Feedback Systems

An Introduction for Scientists and Engineers

SECOND EDITION

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Chapter Two
Feedback Principles

Feedback - it is the fundamental principle that underlies all self-regulating systems, not only machines but also the processes of life and the tides of human affairs.

A. Tustin, "Feedback", *Scientific American*, 1952 [Tus52].

This chapter presents examples that illustrate fundamental properties of feedback: disturbance attenuation, command signal following, robustness to uncertainty, and shaping of behavior. The analysis is based on simple static and dynamical models. After reading this chapter, readers should have some insight into the power of feedback, they should know about transfer functions and block diagrams, and they should be able to design simple feedback systems. The basic concepts described in this chapter are explained in more detail in the remainder of the text, and this chapter can be skipped for readers who prefer to move directly to the more detailed analysis and design techniques.

2.1 Nonlinear Static Models

We will start by capturing the behavior of the process and the controller using static models. Although these models are very simple, they give significant insight about the fundamental properties of feedback: negative feedback increases the range of linearity, it improves command signal following, and it reduces the gain and the effects of disturbances and parameter variations. Moderate positive feedback has the opposite properties: it shrinks the range of linearity and increases the gain of the system. At a critical value the gain becomes infinite and the system behaves like a relay; larger values of the gain gives hysteretic behavior. Although static models give some insight, they cannot capture dynamic phenomena like stability. Positive feedback combined with dynamics often leads to instability and oscillations, as will be discussed toward the end of the chapter.

Consider the closed loop system whose block diagram is shown in Figure 2.1. The closed loop system has a command signal or a reference r that gives the desired system output. The controller C has an input e that is the difference between the reference r and the process output y, and the output of the controller is the control signal u. There is also a load disturbance v at the process input that perturbs the system. Although we will mostly deal with negative feedback, this simple model also permits analysis of positive feedback.

The process P is modeled as a function that is linear for inputs that are less than one in magnitude and saturates for inputs of magnitude larger than one. The

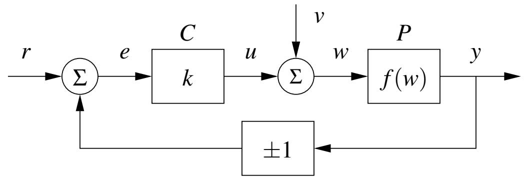


Figure 2.1: Block diagram of simple, static feedback system. The controller is a constant gain $k > 0$ and the process is modeled by a nonlinear function $F(w)$. The process output is y, the control signal is u, the external signals are the reference r, and the load disturbance v. The sign in the lower block indicates whether the feedback is positive (+) or negative (−).

controller is modeled by a constant gain k. Formally the process and the controller are described by the functions

$$y = F(w) = \operatorname{sat}(w) = \begin{cases} -1 & \text{if } w \leq -1, \\ w & \text{if } |w| < 1, \\ 1 & \text{if } w \geq 1, \end{cases} \quad \text{and} \quad u = ke. \tag{2.1}$$

The process is linear for $|w| < 1$, which is called the *linearity region*. In this region we have $y = w$ and the *process gain* is 1. The *controller gain* is k because the controller's output u is k times its input e.

The *open loop system* is the combination of the controller and the process when there is no feedback. Neglecting the disturbance v, it follows from equation (2.1) that the input/output relation for the open loop system is

$$y = F(kr) = \operatorname{sat}(kr). \tag{2.2}$$

It has the gain k and the linearity region $|r| < 1/k$.

Response to Command Signals

To explore how well the system output y can follow the command signal r we assume that the load disturbance v in Figure 2.1 is zero. We will first consider *negative feedback* by setting the gain in the lower block of Figure 2.1 to -1. It follows from Figure 2.1 and equation (2.1) that the closed loop system is described by

$$y = \operatorname{sat}(u), \qquad u = k(r - y). \tag{2.3}$$

Eliminating u in these equations we obtain

$$y = \operatorname{sat}(k(r - y)). \tag{2.4}$$

To find the relation between the reference r and the output y we have to solve an algebraic equation. In the linear range $|k(r-y)| < 1$ we have $y = \frac{k}{k+1} r$. When $|k(r-y)| \geq 1$ the output saturates and we obtain $y = \pm 1$ (depending on the sign of

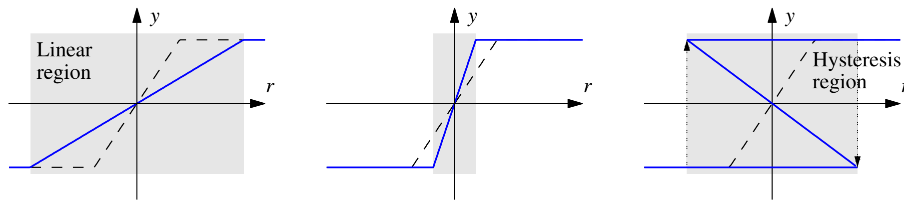


Figure 2.2: Input/output behavior of the system: (a) for large negative feedback (b) positive feedback $k < 1$ and (c) large positive feedback. The solid line is the response of the closed loop system and the dotted line is the response of the open loop system. Redrawn from [SGA18, Figure 20.5].

$k(r-y)$. It can be shown that the overall input/output relationship satisfies

$$y = \text{sat}\Big(\frac{k}{k+1}r\Big) = \begin{cases} -1 & r \le -\frac{k+1}{k}, \\ \frac{k}{k+1}r & |r| < \frac{k+1}{k}, \\ 1 & r \ge \frac{k+1}{k}. \end{cases} \tag{2.5}$$

The linearity range for the closed loop system is $|r| < \frac{k+1}{k}$. Comparing with equation (2.2) we find that negative feedback widens the linear range of the system by a factor of $k+1$ compared to the open loop system. This is illustrated in Figure 2.2a, which shows the input/output relations of the open loop system (dashed) and the closed loop system (solid).

Robustness to Parameter Uncertainty

Next we will investigate the sensitivity of the closed loop system to gain variations. The *sensitivity* of a system describes how changes in the system parameters affect the performance of the system. For the open loop system in the linear range we have $y = kw$ and it thus follows that

$$\frac{dy}{dk} = w = \frac{y}{k}, \quad \Rightarrow \quad \frac{dy}{y} = \frac{dk}{k}. \tag{2.6}$$

The relative change of the output is thus equal to the relative change of the parameter and we say that the sensitivity is 1. Thus, for the open loop system, a change in k of 10% will lead to a change in the output of 10%.

For the closed loop system with an input in the linear range, it follows from equation (2.5) that

$$\frac{dy}{dk} = \frac{r}{k+1} - \frac{kr}{(k+1)^2} = \frac{r}{(k+1)^2} = \frac{y}{k(k+1)},$$

and hence

$$\frac{dy}{y} = \frac{1}{k+1}\frac{dk}{k}. \tag{2.7}$$

A comparison with equation (2.6) shows that negative feedback with gain k reduces the sensitivity to gain variations by a factor of $k+1$. If k is 100, for example, a 10% change in k would lead to less than a 0.1% change in y, so the closed loop system is *much* less sensitive to parameter variation.

This type of analysis can also be used to investigate the effect of positive feedback. If the -1 in the feedback loop in Figure 2.1 is replaced by $+1$, equation (2.5) becomes

$$y = \operatorname{sat}\Big(\frac{k}{-k+1}\,r\Big). \tag{2.8}$$

Notice that the gain of the closed loop system is positive and larger than the open gain for $k < 1$, as shown in Figure 2.2b. The linearity range is $|r| < (1-k)/k$. A comparison with the open loop system in equation (2.2) shows that positive feedback with $k < 1$ shrinks the linearity range by a factor of $1-k$. As k approaches 1 the closed loop gain approaches infinity, the range shrinks to zero, and the system behaves like a relay.

For positive feedback with $k > 1$ it follows from equation (2.8) that the closed loop gain is negative, as shown in Figure 2.2c, and that it approaches -1 as k approaches infinity. Positive feedback with large gains creates an input/output characteristic with multiple output values possible for inputs in the range $|r| < k/(k+1)$ and the closed loop system behaves like a switch with hysteresis. This concept is explored in more detail in Section 2.6, and it is shown that if the process has dynamics then all points where the input/output characteristics has negative slope are unstable.

We will mostly deal with negative feedback but there are systems that employ positive feedback, which is illustrated by the following example.

Example 2.1 The Superregenerative Amplifier

Armstrong constructed a "superregenerative" radio receiver with only one vacuum tube in 1914, when he was still an undergraduate at Columbia University. The superregenerative amplifier can be modeled as an amplifier with open loop gain k and a saturated output, combined with a positive feedback loop, as shown in Figure 2.1. Using equation (2.8), we can compute the gain of the closed loop system to be $k_{\mathrm{cl}} = k/(1-k)$. A very large closed loop gain can be obtained by selecting a feedback gain k that is just below 1. Choosing $k = 0.999$ gives $k_{\mathrm{cl}} = 999$, which is a gain increase of almost three orders of magnitude.

