right. & \\ \qquad \left. + \frac{1}{2}\Big(\frac{\zeta}{\sqrt{\zeta^2-1}} - 1\Big)e^{-\omega_0 t(\zeta + \sqrt{\zeta^2-1})}\right), & \text{if } \zeta > 1, \end{cases} \tag{7.24}$$



(a) Eigenvalues (b) Step responses

Figure 7.8: Step response for a second-order system. Normalized step responses for the system (7.23) for $\zeta = 0$, 0.4, 0.7 (thicker), 1, and 1.2. As the damping ratio is increased, the rise time of the system gets longer, but there is less overshoot. The horizontal axis is in scaled units $\omega_0 t$; higher values of ω_0 result in a faster response (rise time and settling time).

where we have taken $x(0) = 0$. Note that for the lightly damped case ($\zeta < 1$) we have an oscillatory solution at frequency ω_d.

Step responses of systems with $k = 1$ and different values of ζ are shown in Figure 7.8. The shape of the response is determined by ζ, and the speed of the response is determined by ω_0 (included in the time axis scaling): the response is faster if ω_0 is larger.

In addition to the explicit form of the solution, we can also compute the properties of the step response that were defined in Section 6.3. For example, to compute the maximum overshoot for an underdamped system, we rewrite the output as

$$y(t) = k\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t}\sin(\omega_\mathrm{d} t + \varphi)\right), \tag{7.25}$$

where $\varphi = \arccos\zeta$. The maximum overshoot will occur at the first time in which the derivative of y is zero, at which time the fraction of the final value can be shown to be

$$M_\mathrm{p} = e^{-\pi\zeta/\sqrt{1-\zeta^2}}.$$

The rise time is normally defined as the time for the step response to go from p% of its final value to $(100-p)$%. Typical values are $p = 5$ or 10%. An alternative definition is the inverse of the steepest slope: by differentiating equation (7.25) we find after straightforward but tedious calculations that

$$T_\mathrm{r} = \frac{1}{\omega_0} e^{\varphi/\tan\varphi}, \qquad \varphi = \arccos\zeta.$$

Similar computations can be done for the other characteristics of a step response. Table 7.1 summarizes these calculations.

The frequency response for a second-order system can also be computed explicitly and is given by

$$Me^{i\theta} = \frac{k\omega_0^2}{(i\omega)^2 + 2\zeta\omega_0(i\omega) + \omega_0^2} = \frac{k\omega_0^2}{\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega}.$$

Table 7.1: Properties of the step response for a second-order system $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u$ with $0 < \zeta \le 1$.

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Steady-state value	k	k	k	k
Rise time (inverse slope)	$T_\mathrm{r} = e^{\varphi/\tan\varphi}/\omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_\mathrm{p} = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_\mathrm{s} \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$

A graphical illustration of the frequency response is given in Figure 7.9. Notice the resonant peak that increases with decreasing ζ. The peak is often characterized by its *Q-value*, defined as $Q = 1/2\zeta$. The properties of the frequency response for a second-order system are summarized in Table 7.2.

Example 7.6 Drug administration
To illustrate the use of these formulas, consider the two-compartment model for drug administration, described in Section 4.6. The dynamics of the system are

$$\frac{dc}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} c,$$

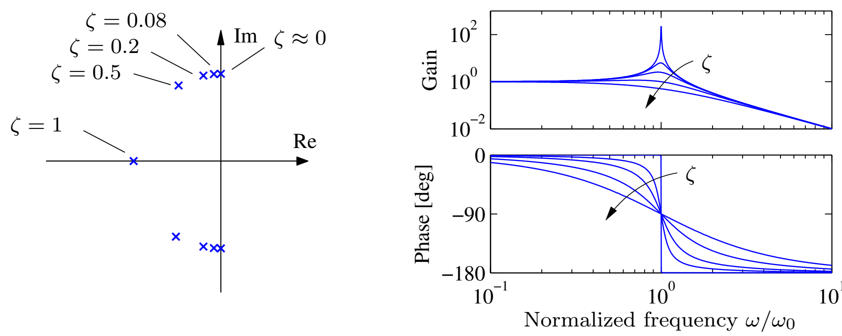
where c_1 and c_2 are the concentrations of the drug in each compartment, k_0, k_1, k_2, and b_0 are parameters of the system, u is the flow rate of the drug into compartment 1, and y is the concentration of the drug in compartment 2. We assume that we can measure the concentrations of the drug in each compartment, and we would like to design a feedback law to maintain the output at a given reference value r.

We choose $\zeta = 1/\sqrt{2}$ to minimize the overshoot and additionally require the rise time to be $T_\mathrm{r} = 10$ min. Using the formulas in Table 7.1, this gives a value for $\omega_0 = 0.22$. We can now compute the gains to place the eigenvalues at this location. Setting $u = -Kx + k_\mathrm{f} r$, the closed loop eigenvalues for the system satisfy

$$\lambda(s) = -0.2 \pm 0.096i.$$

Table 7.2: Properties of the frequency response for a second-order system $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u$ with $0 < \zeta \le 1$.

Property	Value	$\zeta{=}0.1$	$\zeta{=}0.5$	$\zeta{=}1/\sqrt{2}$
Zero frequency gain	M_0	k	k	k
Bandwidth	$\omega_\mathrm{b} = \omega_0\sqrt{1 - 2\zeta^2 + \sqrt{(1-2\zeta^2)^2 + 1}}$	$1.54\,\omega_0$	$1.27\,\omega_0$	ω_0
Resonant peak gain	$M_\mathrm{r} = \begin{cases} k/(2\zeta\sqrt{1-\zeta^2}) & \zeta \le \sqrt{2}/2, \\ \text{N/A} & \zeta > \sqrt{2}/2 \end{cases}$	$5\,k$	$1.15\,k$	k
Resonant frequency	$\omega_\mathrm{mr} = \begin{cases} \omega_0\sqrt{1-2\zeta^2} & \zeta \le \sqrt{2}/2, \\ 0 & \zeta > \sqrt{2}/2 \end{cases}$	ω_0	$0.707\omega_0$	0



(a) Eigenvalues (b) Frequency responses

Figure 7.9: Frequency response of a second-order system (7.23). (a) Eigenvalues as a function of ζ. (b) Frequency response as a function of ζ. The upper curve shows the gain ratio M, and the lower curve shows the phase shift θ. For small ζ there is a large peak in the magnitude of the frequency response and a rapid change in phase centered at $\omega = \omega_0$. As ζ is increased, the magnitude of the peak drops and the phase changes more smoothly between 0° and -180°.