The drawback by using positive feedback is that the system is highly sensitive and that the gain has to be adjusted carefully to avoid oscillations. For example, if the gain k is 0.99 instead of 0.999 (a difference of less than 1%), then the closed loop gain becomes $k_{\mathrm{cl}} = 99$, a difference of 10X (or 1000%). The oscillatory nature of this circuit requires the use of a more advanced (dynamic) model for analysis of the amplifier.

Despite its limitations, this type of amplifier is still used in simple walkie-talkies, garage door openers, and toys. ∇

Load Disturbance Attenuation

Another use of feedback is to reduce the effects of external disturbances, represented by the signal v in Figure 2.1. For the open loop system, the output when $v \neq 0$ is given by

$$y = \text{sat}(kr + v).$$

In the linear region we thus have a gain of 1 between v and y, so that disturbances are passed through with no attenuation.

To investigate the effect of feedback on load disturbances we consider the system in Figure 2.1 with negative feedback and, for simplicity, we set the reference signal r to be zero. The relationship between the load disturbance v and the the output y is given by $y = \text{sat}(v - ky)$, which is again an algebraic equation. In the linear range we get $y = v/(k+1)$ and more generally it can be shown that

$$y = \text{sat}\left(\frac{v}{k+1}\right). \tag{2.9}$$

In the linear region, negative feedback thus reduces the effect of load disturbances by the factor $k+1$. The analysis of the effects of positive feedback is discussed in Exercise 2.1.

Combining these three sets of analyses, we see that negative feedback increases the range of linearity of the system, decreases the sensitivity of the system to parameter uncertainty, and attenuates load disturbances. The trade-off is that the closed loop gain is decreased. Positive feedback has the opposite effect: it can increase the closed loop gain, but at the cost of increased sensitivity and amplification of disturbances.

2.2 Linear Dynamical Models

The analysis in the previous section was based on static models and the dynamics of the process were neglected. We will now introduce a set of concepts and tools to analyze the effects of dynamics. To do this we will introduce block diagrams, linear differential equations, and transfer functions. The block diagram is an abstraction that describes a system as an interconnection of blocks, whose input/output behavior is described by differential equations. The transfer function, which is a function of complex variables, is a convenient representation of the differential equations describing the dynamics of the system. Transfer functions make it possible for us to find the relations between the signals of a complex system represented by block diagrams using simple algebra. The values of the transfer function on the imaginary axis gives the steady state response to sinusoidal signals, which means that the transfer function can be determined experimentally from the steady state response to sinusoidal signals.

Linear Differential Equations and Transfer Functions

In many practical situations, the input/output behavior of a system can be modeled by a linear differential equation of the form

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u, \tag{2.10}$$

where u is the input, y is the output, and the coefficients a_k and b_k are real numbers. The differential equation (2.10) is characterized by two polynomials

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n, \qquad b(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_m, \tag{2.11}$$

where $a(s)$ is the *characteristic polynomial* of the differential equation (2.10). We assume that the polynomials $a(s)$ and $b(s)$ do not have common roots. (The consequences of having common roots is discussed in Section 8.3.)

Equation (2.10) represents a *time invariant* system because if the pair $u(t), y(t)$ satisfies the equation so does $u(t+\tau), y(t+\tau)$. The equation is also *linear* because if $u_1(t), y_1(t)$, and $u_2(t), y_2(t)$ satisfy the equation so does $\alpha u_1(t) + \beta u_2(t), \alpha y_1(t) + \beta y_2(t)$, where α and β are real numbers. Systems that are linear and time invariant are often called *LTI systems*. We can visualize these systems as being characterized by a huge table of corresponding input/output signal pairs. An interesting property of an LTI system is that it can be characterized by a single carefully chosen pair, for example the response of the system to a step input.

The solution to equation (2.10) is the sum of two terms: the general solution to the *homogeneous equation*, which does not depend on the input, and a *particular solution*, which depends on the input. The homogeneous equation associated with equation (2.10) is

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = 0. \tag{2.12}$$

Letting s_k represent the roots of the *characteristic equation* $a(s) = 0$, the solution to equation (2.12) is of the form

$$y(t) = \sum_{k=1}^{n} C_k e^{s_k t} \tag{2.13}$$

if the characteristic equation does not have repeated roots. The numbers $C_1, \ldots, C_n$ can be determined from the initial conditions at $t = 0$.

Since the coefficients a_k are real, the roots of the characteristic equation are either real-valued or occur in complex conjugate pairs. A real root s_k of the characteristic equation corresponds to the exponential function $e^{s_k t}$. This function decreases over time if s_k is negative, is constant if $s_k = 0$, and increases if s_k is positive, as shown in the top row of Figure 2.3. For real roots s_k the parameter $T = 1/s_k$ is called the *time constant*, because it describes how quickly the signal decays.

A complex root $s_k = \sigma \pm i\omega$ corresponds to the time functions

$$e^{\sigma t} \sin(\omega t), \qquad e^{\sigma t} \cos(\omega t),$$

which have oscillatory behavior, as illustrated in the bottom row of Figure 2.3.

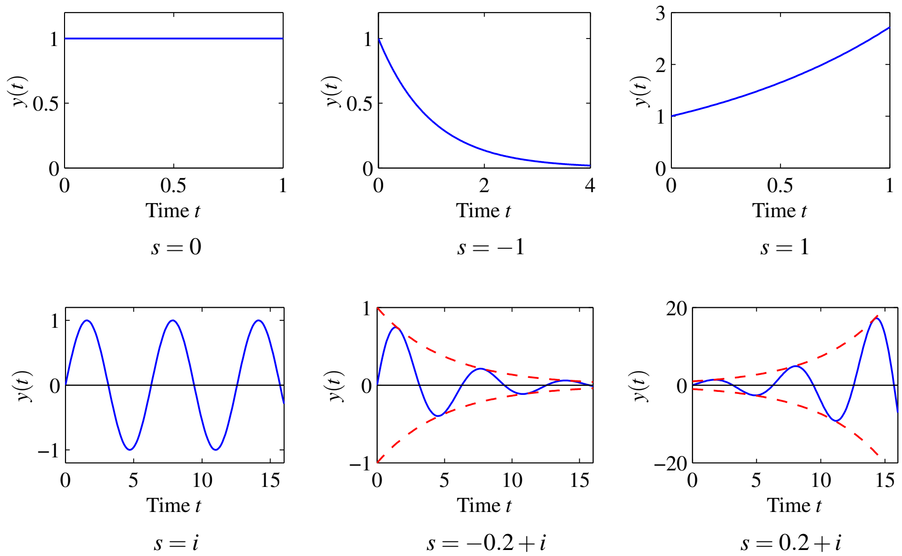


Figure 2.3: Examples of exponential signals. The top row corresponds to exponential signals with a real exponent, and the bottom row corresponds to those with complex exponents. The dashed line in the last two cases denotes the bounding envelope for the oscillatory signals. In each case, if the real part of the exponent is negative then the signal decays, while if the real part is positive then it grows.

The sine terms are shown as solid lines; they have zero crossings with the spacing π/ω. The dashed lines show the envelopes, which correspond to the exponential function $\pm e^{\sigma t}$.

When the characteristic equation (2.13) has repeated roots, the solutions to the homogeneous equation (2.12) take the form

$$y(t) = \sum_{k=1}^{m} C_k(t) e^{s_k t}, \tag{2.14}$$

where $C_k(t)$ is a polynomial with degree less than the multiplicity of the root s_k. The solution (2.14) has $\sum_{k=1}^{m}(\deg C_k + 1) = n$ free parameters.

Having explored the solution to the homogeneous equation, we now turn to the input-dependent part of the solution. The solution to equation (2.10) for an exponential input is of particular interest, as will be shown in the following. We set $u(t) = e^{st}$, where $s \neq s_k$ is a complex number, and investigate if there is a unique particular solution of the form $y(t) = G(s)e^{st}$. Assuming this to be the case, we find

$$\begin{aligned} \frac{du}{dt} &= se^{st}, & \frac{d^2u}{dt^2} &= s^2e^{st}, & \cdots \quad \frac{d^mu}{dt^m} &= s^me^{st} \\ \frac{dy}{dt} &= sG(s)e^{st}, & \frac{d^2y}{dt^2} &= s^2G(s)e^{st}, & \cdots \quad \frac{d^ny}{dt^n} &= s^nG(s)e^{st}. \end{aligned} \tag{2.15}$$

Inserting these expressions into the differential equation (2.10) gives

$$(s^n + a_1 s^{n-1} + \cdots + a_n)G(s)e^{st} = (b_0 s^m + b_1 s^{m-1} + \cdots + b_m)e^{st}$$

and hence

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)}. \tag{2.16}$$

This function is called the *transfer function* of the system. It describes a particular solution to the differential equation for the input e^{st}. Combining this with the solution to the homogeneous equation, we find that a solution to the differential equation (2.10) for the exponential input $u(t) = e^{st}$ is

$$y(t) = \sum_{k=1}^{m} C_k(t)e^{s_k t} + G(s)e^{st}. \tag{2.17}$$

The relation between the transfer function (2.16) and the differential equation (2.10) is clear: the transfer function (2.16) can be obtained by inspection from the differential equation (2.10), and conversely the differential equation can be obtained from the transfer function if the polynomials $a(s)$ and $b(s)$ do not have common factors. The transfer function can thus be regarded as a shorthand notation for the differential equation (2.10). It is a complete characterization of the differential equation even if it was derived as the response to a specific input $u(t) = e^{st}$. We note that the input and the initial conditions must *both* be given to obtain the full solution of the differential equation, also referred to as the *response* of the system.