Choosing $\tilde{k}_1 = -0.2$ and $\tilde{k}_2 = 0.2$, with $K = (\tilde{k}_1, \tilde{k}_2)$ to avoid confusion with the rates $k_{\rm i}$ in the dynamics matrix, gives the desired closed loop behavior. Equation (7.13) gives the feedforward gain $k_{\rm f} = 0.065$. The response of the controller is shown in Figure 7.10 and compared with an open loop strategy involving administering periodic doses of the drug. ∇

Higher-Order Systems

Our emphasis so far has considered only second-order systems. For higher-order systems, eigenvalue assignment is considerably more difficult, especially when trying to account for the many trade-offs that are present in a feedback design.

One of the other reasons why second-order systems play such an important role in feedback systems is that even for more complicated systems the response is often characterized by the *dominant eigenvalues*. To define these more precisely, consider a stable system with eigenvalues λ_j, $j = 1, \ldots, n$. We say that a complex conjugate pair of eigenvalues λ, λ^* is a *dominant pair* if they are the closest pair to the imaginary axis. In the case when multiple eigenvalues pairs are the same distance to the imaginary axis, a second criterion is to look at the relative damping of the system modes. We define the *damping ratio* for a complex eigenvalue λ as

$$\zeta = \frac{-\mathrm{Re}\,\lambda}{|\lambda|}.$$

Given multiple complex conjugate pairs with the same real part, the dominant pair will be the set with the lowest damping ratio.

Assuming that a system is stable, the dominant pair of eigenvalues tends to be the most important element of the response. To see this, assume that we have a system in Jordan form with a simple Jordan block corresponding to the dominant

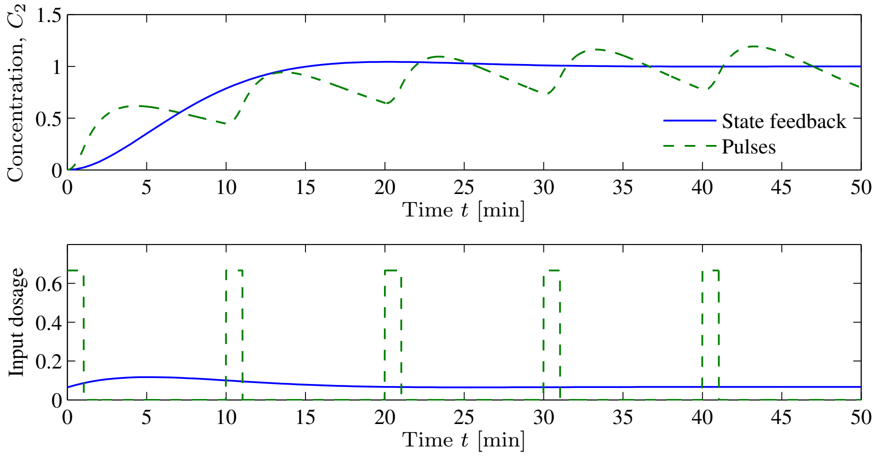


Figure 7.10: Open loop versus closed loop drug administration. Comparison between drug administration using a sequence of doses versus continuously monitoring the concentrations and adjusting the dosage continuously. In each case, the concentration is (approximately) maintained at the desired level, but the closed loop system has substantially less variability in drug concentration.

pair of eigenvalues:

$$\frac{dz}{dt} = \begin{pmatrix} \lambda & & & & \\ & \lambda^* & & & \\ & & J_2 & & \\ & & & \ddots & \\ & & & & J_k \end{pmatrix} z + Bu, \qquad y = Cz.$$

(Note that the state z may be complex because of the Jordan transformation.) The response of the system will be a linear combination of the responses from each of the individual Jordan subsystems. As we see from Figure 7.8, for $\zeta < 1$ the subsystem with the slowest response is precisely the one with eigenvalues closest to the imaginary axis. Hence, when we add the responses from each of the individual subsystems, it is the dominant pair of eigenvalues that will be the primary factor after the initial transients due to the other terms in the solution die out. While this simple analysis does not always hold (e.g., if some non-dominant terms have larger coefficients because of the particular form of the system), it is often the case that the dominant eigenvalues determine the (step) response of the system.

The only formal requirement for eigenvalue assignment is that the system be reachable. In practice there are many other constraints because the selection of eigenvalues has a strong effect on the magnitude and rate of change of the control signal. Large eigenvalues will in general require large control signals as well as fast changes of the signals. The capability of the actuators will therefore impose constraints on the possible location of closed loop eigenvalues. These issues will be discussed in depth in Chapters 12–14.

We illustrate some of the main ideas using the balance system as an example.

Example 7.7 Balance system
Consider the problem of stabilizing a balance system, whose dynamics were given in Example 7.2. The dynamics are given by

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2l^2g/\mu & -cJ_\mathrm{t}/\mu & -\gamma lm/\mu \\ 0 & M_\mathrm{t}mgl/\mu & -clm/\mu & -\gamma M_\mathrm{t}/\mu \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ J_\mathrm{t}/\mu \\ lm/\mu \end{pmatrix},$$

where $M_\mathrm{t} = M + m$, $J_\mathrm{t} = J + ml^2$, $\mu = M_\mathrm{t}J_\mathrm{t} - m^2l^2$ and we have left c and γ nonzero. We use the following parameters for the system (corresponding roughly to a human being balanced on a stabilizing cart):

$$\begin{aligned} M &= 10 \text{ kg}, & m &= 80 \text{ kg}, & c &= 0.1 \text{ N s/m}, \\ J &= 100 \text{ kg m}^2/\text{s}^2, & l &= 1 \text{ m}, & \gamma &= 0.01 \text{ N m s}, \end{aligned} \qquad g = 9.8 \text{ m/s}^2.$$

The eigenvalues of the open loop dynamics are given by $\lambda \approx 0, -0.0011, \pm 2.68$. We have verified already in Example 7.2 that the system is reachable, and hence we can use state feedback to stabilize the system and provide a desired level of performance.