To deal with oscillatory signals, like those shown in the bottom row of Figure 2.3, we allow s to be a complex number. The transfer function G is then a function that maps complex numbers to complex numbers. We let arg represent the argument (phase, angle) of a complex number and $|\cdot|$ the magnitude, and note that the complex response to an input $u = e^{i\omega t} = \cos\omega t + i\sin\omega t$ is given by $G(i\omega t)e^{i\omega t}$. Using just the imaginary parts of the signals, it follows that the particular solution for the input $u = \sin(\omega t) = \operatorname{Im} e^{i\omega t}$ is

$$\begin{aligned} y(t) &= \operatorname{Im}\left(G(i\omega)\, e^{i\omega t}\right) = \operatorname{Im}\left(|G(i\omega)|\, e^{i\arg G(i\omega)}\, e^{i\omega t}\right) \\ &= |G(i\omega)| \operatorname{Im} e^{i(\arg G(i\omega) + \omega t)} = |G(i\omega)| \sin(\omega t + \arg G(i\omega)). \end{aligned}$$

The input is thus amplified by $|G(i\omega)|$ and the phase shift between input and output is $\arg G(i\omega)$. The functions $G(i\omega)$, $|G(i\omega)|$, and $\arg G(i\omega)$ are called the *frequency response*, *gain*, and *phase*. Gain and phase are also called *magnitude* and *angle*.

When the input and the output are constant, $u(t) = u_0$ and $y(t) = y_0$, the differential equation (2.10) has the particular solution $y(t) = (b_n/a_n)u_0 = G(0)u_0$, obtained by setting $s = 0$. The input is thus amplified by the factor $G(0)$, which is therefore called the *zero frequency gain* (or sometimes the *static gain*). If the differential equation is stable the solution will converge to $G(0)u_0$ as t goes to infinity.

The full response to an exponential input is the sum of a particular solution and a solution to the homogeneous equation that is determined by the initial conditions,

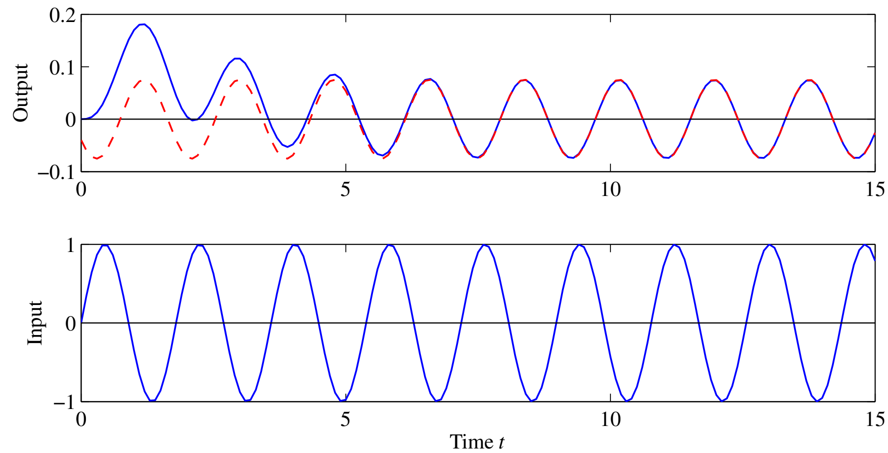


Figure 2.4: Two responses of a linear time-invariant system to a sinusoidal input. The dashed line shows the output when the initial conditions are chosen so that the output is purely sinusoidal. The solid line shows the response response for the initial conditions $y(0) = 0$ and $y'(0) = 0$. The transfer function $G(s) = 1/(s+1)^2$.

as given in equation (2.17). An illustration is given in Figure 2.4 for the transfer function $G(s) = 1/(s+1)^2$. The dashed line, which is a pure sine wave, is the solution obtained when all C_k in equation (2.17) are zero. The solid line shows the response obtained when the C_k are chosen so that $y(0)$ and its derivatives $y^{(k)}(0)$, $k = 1, \ldots, n-1$ are all zero. Since all roots of the characteristic equation have negative real parts, the solution to the homogeneous equation (2.14) goes to zero as $t \to \infty$ and the general solution converges to the particular solution.

The transfer function has many interpretations that can be exploited for insight, analysis, and design. The roots s_k of the characteristic equation $a(s) = 0$ are called *poles* of the transfer function: the transfer function is infinite for $s = s_k$. The poles s_k appear as exponents in the general solution to the homogeneous equation, as seen in equations (2.13) and (2.14). Systems with poles that are "lightly damped" ($\mathrm{Re}(s_k)$ is negative but close to zero) can exhibit resonances when a sinusoidal input is applied whose frequency is near the imaginary part of s_k.

The roots s_j of the polynomial $b(s)$ are called *zeros* of the transfer function. The reason is that if $b(s_j) = 0$ it follows that $G(s_j) = 0$, and the particular solution for the input $e^{s_k t}$ is then zero. A system theoretic interpretation is that the transmission of the exponential signal $e^{s_j t}$ is blocked by the zero $s = s_j$, which is therefore also called a *transmission zero*.

The transfer function can also convey a great deal of intuition: $G(0)$ is the zero frequency gain for constant inputs and the frequency response $G(i\omega)$ captures the steady state response to sinusoidal functions. The frequency response of a stable system can be determined experimentally by exploring the steady state response of a system to sinusoidal signals. This is an alternative or a complement to physical modeling. A more elaborate treatment of transfer functions and the frequency Ⓐ

response will be given in Chapter 9.

Stability: The Routh–Hurwitz Criterion

When using feedback there is always the danger that the system may become unstable, and it is therefore important to have a stability criterion. The differential equation (2.10) is called *stable* if all solutions of the homogeneous equation (2.12) go to zero for any initial condition. It follows from equation (2.14) that this requires that all the roots of the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

have negative real parts.

It can often be difficult to analytically compute the roots of a high-order polynomial. The *Routh–Hurwitz criterion* is a stability criterion that does not require explicit calculation of the roots, because it gives conditions in terms of the coefficients of the characteristic polynomial.

We illustrate the Routh–Hurwitz criterion by describing it for low-order differential equations. A first-order differential equation is stable when the coefficient a_1 of the characteristic polynomial is positive, since the root of the characteristic polynomial will be $s = -a_1 < 0$. A second-order polynomial has the roots

$$s = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_2}\right),$$

and it is easy to verify that the real parts of the roots are both negative if and only if $a_1 > 0$ and $a_2 > 0$. A third order differential equation is more complicated, but the roots can be shown to have negative real parts if and only if

$$a_1, a_2, a_3 > 0, \quad \text{and} \quad a_1 a_2 > a_3. \tag{2.18}$$

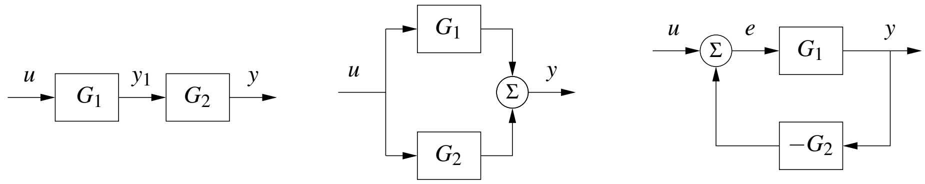
The corresponding conditions for a fourth order differential equation are

$$a_1, a_2, a_3, a_4 > 0, \qquad a_1 a_2 > a_3, \quad \text{and} \quad a_1 a_2 a_3 > a_1^2 a_4 + a_3^2. \tag{2.19}$$

The Routh–Hurwitz criterion [Gan60] gives similar conditions for arbitrarily high order polynomials. Stability of a linear differential equation can thus be investigated just by analyzing the signs of various combinations of the coefficients of the characteristic polynomial. Ⓗ

Block Diagrams and Transfer Functions

As we saw already in Chapter 1, control systems are often described using block diagrams, such as the ones shown in Figures 1.1 and 1.4. If the behavior of the blocks are represented by transfer functions, the transfer function of a system can be obtained simply by algebraic manipulations. It follows from equation (2.17) that the transfer function can be derived from the particular solution for the input e^{st}. To derive the transfer function for a system composed of several blocks, we assume that the input signal is an exponential $u(t) = e^{st}$ and compute the corresponding particular solutions for all blocks.