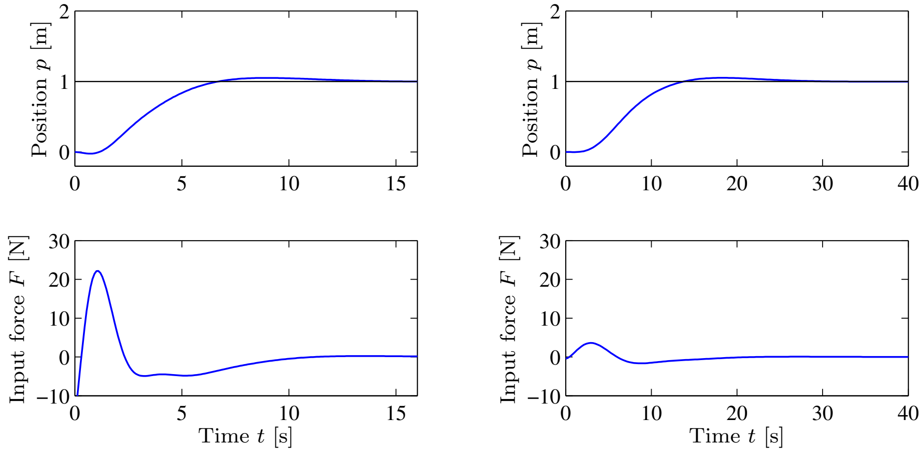
To decide where to place the closed loop eigenvalues, we note that the closed loop dynamics will roughly consist of two components: a set of fast dynamics that stabilize the pendulum in the inverted position and a set of slower dynamics that control the position of the cart. For the fast dynamics, we look to the natural period of the pendulum (in the hanging-down position), which is given by $\omega_0 = \sqrt{mgl/(J + ml^2)} \approx 2.1$ rad/s. To provide a fast response we choose a damping ratio of $\zeta = 0.5$ and try to place the first pair of eigenvalues at $\lambda_{1,2} \approx -\zeta\omega_0 \pm i\omega_0 \approx -1 \pm 2i$, where we have used the approximation that $\sqrt{1 - \zeta^2} \approx 1$. For the slow dynamics, we choose the damping ratio to be 0.7 to provide a small overshoot and choose the natural frequency to be 0.5 to give a rise time of approximately 5 s. This gives eigenvalues $\lambda_{3,4} = -0.35 \pm 0.35i$.

The controller consists of feedback on the state and a feedforward gain for the reference input. The feedback gain is given by

$$K = \begin{pmatrix} -15.6 & 1730 & -50.1 & 443 \end{pmatrix},$$

which can be computed using Theorem 7.3 or using the MATLAB `place` command. The feedforward gain is $k_\mathrm{f} = -1/(C(A - BK)^{-1}B) = -15.6$. The step response for the resulting controller (applied to the linearized system) is given in Figure 7.11a. While the step response gives the desired characteristics, the input required (lower left) is excessively large, almost three times the force of gravity at its peak.

To provide a more realistic response, we can redesign the controller to have slower dynamics. We see that the peak of the input force occurs on the fast time scale, and hence we choose to slow this down by approximately a factor of 3, leaving the damping ratio unchanged. We also slow down the second set of eigenvalues, with the intuition that we should move the position of the cart more slowly than we stabilize the pendulum dynamics. Leaving the damping ratio for the slow dynamics



(a) $\lambda_{1,2} = -1 \pm 2i$

(b) $\lambda_{1,2} = -0.33 \pm 0.66i$

Figure 7.11: State feedback control of a balance system. The step response of a controller designed to give fast performance is shown in (a). Although the response characteristics (upper left) look very good, the input magnitude (lower left) is very large. Also note that the force is negative initially. A less aggressive controller is shown in (b). Here the response time is slowed down, but the input magnitude is much more reasonable. Both step responses are applied to the linearized dynamics.

unchanged at 0.7 and changing the frequency to 1 (corresponding to a rise time of approximately 10 s), the desired eigenvalues become

$$\lambda = \{-0.33 \pm 0.66i,\ -0.18 \pm 0.18i\}.$$

The performance of the resulting controller is shown in Figure 7.11b. ∇

As we see from this example, it can be difficult to decide where to place the eigenvalues using state feedback. This is one of the principal limitations of this approach, especially for systems of higher dimension. Optimal control, such as the linear quadratic regulator problem discussed in Section 7.5, is one approach that is available. One can also focus on the frequency response for performing the design, which is the subject of Chapters 9–13.

7.4 Integral Action

Controllers based on state feedback achieve the correct steady-state response to command signals by careful calibration of the gain k_f. However, one of the primary uses of feedback is to allow good performance in the presence of uncertainty and hence requiring that we have an *exact* model of the process is undesirable. An alternative to calibration is to make use of integral feedback, in which the controller uses an integrator to provide zero steady-state error. The basic concept of integral feedback was introduced in Section 1.6 and discussed briefly in Sections 2.3 and 2.4; here we provide a more complete description and analysis.

System Augmentation

The basic approach in integral feedback is to create a state within the controller that computes the integral of the error signal, which is then used as a feedback term. We do this by augmenting the description of the system with a new state z, which is the integral of the difference between the the actual output y and desired output r. The augmented state equations become

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} Ax + Bu \\ y - r \end{pmatrix} = \begin{pmatrix} Ax + Bu \\ Cx - r \end{pmatrix}. \tag{7.26}$$

Note that if we find a controller that stabilizes the system, then we will necessarily have $\dot{z} = 0$ in steady state and hence $y = r$ in steady state.

Given the augmented system, we design a state space controller in the usual fashion, with a control law of the form

$$u = -Kx - k_\mathrm{i} z + k_\mathrm{f} r, \tag{7.27}$$

where K is the usual state feedback term, k_i is the integral term, and k_f is used to set the nominal input for the desired steady state. The resulting equilibrium point for the system is given by

$$x_\mathrm{e} = -(A - BK)^{-1}B(k_\mathrm{f} r - k_\mathrm{i} z_\mathrm{e}), \qquad Cx_\mathrm{e} = r,$$

which comes from setting the right hand side of equation (7.26) to zero and substituting u from equation (7.27). Note that the value of z_e is not specified but rather will automatically settle to the value that makes $\dot{z} = y - r = 0$, which implies that at equilibrium the output will equal the reference value. This holds independently of the specific values of A, B, and K as long as the system is stable (which can be done through appropriate choice of K and k_i).

The final control law is given by

$$u = -Kx - k_\mathrm{i} z + k_\mathrm{f} r, \qquad \frac{dz}{dt} = y - r,$$

where we have now included the dynamics of the integrator as part of the specification of the controller. This type of control law is known as a *dynamic compensator* since it has its own internal dynamics. The following example illustrates the basic approach.

Example 7.8 Cruise control

Consider the cruise control example introduced in Section 1.5 and considered further in Example 6.11 (see also Section 4.1). The linearized dynamics of the process around an equilibrium point v_e, u_e are given by

$$\frac{dx}{dt} = -ax - b_\mathrm{g}\theta + bw, \qquad y = v = x + v_\mathrm{e},$$

where $x = v - v_\mathrm{e}$, $w = u - u_\mathrm{e}$, m is the mass of the car, and θ is the angle of the road. The constants a, b, and b_g depend on the properties of the car and are given in Example 6.11.

If we augment the system with an integrator, the system dynamics become

$$\frac{dx}{dt} = -ax - b_g\theta + bw, \qquad \frac{dz}{dt} = y - v_r = v_e + x - v_r,$$

or, in state space form,

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} w + \begin{pmatrix} -b_g \\ 0 \end{pmatrix}\theta + \begin{pmatrix} 0 \\ v_e - v_r \end{pmatrix}.$$

Note that when the system is at equilibrium, we have that $\dot{z} = 0$, which implies that the vehicle speed $v = v_e + x$ should be equal to the desired reference speed v_r. Our controller will be of the form

$$\frac{dz}{dt} = y - v_r, \qquad w = -k_p x - k_i z + k_f v_r,$$

and the gains k_p, k_i, and k_f will be chosen to stabilize the system and provide the correct input for the reference speed.

Assume that we wish to design the closed loop system to have the characteristic polynomial

$$\lambda(s) = s^2 + a_1 s + a_2.$$

Setting the disturbance $\theta = 0$, the characteristic polynomial of the closed loop system is given by

$$\det\bigl(sI - (A - BK)\bigr) = s^2 + (bk_p + a)s + bk_i,$$

and hence we set

$$k_p = \frac{a_1 - a}{b}, \qquad k_i = \frac{a_2}{b}, \qquad k_f = -1/\bigl(C(A - BK)^{-1}B\bigr) = \frac{a_1}{b}.$$

The resulting controller stabilizes the system and hence brings $\dot{z} = y - v_r$ to zero, resulting in perfect tracking. Notice that even if we have a small error in the values of the parameters defining the system, as long as the closed loop eigenvalues are still stable, then the tracking error will approach zero. Thus the exact calibration required in our previous approach (using k_f) is not needed here. Indeed, we can even choose $k_f = 0$ and let the feedback controller do all of the work. However, k_f does influence the transient response to reference signals and setting it properly will generally give a more favorable response.

Integral feedback can also be used to compensate for constant disturbances. Figure 7.12 shows the results of a simulation in which the car encounters a hill with angle $\theta = 4^\circ$ at $t = 5$ s. The steady-state values of the throttle for a state feedback controller and a controller with integral action are very close, but the corresponding values of the car velocity are quite different. The reason for this is that the zero frequency gain from throttle to velocity is $-b/a = 130$ is high. The stability of the system is not affected by this external disturbance, and so we once again see that the car's velocity converges to the reference speed. This ability to handle constant disturbances is a general property of controllers with integral feedback (see Exercise 7.15). ∇

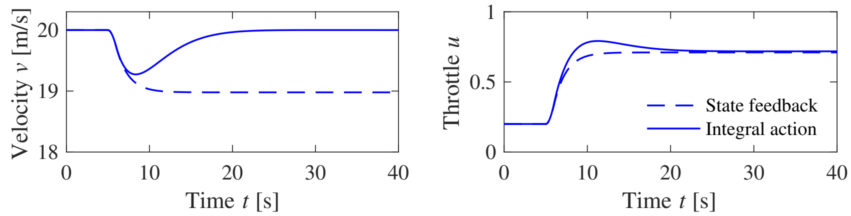


Figure 7.12: Velocity and throttle for a car with cruise control based on state feedback (dashed) and state feedback with integral action (solid). The controller with integral action is able to adjust the throttle to compensate for the effect of the hill and maintain the speed at the reference value of $v_\mathrm{r} = 20$ m/s. The controller gains are $k_\mathrm{p} = 0.5$ and $k_\mathrm{i} = 0.1$.