(a) $G_{yu}(s) = G_2(s)G_1(s)$ (b) $G_{yu}(s) = G_1(s) + G_2(s)$ (c) $G_{yu}(s) = \dfrac{G_1(s)}{1 + G_1(s)G_2(s)}$

Figure 2.5: Interconnections of linear systems. Series (a), parallel (b) and feedback (c) connections are shown. The transfer functions for the composite systems can be derived by algebraic manipulations assuming exponential functions for all signals.

Consider for example the system in Figure 2.5a, which is a series connection of two systems with the transfer functions $G_1(s)$ and $G_2(s)$. Let the input of the system be $u(t) = e^{st}$ The output of the first block is then $y_1(t) = G_1(s)e^{st}$, which is also an exponential, and the output of the second system is $y(t) = G_2(s)y_1(s) = G_2(s)G_1(s)e^{st} = G_2(s)G_1(s)u(t)$. The transfer function of the system is thus $G_{yu}(s) = G_2(s)G_1(s)$, where we use the convention that the right subscript is the input and the left subscript is the output, so that $y = G_{yu}u$.

Next we will consider parallel connections of systems as shown in Figure 2.5b. Assuming that the input is $u(t) = e^{st}$, the exponential outputs of the blocks are $y_1(t) = G_1(s)e^{st}$ and $y_2(t) = G_s(s)e^{st}$. The output of the system is then

$$y(t) = \big(G_1(s)e^{st} + G_2(s)e^{st}\big) = \big(G_1(s) + G_2(s)\big)\, e^{st},$$

and the transfer function of a parallel connection of systems with the transfer functions $G_{(}s)$ and $G_2(s)$ is thus $G_{yu}(s) = G_1(s) + G_2(s)$.

Finally we will consider the feedback connection shown in Figure 2.5c. If the input $u(t) = e^{st}$ is an exponential we find

$$y(t) = G_1(s)e(t) = G_1(s)\big(u(t) - G_2(s)y(t)\big) = G_1(s)\big(e^{st} - G_2(s)y(t)\big).$$

Solving for $y(t)$ gives

$$y(t) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}\, e^{st} = G_{yu}(s)\, e^{st}.$$

The transfer function of a feedback connection of systems with the transfer functions $G_1(s)$ and $G_2(s)$ is thus

$$G_{yu}(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}. \tag{2.20}$$

By using polynomials and transfer functions the relations between signals in a feedback system can thus be obtained by algebra. With some practice the transfer functions can often be obtained by inspection, as we explore in more detail in Chapter 9.

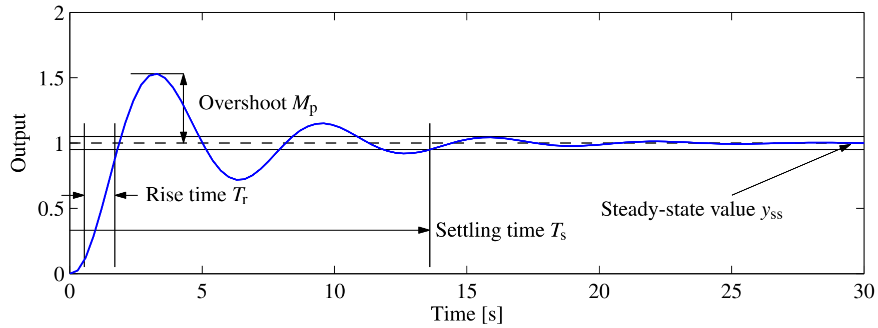


Figure 2.6: Sample step response. The rise time T_r, overshoot M_p, settling time T_s, and steady-state value y_{ss} describe important performance properties of the signal.

Computations Using Transfer Functions

Many software packages for control system analysis and design permit direct manipulation of transfer functions. In MATLAB the transfer function

$$G(s) = \frac{s+1}{(s^2+5s+6)}$$

can be created by the commands `s=tf('s')` and `G=(s+1)/(s^2+5*s+6)`. Given two transfer functions `G1` and `G2`, we can form series, parallel, and feedback interconnections using the commands `Gs = series(G1, G2)`, `Gp = parallel(G1, G2)`, and `Gf = feedback(G1, G2)` (by default, MATLAB's `feedback()` command uses negative feedback).

Software packages can also be used to compute the response of a linear input/output system, represented by its transfer function, to different types of inputs. A common input that is used for performance characterization is a signal that is 0 for $t \leq 0$ and then 1 for $t > 0$. This type of input is called a "step input" and the response of the system to a step input is called the *step response* of the system. A typical step response for a linear system is shown in Figure 2.6. Some standard features of a step response are the rise time T_r, settling time T_s, overshoot M_p, and steady state value y_{ss}, as illustrated in the figure. The step response for a transfer function G is generated by the MATLAB command `y=step(G)`. If we want to specify the simulation time interval explicitly, we can instead use the command `y=step(G,T)`. The response to a specific input signal can be generated by `y=lsim(G,u,T)`. Having a transfer function, it is thus very easy to generate time responses.

A detailed presentation of transfer functions will be given in Chapter 9, where we will see that transfer functions can also be used to represent systems with time delays and systems described by partial differential equations.

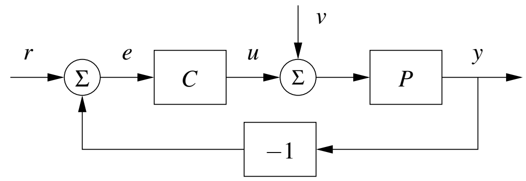


Figure 2.7: Block diagram of a simple feedback system. The controller transfer function is $C(s)$ and the process transfer function is $P(s)$. The process output is y, the external signals are the reference r and the load disturbance v.

2.3 Using Feedback to Improve Disturbance Attenuation

Reducing the effects of disturbances is a primary use of feedback. It was used by James Watt to make steam engines run at constant speed in spite of varying load and by electrical engineers to make generators driven by water turbines deliver electricity with constant frequency and voltage. Feedback is commonly used to alleviate effects of disturbances in the process industry, for machine tools, and for engine and cruise control in cars. The human body exploits feedback to keep body temperature, blood pressure, and other important variables constant. For example the pupillary reflex guarantees that the light intensity of the retina is reasonably constant in spite of large variations in the ambient light intensity. Keeping variables close to a desired, constant reference value in spite of disturbances is called a *regulation problem*.

To discuss disturbance attenuation we consider the system shown in Figure 2.7. Since we will focus on the effects of a load disturbance v we will assume for now that the reference r is zero. To derive the transfer functions from the disturbance input v to the process output y, which we write as G_{yv}, we assume that the disturbance is an exponential function $v = e^{st}$. Applying block diagram algebra to Figure 2.7 gives

$$y(t) = P(s)e^{st} - P(s)C(s)y(t) \quad \Longrightarrow \quad y(t) = \frac{P(s)}{1+P(s)C(s)}\, e^{st}.$$

The transfer function relating the output y to the load disturbance v is thus

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)}. \tag{2.21}$$

To explore the use of feedback to improve disturbance attenuation, we will focus on a simple process modeled by the first order differential equation

$$\frac{dy}{dt} + ay = bu, \qquad a > 0, \quad b > 0.$$

The corresponding transfer function is

$$P(s) = \frac{b}{s+a}. \tag{2.22}$$

This model is a reasonable approximation for a physical process if the storage of mass, momentum, or energy can be captured by a single state variable. Typical examples are the velocity of a car on a road, the angular velocity of a rotating system, and the fluid level of a tank.

Proportional Control

We will first investigate the case of proportional control, when the control signal is proportional to the output error: $u = k_p e$, as introduced already in Section 1.6. The controller transfer function is then $C(s) = k_p$. The process transfer function is given by equation (2.22) and the effect of the disturbance on the output is then described by the transfer function (2.21):

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{b/(s+a)}{1+bk_p/(s+a)} = \frac{b}{s+(a+bk_p)}.$$

The relation between the disturbance v and the output y is thus given by the differential equation

$$\frac{dy}{dt} + (a+bk_p)y = bv.$$

The closed loop system is stable if $a + bk_p > 0$. A constant disturbance $v = v_0$ then gives an output that exponentially approaches the value

$$y_0 = G_{yv}(0)v_0 = \frac{b}{a+bk_p} v_0$$

with the time constant $T = 1/(a+bk_p)$. Without feedback, $k_p = 0$ and for a constant disturbance v_0, the output will instead approach bv_0/a. The effect of the disturbance is thus reduced if $k_p > 0$.

We have thus shown that a constant disturbance gives an error that can be reduced by feedback using a proportional controller. The error decreases with increasing controller gain. Figure 2.8a shows the responses for a few values of the controller gain k_p.

Proportional-Integral (PI) Control

The PI controller, introduced in Section 1.6, is described by

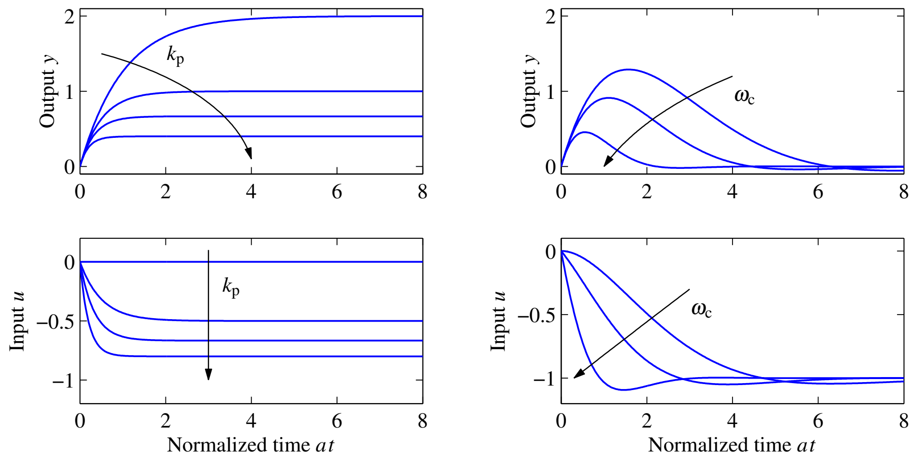
$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)d\tau. \qquad (2.23)$$

To determine the transfer function of the controller we differentiate to obtain

$$\frac{du}{dt} = k_p \frac{de}{dt} + k_i e,$$

and we find that the transfer function is $C(s) = k_p + k_i/s$. To investigate the effect of the disturbance v on the output we use the block diagram in Figure 2.7, and the transfer function from v to y is

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{bs}{s^2+(a+bk_p)s+bk_i}. \qquad (2.24)$$



(a) Proportional control (b) Proportional-integral (PI) control

Figure 2.8: Step responses for a first-order, closed loop system with proportional control (a) and PI control (b). The process transfer function is $P = 2/(s+1)$. The controller gains for proportional control are $k_p = 0$, 0.5, 1, and 2. The PI controller is designed using equation (2.28) with $\zeta_c = 0.707$ and $\omega_c = 0.707$, 1, and 2, which gives the controller parameters $k_p = 0$, 0.207, and 0.914 and $k_i = 0.25$, 0.50, and 2.

Using the relationship between transfer functions and differential equations given by equations (2.10) and (2.16), it follows that the relation between the load disturbance and the output is given by the differential equation

$$\frac{d^2y}{dt^2} + (a + bk_p)\frac{dy}{dt} + bk_i y = b\frac{dv}{dt}. \tag{2.25}$$

Notice that since the disturbance enters as a derivative on the right hand side, a constant disturbance gives no steady state error. The same conclusion can be drawn from the observation that $G_{yv}(0) = 0$. This is consistent with the discussion of integral action and steady state error in Section 1.6.

To find suitable values of the controller parameters k_p and k_i, we consider the characteristic polynomial of the differential equation (2.25),

$$a_{cl}(s) = s^2 + (a + bk_p)s + bk_i. \tag{2.26}$$

We can assign arbitrary roots to the characteristic polynomial by choosing the controller gains k_p and k_i. The most common case is that we assign complex roots that give the characteristic polynomial

$$(s + \sigma_d + i\,\omega_d)(s + \sigma_d - i\,\omega_d) = s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2. \tag{2.27}$$

By construction, this polynomial has roots at $s = -\sigma_d \pm i\,\omega_d$. The general solution to the homogeneous equation is then a linear combination of the terms

$$e^{-\sigma_d t}\sin(\omega_d t), \qquad e^{-\sigma_d t}\cos(\omega_d t),$$

which are damped sine and cosine functions, as shown in the lower middle plot in Figure 2.3. The coefficient σ_d determines the decay rate and the parameter ω_d gives the frequency of the decaying oscillation. Identifying coefficients of equal powers of s in the polynomials (2.26) and (2.27) gives

$$k_p = \frac{2\sigma_d - a}{b}, \qquad k_i = \frac{\sigma_d^2 + \omega_d^2}{b}. \tag{2.28}$$

We can thus choose the controller gains to give a desired closed lop response.

Instead of parameterizing the closed loop system in terms of σ_d and ω_d it is common practice to use the *undamped natural frequency* $\omega_c = \sqrt{\sigma_d^2 + \omega_d^2}$ and the *damping ratio* $\zeta_c = \sigma_d/\omega_c$. The closed loop characteristic polynomial is then

$$a_{cl}(s) = s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2 = s^2 + 2\zeta_c\omega_c s + \omega_c^2.$$

This parameterization has the advantage that ζ_c, which is in the range $[-1,1]$, determines the shape of the response and ω_c gives the response speed.

Figure 2.8b shows the output y and the control signal u for $\zeta_c = 1/\sqrt{2} = 0.707$ and different values of the design parameter ω_c. Proportional control gives a steady-state error that decreases with increasing controller gain k_p. With PI control the steady-state error is zero. Both the decay rate and the peak error decrease when the design parameter ω_c is increased. Larger controller gains give smaller errors and control signals that react more quickly to the disturbance.

With the controller parameters (2.28), the transfer function (2.24) from disturbance v to process output y becomes

$$G_{yv}(s) = \frac{P(s)}{1 + P(s)C(s)} = \frac{bs}{s^2 + 2\zeta_c\omega_c s + \omega_c^2}.$$

For efficient attenuation of disturbances, it is desirable that $|G_{yv}(i\omega)|$ is small for all ω. For small values of ω we have $|G_{yv}(i\omega)| \approx b\omega/\omega_c^2$, while for large ω we have $|G_{yv}(i\omega)| \approx b/\omega$. The largest value of $|G_{yv}(i\omega)|$ is $b/(2\zeta_c\omega_c)$ for $\omega = \omega_c$. It thus follows that a large value of ω_c gives good load disturbance attenuation.

In summary, we find that the analysis gives a simple way to find the parameters of PI controllers for processes whose dynamics can be approximated by a first-order system. The technique can be generalized to more complicated systems but the controller will be more complex. To achieve the benefits of large control gains the model must be accurate over wide frequency ranges, as will be discussed next.

Unmodeled Dynamics

The analysis we have made so far indicates that there are no limits to the performance that can be achieved. Figure 2.8b shows that arbitrarily fast response can be obtained simply by making ω_c sufficiently large. In reality there are of course limits on what is achievable. One reason is that the controller gains increase with ω_c: the proportional gain is $k_p = (2\zeta_c\omega_c - a)/b$ and the integral gain is $k_i = \omega_c^2/b$. A large value of ω_c thus gives large controller gains and the control signal may saturate. Another reason is that the model (2.22) is a simplification: it is only valid in a given frequency range. If the model is instead

$$P(s) = \frac{b}{(s+a)(1+sT)}, \tag{2.29}$$

where the term $1+sT$ represents the dynamics of sensors, actuators, or other dynamics that were neglected when deriving equation (2.22)—so-called *unmodeled dynamics*—the closed loop characteristic polynomial for the closed loop system becomes

$$a_{\mathrm{cl}} = s(s+a)(1+sT) + k_{\mathrm{p}}s + k_{\mathrm{i}} = s^3T + s^2(1+aT) + 2\zeta_{\mathrm{c}}\omega_{\mathrm{c}}s + \omega_{\mathrm{c}}^2.$$

It follows from the Routh–Hurwitz criterion (2.18) that the closed loop system is stable if $\omega_{\mathrm{c}}^2 T < 2\zeta_{\mathrm{c}}\omega_{\mathrm{c}}(1+aT)$ or if

$$\omega_{\mathrm{c}}T < 2\zeta_{\mathrm{c}}(1+aT).$$

The frequency ω_{c} and the achievable response time are thus limited by the unmodeled dynamics represented by T, which typically is smaller than the time constant $1/a$ of the process. When models are developed for control it is therefore important to also consider the unmodeled dynamics.

The fact that unmodeled dynamics limit the performance of a feedback system is an important property and must be considered during the system design. It is common to use simplified models when designing components of complex systems and if the unmodeled dynamics of those components (or the other subsystems they interact with) are not properly taken into account, the implementation of the system can display poor behavior (of which instability is one extreme example). As we shall see in later chapters, it is the ability to reason about the effects of uncertainty that makes control theory a particularly powerful mathematical tool for systems design.

2.4 Using Feedback to Follow Command Signals

Another major application of feedback is to make a system output follow a reference value, which is called the *servo problem*. Cruise control, steering of a car, and tracking a satellite with an antenna or a star with a telescope are some examples. Other examples are high performance audio amplifiers, machine tools, and industrial robots.