Reachability of the Augmented System

Eigenvalue assignment requires that the augmented system (7.26) is reachable. To explore this we compute the reachability matrix of the augmented system:

$$W_\mathrm{r} = \begin{pmatrix} B & AB & \dots & A^n B \\ 0 & CB & \dots & CA^{n-1}B \end{pmatrix}.$$

To find the conditions for W_r to be of full rank, the matrix will be transformed by making column operations. Let a_k be the coefficients of the characteristic polynomial of the matrix A:

$$\lambda_A(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n.$$

Multiplying the first column by a_n, the second by a_{n-1}, through multiplication of the (n-1)th column by a_1 and then adding these to the last column of the matrix W_r, it follows from the Cayley–Hamilton theorem (Exercise 7.3) that the transformed matrix becomes

$$W_\mathrm{r} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B & 0 \\ 0 & CB & \cdots & CA^{n-2}B & b_n \end{pmatrix},$$

where

$$b_n = C(A^{n-1}B + a_1 A^{n-2}B + \ldots + a_{n-1}B). \tag{7.28}$$

If the matrix A is invertible, implying that there are no eigenvalues at the origin, then we can rewrite the formula for b_n as

$$b_n = CA^{-1}(A^n + a_1 A^{n-1} + \ldots + a_{n-1}A)B = -a_n CA^{-1}B,$$

where the final equality follows from a second application of the Cayley–Hamilton theorem. As long as the coefficient $b_n \neq 0$, then the system is reachable and it is possible to assign the eigenvalues of the augmented system to arbitrary values.

We will see in Chapter 9 that the coefficient b_n can be identified with a coefficient of the transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}.$$

The condition for reachability is thus that the original system does not contain a pure derivative in the input/output response.

7.5 Linear Quadratic Regulators

As an alternative to selecting the closed loop eigenvalue locations to accomplish a certain objective, the gains for a state feedback controller can instead be chosen by attempting to optimize a cost function. This can be particularly useful in helping balance the performance of the system with the magnitude of the inputs required to achieve that level of performance.

The linear quadratic regulator (LQR) problem is one of the most common optimal control problems. Given a multi-input linear system

$$\frac{dx}{dt} = Ax + Bu, \qquad x \in \mathbb{R}^n,\, u \in \mathbb{R}^p$$

with initial condition $x(0) = x_0$, we attempt to minimize the quadratic cost function

$$J(x_0) = \int_0^{t_\mathrm{f}} \left(x^T Q_x x + u^T Q_u u\right)\, dt + x^T(t_\mathrm{f}) Q_\mathrm{f} x(t_\mathrm{f}), \tag{7.29}$$

where $Q_x \succeq 0$, $Q_u \succ 0$ and $Q_\mathrm{f} \succeq 0$ are symmetric, positive (semi-) definite matrices of the appropriate dimensions. This cost function represents a trade-off between the deviation of the state from the origin and the cost of the control input. By choosing the matrices Q_x, Q_u, and Q_f we can balance the rate of convergence of the solutions with the cost of the control.

The solution to the LQR problem is given by a linear control law of the form

$$u = -Kx, \quad K = Q_u^{-1} B^T S, \tag{7.30}$$

where $S \in \mathbb{R}^{n\times n}$ is a positive definite, symmetric matrix given by

$$-\frac{dS}{dt} = A^T S + SA - SBQ_u^{-1}B^T S + Q_x, \quad S(t_\mathrm{f}) = Q_\mathrm{f}. \tag{7.31}$$

This differential equation, called the *Riccati differential equation*, is integrated backwards in time starting with $S(t_\mathrm{f}) = Q_\mathrm{f}$. The minimal cost function, representing the optimal cost, is given by

$$\min_u \int_0^{t_\mathrm{f}} \left(x^T Q_x x + u^T Q_u u\right)\, dt + x^T(t_\mathrm{f}) Q_\mathrm{f} x(t_\mathrm{f}) = x^T(0) S(0) x(0). \tag{7.32}$$

The matrices A, B, Q_x, Q_u, and K may depend on time. A solution to the optimal control problem exists if the Riccati equation has a unique positive solution. The LQR approach is particularly well suited when linearizing around a trajectory, as will be done later in Section 8.5.

The LQR problem is simplified significantly if the time horizon is infinite and all matrices are constants, in which case S is a constant matrix given by the steady-state solution of (7.31):

$$A^T S + SA - SBQ_u^{-1}B^T S + Q_x = 0. \tag{7.33}$$

This equation is called the *algebraic Riccati equation*. If the system is reachable, it can be shown that there is a unique positive definite matrix S satisfying equation (7.33) that makes the closed loop system stable. The feedback gain

$K = Q_u^{-1} B^T S$ is then also a constant matrix. The MATLAB command `lqr` returns K, S, and the dynamics matrix $E = A - BK$ of the closed loop system.

A key question in LQR design is how to choose the weights Q_x, Q_u, and Q_{f}. To guarantee that a solution exists, we must have $Q_x \succeq 0$ and $Q_u \succ 0$. In addition, there are certain "observability" conditions on Q_x that limit its choice. Here we assume $Q_x \succ 0$ to ensure that solutions to the algebraic Riccati equation always exist. To choose specific values for the cost function weights Q_x and Q_u, we must use our knowledge of the system we are trying to control. A particularly simple choice is to use diagonal weights

$$Q_x = \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{pmatrix}, \qquad Q_u = \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_n \end{pmatrix}.$$

For this choice of Q_x and Q_u, the individual diagonal elements describe how much each state and input (squared) should contribute to the overall cost. Hence, we can take states that should remain small and attach higher weight values to them. Similarly, we can penalize an input versus the states and other inputs through choice of the corresponding input weight ρ.