To illustrate command signal following we will consider the system in Figure 2.7 where the process is a first-order system and the controller is a PI controller with proportional gain k_{p} and integral gain k_{i}. The transfer functions of the process and the controller are

$$P(s) = \frac{b}{s+a}, \qquad C(s) = \frac{k_{\mathrm{p}}s + k_{\mathrm{i}}}{s}. \tag{2.30}$$

Since we will focus on following the command signal r, we will neglect the load disturbance and set $v = 0$. Applying block diagram algebra to the system in Fig-

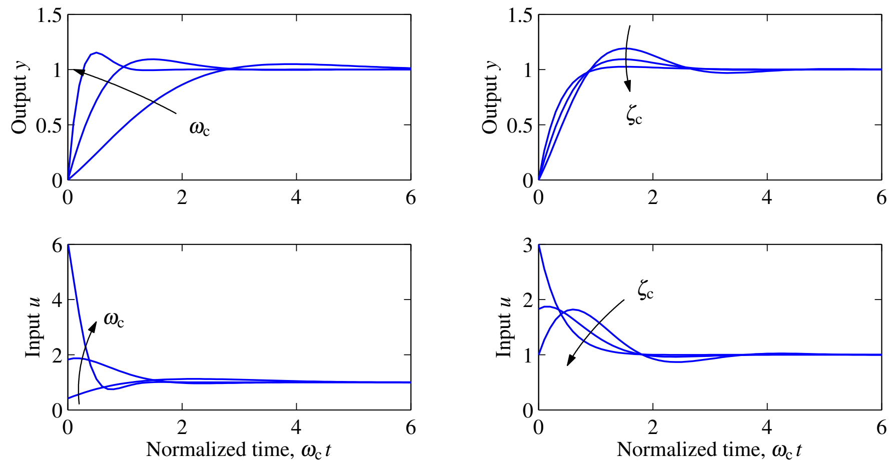


Figure 2.9: Responses to a unit step change in the command signal for different values of the design parameters ω_c and ζ_c. The left figure shows responses for fixed $\zeta_c = 0.707$ and $\omega_c = 1$, 2, and 5. The right figure shows responses for $\omega_c = 2$ and $\zeta_c = 0.5$, 0.707, and 1. The process parameters are $a = b = 1$. The initial value of the control signal is k_p.

ure 2.7, we find that the transfer function from the command signal r to the output y is

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{bk_p s + bk_i}{s^2 + (a+bk_p)s + bk_i}. \tag{2.31}$$

Since $G_{yr}(0) = 1$ it follows that $r = y$ when r and y are constant, independent of the values of the parameters a and b, as long as the closed loop is stable. The steady state output is thus equal to the reference, a consequence of the integral action in the controller.

To determine suitable values of the controller parameters k_p and k_i, we proceed as in Section 2.3 by choosing controller parameters that make the closed-loop characteristic polynomial

$$a_{cl}(s) = s^2 + (a+bk_p)s + bk_i \tag{2.32}$$

equal to $s^2 + 2\zeta_c\omega_c s + \omega_c^2$ with $\zeta_c > 0$ and $\omega_c > 0$. Identifying coefficients of equal powers of s in these polynomials gives

$$k_p = \frac{2\zeta_c\omega_c - a}{b}, \qquad k_i = \frac{\omega_c^2}{b}, \tag{2.33}$$

which is equivalent to equation (2.28). Notice that integral gain increases with the square of ω_c. Figure 2.9 shows the output signal y and the control signal u for different values of the design parameters ζ_c and ω_c. The response time decreases with increasing ω_c and the initial value of the control signal also increases because it takes more effort to move rapidly. The overshoot decreases with increasing ζ_c. For $\omega_c = 2$, the design choice $\zeta_c = 1$ gives a short settling time and a response without overshoot.

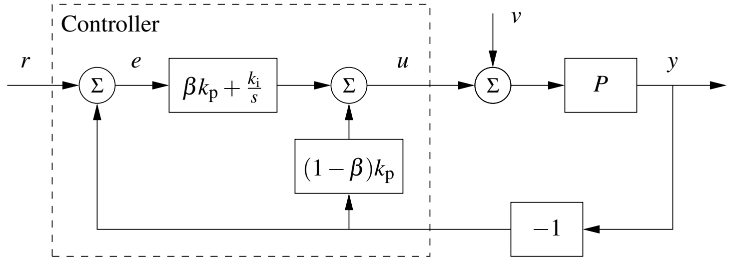


Figure 2.10: Block diagram of a closed-loop system with a PI controller having an architecture with two degrees of freedom.

It is desirable that the output y will track the reference signal r for time-varying references. This means that we would like the transfer function $G_{yr}(s)$ to be close to 1 for large frequency ranges. With the controller parameters (2.33), it follows from equation (2.31) that

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{(2\zeta_c\omega_c - a)s + \omega_c^2}{s^2 + 2\zeta_c\omega_c s + \omega_c^2}.$$

Since $G_{yr}(0) = 1$, tracking of constant inputs is perfect. In addition, if $s = i\omega$ is smaller in magnitude than ω_c, then we see that $G_{yr}(s)$ will be very close to one. The frequency ω_c thus determines the upper bound of the frequency of input signals that can be tracked with small error, and this bound is referred to as the *bandwidth* of the closed loop system. The frequency response of G_{yr} thus provides a quantitative representation of the tracking abilities.

Controllers with Two Degrees of Freedom

The control law in Figure 2.7 has *error feedback* because the control signal u is generated from the error $e = r - y$. With proportional control, a step in the reference signal r gives an immediate step change in the control signal u. This rapid reaction can be advantageous, but it may give large overshoot, which can be avoided by a replacing the PI controller in equation (2.23) with a controller of the form

$$u(t) = k_p\big(\beta r(t) - y(t)\big) + k_i \int_0^t \big(r(\tau) - y(\tau)\big)\, d\tau. \tag{2.34}$$

In this modified PI algorithm, the proportional action only acts on the fraction β of the reference signal. The signal transmissions from reference r to u and from output y to u can be represented by the transfer functions

$$C_{ur}(s) = \beta k_p + \frac{k_i}{s}, \qquad C_{uy}(s) = k_p + \frac{k_i}{s} = C(s). \tag{2.35}$$

The controller (2.34) is called a controller with *two degrees of freedom* since the transfer functions $C_{ur}(s)$ and $C_{uy}(s)$ are different.

A block diagram of a closed loop system with a PI controller having two degrees of freedom is shown in Figure 2.10. Let the process transfer function be

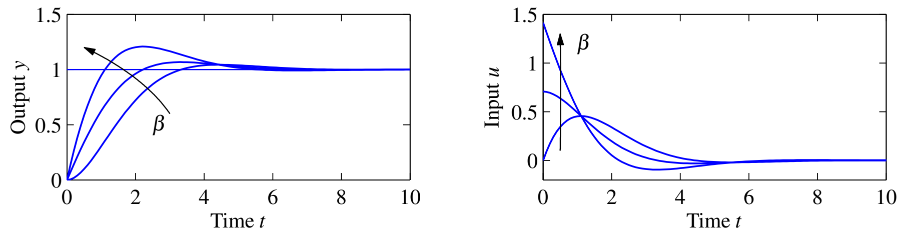


Figure 2.11: Response to a step change in the command signal for a system with a PI controller having two degrees of freedom. The process transfer function is $P(s) = 1/s$ and the controller gains are $k_{\mathrm{p}} = 1.414$, $k_{\mathrm{i}} = 1$, and $\beta = 0$, 0.5, and 1.

$P(s) = b/(s+a)$. The transfer functions from reference r and disturbance v to output y are

$$G_{yr}(s) = \frac{b\beta k_{\mathrm{p}} s + bk_{\mathrm{i}}}{s^2 + (a + bk_{\mathrm{p}})s + bk_{\mathrm{i}}}, \qquad G_{yv}(s) = \frac{bs}{s^2 + (a + bk_{\mathrm{p}})s + bk_{\mathrm{i}}}. \tag{2.36}$$

Comparing with the corresponding transfer function for a controller with error feedback in equations (2.24) and (2.31), we find that the responses to the load disturbances is the same but the response to reference signals is different.

A simulation of the closed loop system for $a = 0$ and $b = 1$ is shown in Figure 2.11. The figure shows that the parameter β has a significant effect on the responses. Comparing the system with error feedback ($\beta = 1$) to the system with smaller values of β we find that using a system with two degrees of freedom gives less overshoot and gentler control actions.

The example shows that command signal response can be improved by using a controller architecture having two degrees of freedom. In Section 12.4 we will further show that the responses to command signals and disturbances can be completely separated by using a more general system architecture. To use a system with two degrees of freedom both the reference signal r and the output signal y must be measured. There are situations where only the error signal $e = r - y$ can be measured; typical examples are DVD players, optical memories, and atomic force microscopes. In these cases, only single degree of freedom (error feedback) controllers can be used.