Example 7.9 Vectored thrust aircraft

Consider the original dynamics of the system (3.28), written in state space form as

$$\frac{dz}{dt} = \begin{pmatrix} z_4 \\ z_5 \\ z_6 \\ -\frac{c}{m} z_4 \\ -g - \frac{c}{m} z_5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{F_1}{m}\cos\theta - \frac{F_2}{m}\sin\theta \\ \frac{F_1}{m}\sin\theta + \frac{F_2}{m}\cos\theta \\ \frac{r}{J} F_1 \end{pmatrix}$$

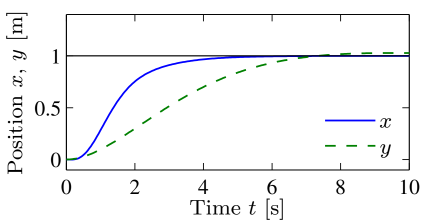
(see also Example 6.4). The system parameters are $m = 4$ kg, $J = 0.0475\ \mathrm{kg\,m^2}$, $r = 0.25$ m, $g = 9.8\ \mathrm{m/s^2}$, and $c = 0.05$ N s/m, which correspond to a scaled model of the system. The equilibrium point for the system is given by $F_1 = 0$, $F_2 = mg$, and $z_{\mathrm{e}} = (x_{\mathrm{e}}, y_{\mathrm{e}}, 0, 0, 0, 0)$. To derive the linearized model near an equilibrium point, we compute the linearization according to equation (6.35):

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & -c/m & 0 & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/m \\ r/J & 0 \end{pmatrix},$$

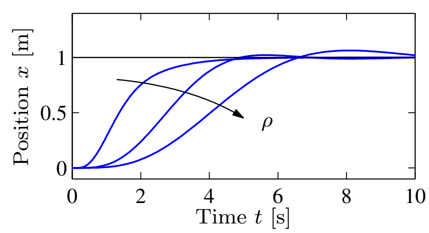
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Letting $\xi = z - z_{\mathrm{e}}$ and $v = F - F_{\mathrm{e}}$, the linearized system is given by

$$\frac{d\xi}{dt} = A\xi + Bv, \qquad y = C\xi.$$



(a) Step response in x and y



(b) Effect of control weight ρ

Figure 7.13: Step response for a vectored thrust aircraft with an LQR controller. The plot in (a) shows the x and y positions of the aircraft when it is commanded to move 1 m in each direction. In (b) the x motion is shown for control weights $\rho = 1,\ 10^2,\ 10^4$. A higher weight of the input term in the cost function causes a more sluggish response.

It can be verified that the system is reachable.

To compute a linear quadratic regulator for the system, we write the cost function as

$$J = \int_0^\infty (\xi^T Q_\xi \xi + v^T Q_v v)\, dt,$$

where $\xi = z - z_e$ and $v = F - F_e$ again represent the local coordinates around the desired equilibrium point (z_e, F_e). We begin with diagonal matrices for the state and input costs:

$$Q_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Q_v = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}.$$

This gives a control law of the form $v = -K\xi$, which can then be used to derive the control law in terms of the original variables:

$$F = v + F_e = -K(z - z_e) + F_e.$$

As computed in Example 6.4, the equilibrium points have $F_e = (0, mg)$ and $z_e = (x_e, y_e, 0, 0, 0, 0)$. The response of the controller to a step change in the desired position is shown in Figure 7.13a for $\rho = 1$. The response can be tuned by adjusting the weights in the LQR cost. Figure 7.13b shows the response in the x direction for different choices of the weight ρ. ∇

Linear quadratic regulators can also be designed for discrete-time systems, as illustrated by the following example.

Example 7.10 Web server control

Consider the web server example given in Section 4.4, where a discrete-time model for the system was given. We wish to design a control law that sets the server parameters so that the average server processor load is maintained at a desired

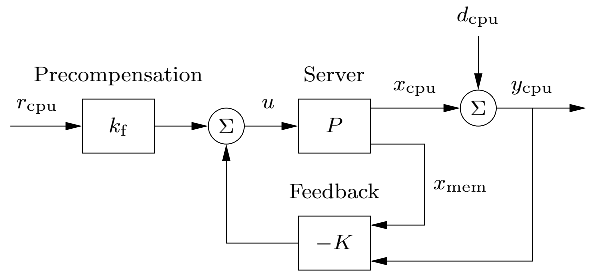


Figure 7.14: Feedback control of a web server. The controller sets the values of the web server parameters based on the difference between the nominal parameters (determined by $k_{\text{f}} r_{\text{cpu}}$) and the current load y_{cpu}. The disturbance d_{cpu} represents the load due to other processes running on the server. Note that the measurement is taken after the disturbance so that we measure the total load on the server.

level. Since other processes may be running on the server, the web server must adjust its parameters in response to changes in the load.

A block diagram for the control system is shown in Figure 7.14. We focus on the special case where we wish to control only the processor load using both the `KeepAlive` and `MaxClients` parameters. We also include a "disturbance" on the measured load that represents the use of the processing cycles by other processes running on the server. The system has the same basic structure as the generic control system in Figure 7.5, with the variation that the disturbance enters after the process dynamics.

The dynamics of the system are given by a set of difference equations of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y_{\text{cpu}}[k] = x_{\text{cpu}}[k] + d_{\text{cpu}}[k],$$

where $x = (x_{\text{cpu}}, x_{\text{mem}})$ is the state of the web server, $u = (u_{\text{ka}}, u_{\text{mc}})$ is the input, d_{cpu} is the processing load from other processes on the computer, and y_{cpu} is the total processor load. The matrices $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$ are described in Section 4.4.

We choose our controller to be a feedback controller of the form

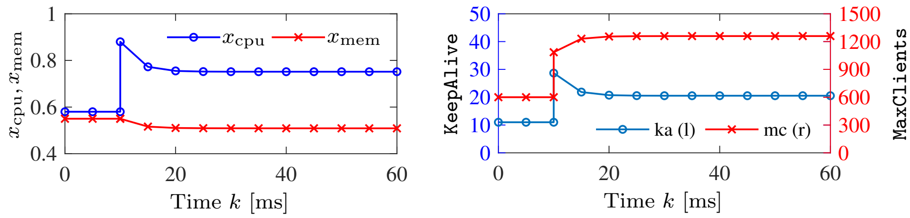
$$u = -K \begin{pmatrix} y_{\text{cpu}} \\ x_{\text{mem}} \end{pmatrix} + k_{\text{f}} r_{\text{cpu}},$$

where r_{cpu} is the desired processor load. Note that we have used the measured processor load y_{cpu} instead of the CPU state x_{cpu} to ensure that we adjust the system operation based on the actual load. (This modification is necessary because of the nonstandard way in which the disturbance enters the process dynamics.)

The feedback gain matrix K can be chosen by any of the methods described in this chapter. Here we use a linear quadratic regulator, with the cost function given by

$$Q_x = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q_u = \begin{pmatrix} 1/50^2 & 0 \\ 0 & 1/1000^2 \end{pmatrix}.$$

The cost function for the state Q_x is chosen so that we place more emphasis on the processor load versus the memory usage. The cost function for the inputs Q_u is



(a) System state (b) System inputs

Figure 7.15: Web server with LQR control. The plot in (a) shows the state of the system under a change in external load applied at $k = 10$ ms. The corresponding web server parameters (system inputs) are shown in (b). The controller is able to reduce the effect of the disturbance by approximately 40%.

chosen so as to normalize the two inputs, with a `KeepAlive` timeout of 50 s having the same weight as a `MaxClients` value of 1000. These values are squared since the cost associated with the inputs is given by $u^T Q_u u$. Using the dynamics in Section 4.4 and the `dlqr` command in MATLAB, the resulting gains become

$$K = \begin{pmatrix} -22.3 & 10.1 \\ 382.7 & 77.7 \end{pmatrix}.$$

As in the case of a continuous-time control system, the feedforward gain k_{f} is chosen to yield the desired operating point for the system. Setting $x[k+1] = x[k] = x_{\mathrm{e}}$, the steady-state equilibrium point and output for a given reference input r are given by

$$x_{\mathrm{e}} = (A - BK)x_{\mathrm{e}} + Bk_{\mathrm{f}}r, \qquad y_{\mathrm{e}} = Cx_{\mathrm{e}}.$$

This is a matrix equation in which k_{f} is a column vector that sets the two input values based on the desired reference. Since we have two inputs, we can set both the desired CPU load $y_{\mathrm{cpu,e}}$ and the desired memory usage $x_{\mathrm{mem,e}}$. If we take the desired equilibrium state to be of the form $x_{\mathrm{e}} = (r, 0)$, where we choose the desired value of memory usage to be zero to make as much memory as possible available for other tasks, then we must solve

$$\begin{pmatrix} r \\ 0 \end{pmatrix} = (A - BK - I)^{-1} Bk_{\mathrm{f}}r.$$

Solving this equation for k_{f}, we obtain

$$k_{\mathrm{f}} = \Bigl(\bigl((A - BK - I)^{-1}B\bigr)\Bigr)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 49.3 \\ 539.5 \end{pmatrix}.$$

The dynamics of the closed loop system are illustrated in Figure 7.15. We apply a change in load of $d_{\mathrm{cpu}} = 0.3$ at time $t = 10$ s, forcing the controller to adjust the operation of the server to attempt to maintain the desired load at 0.57. Note that both the `KeepAlive` and `MaxClients` parameters are adjusted. Although the load is decreased, it remains approximately 0.2 above the desired steady state. ∇

7.6 Further Reading

The importance of state models and state feedback was discussed in the seminal paper by Kalman [Kal60], where the state feedback gain was obtained by solving an optimization problem that minimized a quadratic loss function. The notions of reachability and observability (Chapter 8) are also due to Kalman [Kal61b] (see also [Gil63, KHN63]). Kalman defines controllability and reachability as the ability to reach the origin and an arbitrary state, respectively [KFA69]. Reachability is also used in graph theory as the ability to get from one vertex to another. We note that in most textbooks the term "controllability" is used instead of "reachability," but we prefer the latter term because it is more descriptive of the fundamental property of being able to reach arbitrary states. The result that the eigenvalues of a reachable linear system could be placed in arbitrary positions was first realized by J. Bertram in 1959 [KFA69], who worked in a control group at IBM Research led by Kalman. Bertram's results were based on root-locus analysis; an analytical proof was given in 1960 [Ris60]. Most undergraduate textbooks on control contain material on state space systems, including, for example, Franklin, Powell, and Emami-Naeini [FPEN05] and Ogata [Oga01]. Friedland's textbook [Fri04] covers the material in the previous, current, and next chapter in considerable detail, including the topic of optimal control.

Exercises

7.1 (Double integrator) Consider the double integrator. Find a piecewise constant control strategy that drives the system from the origin to the state $x = (1, 1)$.

7.2 (Reachability from nonzero initial state) Extend the argument in Section 7.1 to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state.

7.3 (Cayley–Hamilton theorem) Let $A \in \mathbb{R}^{n\times n}$ be a matrix with characteristic polynomial $\lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. Show that the matrix A satisfies

$$\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0,$$

where the zero on the right hand side represents a matrix of elements with all zeros. Use this result to show that A^n can be written in terms of lower order powers of A and hence any matrix polynomial in A can be rewritten using terms of order at most $n - 1$.

7.4 (Unreachable systems) Consider a system with the state x and z described by the equations

$$\frac{dx}{dt} = Ax + Bu, \qquad \frac{dz}{dt} = Az + Bu.$$

If $x(0) = z(0)$ it follows that $x(t) = z(t)$ for all t regardless of the input that is applied. Show that this violates the definition of reachability and further show that the reachability matrix W_r is not full rank.

7.5 (Rear-steered bicycle) A simple model for a bicycle was given by equation (4.5) in Section 4.2. A model for a bicycle with rear-wheel steering is obtained by reversing the sign of the velocity in the model. Determine the conditions under which this systems is reachable and explain any situations in which the system is not reachable.

7.6 (Characteristic polynomial for reachable canonical form) Show that the characteristic polynomial for a system in reachable canonical form is given by equation (7.7) and that

$$\frac{d^n z_k}{dt^n} + a_1 \frac{d^{n-1} z_k}{dt^{n-1}} + \cdots + a_{n-1} \frac{dz_k}{dt} + a_n z_k = \frac{d^{n-k} u}{dt^{n-k}},$$

where z_k is the kth state.