2.5 Using Feedback to Provide Robustness

Feedback can be used to make good systems from imprecise components. Black's invention of the feedback amplifier for the telephone network is an early example [Bla77]. Black used negative feedback to design extremely good amplifiers with linear characteristics from components with nonlinear and time-varying properties. Since signals are transmitted over long distances they must be amplified. At the time, the thermionic valve—invented by Lee de Forest in 1906—was the only available technology for amplifying electric signals until the transistor was in invented in 1947. Vacuum tubes were the key to develop radio, telephony, and electronics in the first half of the 20th century. They are still used by hi-fi aficionados in high quality audio amplifiers.

Vacuum tubes can give high gain but they have nonlinear and time varying input/output characteristics which distorts the transmitted signals. Bode [Bod60] expressed the problem as follows:

> Most of you with hi-fi systems are no doubt proud of the quality of your amplifiers, but I doubt whether many of you would care to listen to the sound after the signal had gone in succession through several dozen or several hundred even of your fine amplifiers.

The effect is illustrated in Exercise 2.10.

Black's idea to develop a good amplifier was to close a loop with negative feedback around the tube amplifier. In this way he could obtain a closed loop system with a linear input/output relation having constant gain. The general recipe is to localize the nonlinearities and the source of process variations, and to close feedback loops around them. Ⓗ

Reducing Effects of Parameter Variations and Nonlinearities

Consider an amplifier with a static, nonlinear input/output relation with considerable variability, as illustrated in Figure 2.12a. The nominal input/output characteristics is shown as a dashed bold line and examples of variations as thin lines. The nonlinearity in the figure is given by

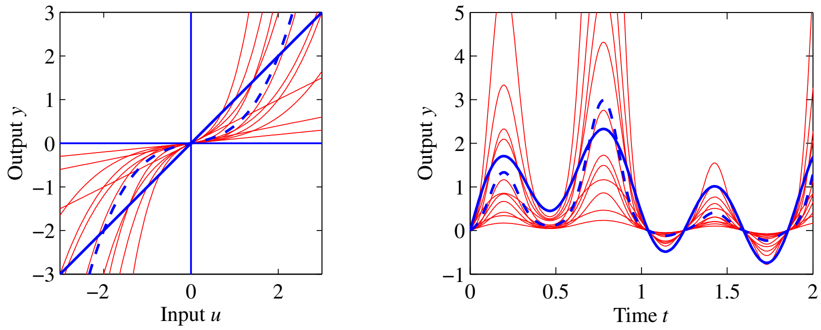
$$y = F(u) = \alpha(u + \beta u^3), \qquad -3 \leq u \leq 3. \tag{2.37}$$

The nominal values corresponding to the dashed line are $\alpha = 0.2$ and $\beta = 1$. The variations of the parameters α and β are in the ranges $0.1 \leq \alpha \leq 0.5$, $0 \leq \beta \leq 2$. The responses of the system to the input

$$u(t) = \sin(t) + \sin(\pi t) + \sin(\pi^2 t). \tag{2.38}$$

are shown in Figure 2.12b. The desired response $y = u$ is shown as a solid bold line and responses for a range of parameters are shown with thin lines. The nominal response of the nonlinear system is shown as a dashed bold line. It is distorted due to the nonlinearity. Notice in particular the heavy distortion both for small and large signal amplitudes.

The behavior of the system is clearly not satisfactory, but it can be improved significantly by introducing feedback. A block diagram of a system with a simple integral controller is shown in Figure 2.13, where the reference input is now taken as r. Figure 2.14 shows the behavior of the closed loop system with the same parameter variations as in Figure 2.12. The input/output plot in Figure 2.14a is a scatter plot of the inputs and the outputs of the feedback system. The input/output relation is practically linear and close to the desired response. There is some variability because of the dynamics introduced by the feedback. Figure 2.14b shows



(a) I/O relationships (b) Output signals

Figure 2.12: Responses of a static nonlinear system. The left figure shows the input/output relations of the open-loop systems and the right figure shows responses to the input signal (2.38). The ideal response is shown solid bold lines. The nominal response of the nonlinear system is shown using dashed bold lines and the responses for different parameter values are shown using thin lines. Notice the large variability in the responses.

the responses to the reference signal; notice the dramatic improvement compared with Figure 2.12b. The tracking error is shown in Figure 2.14c.

Analysis

Analysis of the closed loop system is difficult because it is nonlinear. We can, however, obtain significant insight by using approximations. We first observe that the system is linear when $\beta = 0$. In other situations we can thus approximate the nonlinear function by a straight line around an operating point $u = u_0$. The slope of the nonlinear function at $u = u_0$ is $f'(u_0)$ and we will approximate the process with a linear system with the gain $f'(u_0)$. The transfer functions of the process and the controller are

$$P(s) = f'(u_0) = \alpha(1 + 3\beta u_0^2) = b, \qquad C(s) = \frac{k_\mathrm{i}}{s}, \tag{2.39}$$

where u_0 denotes the operating condition. It follows from equation (2.21) that the transfer functions relating the output y and the error e to the reference signal r are

$$G_{yr}(s) = \frac{bk_\mathrm{i}}{s + bk_\mathrm{i}}, \qquad G_{er}(s) = 1 - G_{yr} = \frac{s}{s + bk_\mathrm{i}}. \tag{2.40}$$

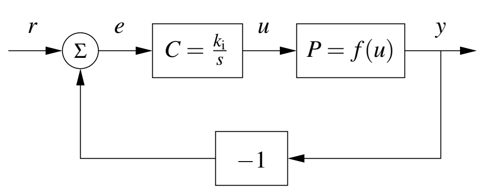
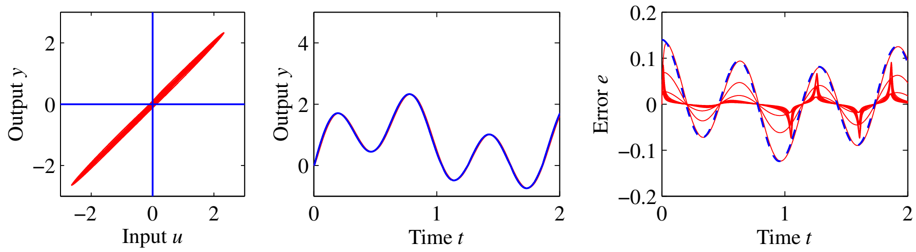


Figure 2.13: Block diagram of a nonlinear system with integral feedback.



(a) I/O relationships (b) Output signals (c) Error

Figure 2.14: Responses of the systems with integral feedback ($k_i = 1000$). The left figure shows the input/output relationships for the open-loop systems, and the center figure shows responses to the input signal (2.38) (compare to the corresponding responses in Figure 2.12a and b). The right figure shows the individual errors (solid lines) and the approximate error given by equation (2.42) (dashed line).

The closed loop system is a first-order system with the pole $s = -bk_i$. The process gain $b = \alpha(1+3\beta u_0^2)$ depends on the values of α, β, and u_0, and its smallest value is 0.1. If the integral gain is chosen as $k_i = 1000$, the smallest value of the closed loop pole is 100 rad/s, which is fast compared to the high frequency component 9.9 rad/s of the input signal. It follows from equation (2.40) that the error $e(t)$ is given by the differential equation

$$\frac{de}{dt} = -bk_i e + \frac{dr}{dt}, \qquad \frac{dr}{dt} = \cos(t) + \pi\cos(\pi t) + \pi^2\cos(\pi^2 t). \tag{2.41}$$

The fast frequency component of the input (2.38) has the frequency $\pi^2 = 9.86$; it is slower than the process dynamics for all parameter variations. Neglecting the term de/dt in equation (2.41) gives

$$e(t) \approx \frac{1}{bk_i}\frac{dr}{dt} \approx \frac{\pi^2}{bk_i}\cos(\pi^2 t). \tag{2.42}$$

An estimate of the largest error $e(t) \approx 0.1\cos(\pi^2 t)$ is obtained for the smallest value of $b = 0.1$. It is shown as a dashed line in Figure 2.14c, and we see that it gives a good estimate of the maximum error across the uncertain parameter space.

This analysis is based on the assumption that the amplifier can be modeled by a constant gain. The closed loop system is however a dynamic system because the controller is an integrator. It follows from equation (2.40) that the closed loop dynamics have the time constant $T_{cl} = 1/(bk_i)$. If the amplifier has dynamics, its time constant must thus be small compared to T_{cl} in order to provide good tracking. It follows that the largest admissible integral gain k_i is determined by the unmodeled dynamics.

This example illustrates that feedback can be used to design an amplifier that has practically linear input/output relation even if the basic amplifier is nonlinear with strongly varying characteristics.