7.7 (Reachability matrix for reachable canonical form) Consider a system in reachable canonical form. Show that the inverse of the reachability matrix is given by

$$\tilde{W}_\mathrm{r}^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ & 1 & a_1 & \cdots & a_{n-2} \\ & & 1 & \ddots & \vdots \\ & 0 & & \ddots & a_1 \\ & & & & 1 \end{pmatrix}.$$

7.8 (Non-maintainable equilibrium points) Consider the normalized model of a pendulum on a cart

$$\frac{d^2 x}{dt^2} = u, \qquad \frac{d^2\theta}{dt^2} = -\theta + u,$$

where x is cart position and θ is pendulum angle. Can the angle $\theta = \theta_0$ for $\theta_0 \neq 0$ be maintained?

7.9 (Eigenvalue assignment) Consider the system

$$\frac{dx}{dt} = Ax + Bu = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} a-1 \\ 1 \end{pmatrix} u,$$

with $a = 1.25$. Design a state feedback that gives $\det(sI - BK) = s^2 + 2\zeta_\mathrm{c}\omega_\mathrm{c} s + \omega_\mathrm{c}^2$, where $\omega_\mathrm{c} = 5$, and $\zeta_\mathrm{c} = 0.6$.

7.10 (Eigenvalue assignment for unreachable system) Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_\mathrm{f} r.$$

Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

7.11 (Motor drive) Consider the normalized model of the motor drive in Exercise 3.7. Using the following normalized parameters,

$$J_1 = 10/9, \qquad J_2 = 10, \qquad c = 0.1, \qquad k = 1, \qquad k_I = 1,$$

verify that the eigenvalues of the open loop system are $0, 0, -0.05 \pm i$. Design a state feedback that gives a closed loop system with eigenvalues -2, -1, and $-1 \pm i$. This choice implies that the oscillatory eigenvalues will be well damped and that the eigenvalues at the origin are replaced by eigenvalues on the negative real axis. Simulate the responses of the closed loop system to step changes in the reference signal for θ_2 and a step change in a disturbance torque on the second rotor.

7.12 (Whipple bicycle model) Consider the Whipple bicycle model given by equation (4.8) in Section 4.2. Using the parameters from the companion web site, the model is unstable at the velocity $v_0 = 5$ m/s and the open loop eigenvalues are -1.84, -14.29, and $1.30 \pm 4.60i$. Find the gains of a controller that stabilizes the bicycle and gives closed loop eigenvalues at -2, -10, and $-1 \pm i$. Simulate the response of the system to a step change in the steering reference of 0.002 rad.

7.13 (Dominant eigenvalues) Consider the following two linear systems:

$$\Sigma_1 : \begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -1.1 & -0.1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} 1.01 & 0.11 \end{pmatrix} x, \end{aligned} \qquad \Sigma_2 : \begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -1.1 & -0.1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} 1.1 & 1.01 \end{pmatrix} x. \end{aligned}$$

Show that although both systems have the same eigenvalues, the step responses of the two systems are dominated by different sets of eigenvalues.

7.14 Consider the second-order system

$$\frac{d^2y}{dt^2} + 0.5\frac{dy}{dt} + y = a\frac{du}{dt} + u.$$

Let the initial conditions be zero.

(a) Show that the initial slope of the unit step response is a. Discuss what it means when $a < 0$.

(b) Show that there are points on the unit step response that are invariant with a. Discuss qualitatively the effect of the parameter a on the solution.

(c) Simulate the system and explore the effect of a on the rise time and overshoot.

7.15 (Integral feedback for rejecting constant disturbances) Consider a linear system of the form

$$\frac{dx}{dt} = Ax + Bu + Fd, \qquad y = Cx,$$

where u is a scalar and v is a disturbance that enters the system through a disturbance vector $F \in \mathbb{R}^n$. Assume that the matrix A is invertible and the zero frequency gain $CA^{-1}B$ is nonzero. Show that integral feedback can be used to compensate for a constant disturbance by giving zero steady-state output error even when $d \neq 0$.

7.16 (Bryson's rule) Bryson and Ho [BH75] have suggested the following method for choosing the matrices Q_x and Q_u in equation (7.29). Start by choosing Q_x and Q_u as diagonal matrices whose elements are the inverses of the squares of the maxima of the corresponding variables. Then modify the elements to obtain a compromise among response time, damping, and control effort. Apply this method to the motor drive in Exercise 7.11. Assume that the largest values of the φ_1 and φ_2 are 1, the largest values of $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are 2, and the largest control signal is 10. Simulate the closed loop system for $\varphi_2(0) = 1$ and all other states are initialized to 0. Explore the effects of different values of the diagonal elements for Q_x and Q_u.

7.17 (LQR proof) Use the Riccati equation (7.31) and the relation

$$x^T(t_\mathrm{f})Q_\mathrm{f}x(t_\mathrm{f}) - x^T(0)S(0)x(0) = \int_0^{t_\mathrm{f}} \Big(\dot{x}^T(t)S(t)x(t) + x^T\dot{S}(t)x(t) + x^T(t)S(t)\dot{x}(t)\Big)dt$$

to show that the cost function for the linear quadratic regulator problem can be written as

$$\int_0^{t_\mathrm{f}} \Big(x^T(t)Q_x x(t) + u^T(t)Q_u u(t)\Big)dt + x^T(t_\mathrm{f})Q_\mathrm{f}x(t_\mathrm{f})$$
$$= x^T(0)S(0)x(0) + \int_0^{t_\mathrm{f}} \Big(u(t) + Q_u^{-1}B^T S(t)x(t)\Big)^T Q_u \Big(u(t) + Q_u^{-1}B^T S(t)x(t)\Big)dt,$$

from which it follows that the control law $u(t) = -Kx(t) = -Q_u^{-1}B^T S(t)x(t)$ is optimal. Does the proof hold when all matrices depend on time?