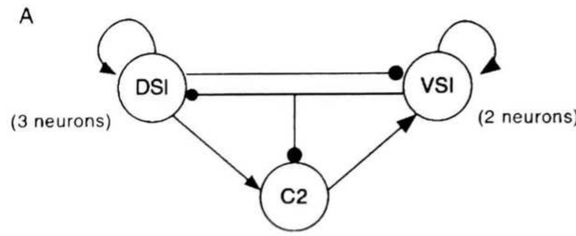


Figure 2.15: Schematic diagram of the neural network that controls swimming motions in the marine mollusk Tritonia, which has both positive and negative feedback [Wil99]. An excitatory connection (positive feedback) is denoted with a line ending with an arrow, an inhibitory interaction (negative feedback) is denoted with an arrow ending with a circle. (Figure adapted from [Wil99].)

2.6 Positive Feedback

Most of this book is focused on negative feedback because of its amazingly good properties, which have been illustrated in the previous sections. In this section we will briefly discuss positive feedback, which has complementary properties. In spite of this, positive feedback has found good use in several contexts.

Systems with negative feedback can be well understood by linear analysis. To understand systems with positive feedback it is necessary to consider nonlinear effects, because without the nonlinearities the instability caused by positive feedback will grow without bound. The nonlinear elements can create interesting and useful effects by limiting the signals.

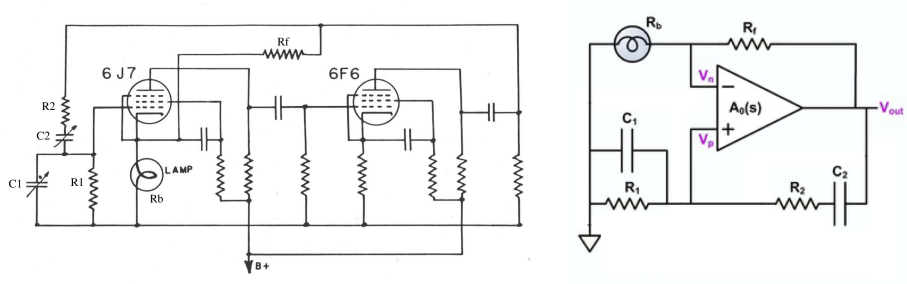
Positive feedback is common in many settings. Encouraging a student or a coworker when they have performed well encourages them do to even better. In biology, it is standard to distinguish inhibitory connections (negative feedback) from excitatory feedback (positive feedback) as illustrated in Figure 2.15. Neurons use a combination of positive and negative feedback to generate spikes.

Positive feedback may cause instabilities. Exponential growth, where the rate of change of a quantity x is proportional to x,

$$\frac{dx}{dt} = \alpha x,$$

is a typical example, which results in exponential growth $x(t) = e^{\alpha t}$. In nature, exponential growth of a species is limited by the finite amount of food. Another common example is when a microphone is placed close to a speaker in public address systems, resulting in a howling noise. Positive feedback can create stampedes in cattle herds, runs on banks, and boom-bust behavior. In all these cases there is exponential growth that is finally limited by finite resources.

The notions of positive and negative feedback are clear if the feedback is static, as we saw for example in Section 2.1. If the feedback is dynamic its action can change from positive to negative depending on the frequency of the signals and hence more care is required. Use of positive feedback will be illustrated by a few examples.



(a) Hewlett's oscillator (b) Operational amplifier version

Figure 2.16: Circuit diagrams of William Hewlett's oscillator. (a) Original system with vacuum tubes. (b) Equivalent realization with an operational amplifier.

Hewlett's Oscillator

William Hewlett used positive and negative feedback very cleverly to design a stable oscillator in his master thesis from Stanford University in 1939. The oscillator was the first product made by Hewlett-Packard, the company that Hewlett founded with David Packard in 1939 [Pac13].

Electronic circuits in the 1930s and 1940s were based on vacuum tube technology. The simplest vacuum tube amplifier has three electrodes: a cathode, grid, and anode enclosed in a glass tube with vacuum. The cathode, which is heated with a filament, emits free electrons. A current is created by applying a high positive voltage between the anode and the cathode. The current can be regulated by changing the voltage on a grid positioned between the anode and the cathode. The current depends on the voltage difference between the grid and the cathode, $V_g - V_c$. Increasing this voltage difference increases the current. The vacuum tube amplifier can be regarded as a valve for controlling a current by applying a voltage to the grid.

A schematic diagram of Hewlett's oscillator is shown in Figure 2.16a. Signals are amplified by two vacuum tubes and there are two feedback loops. One loop provides positive feedback from the anode of the second tube to the grid of the first tube via the network R_1, C_1, R_2, C_2. The second feedback loop provides negative feedback from the output of the second tube to the cathode of the first tube via the resistor R_f and the lamp which has resistance R_b. With a proper gain the positive feedback loop generates an oscillation with the frequency $\omega = 1/\sqrt{R_1 R_2 C_1 C_2}$. The gain is given by the negative feedback loop from the anode of the second loop to the cathode of the first loop, through the resistor R_f and the lamp R_b. This loop has nonlinear gain because the resistance R_b of the lamp increases with increasing temperature. An increase of the amplitude of V_out increases the current through the lamp, which reduces the gain. The result is that an oscillation with stable amplitude and frequency is obtained.

The feedback loops are more clearly visible in the implementation of the oscillator based on an operational amplifier, shown in Figure 2.16b.

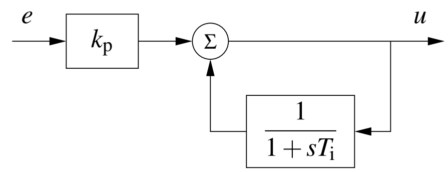


Figure 2.17: Implementation of integral action by *positive feedback*.

Implementation of Integral Action by Positive Feedback

Early feedback controllers made of use of integral action that was implementing by using positive feedback around a system with first order dynamics, as shown in the block diagram of Figure 2.17. Intuitively the system can be explained as follows. Proportional feedback typically gives a steady state error. This can be overcome by adding a bias signal that cancels the steady state error. In Figure 2.17 the bias is estimated by low pass filtering the control signal and adding it back into to the signal path. This serves to compensate for any error that is present.

The circuit can be understood better by a little analysis. Using block diagram algebra we find that the transfer function of the system is

$$G_{ue} = \frac{k_\mathrm{p}}{1 - 1/(1+sT)} = k_\mathrm{p} + \frac{k_\mathrm{p}}{sT},$$

which is a transfer function of a PI controller. This way of implementing integral action is still used in many industrial regulators.

Positive Feedback Combined with Saturation

Systems with interesting and useful properties can be obtained by combining linear and nonlinear components with positive feedback. In this section we consider an example of a simple form of memory implemented using a feedback circuit.

Consider the system in Figure 2.18, which consists of a linear block with first-order dynamics and a nonlinear block with positive feedback. Assume that the nonlinearity is

$$y = F(x) = \frac{x}{1+|x|}, \quad \text{which gives} \quad x = F^{-1}(y) = \frac{y}{1-|y|}.$$

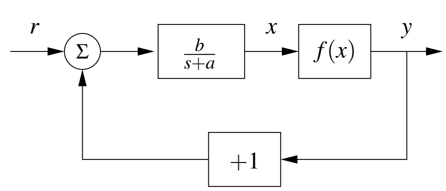
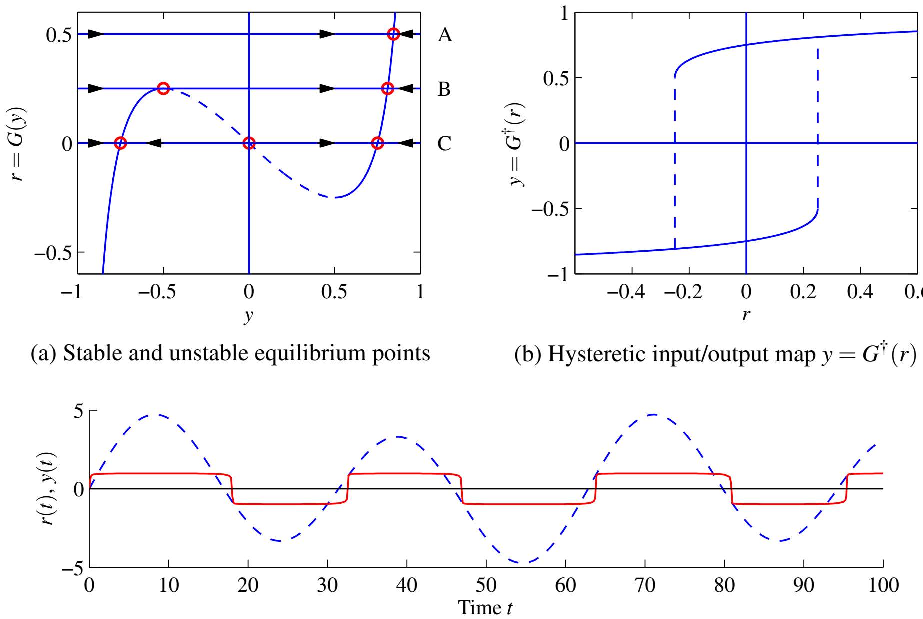


Figure 2.18: Block diagram of system with positive feedback and saturation. The parameters are $a = 1$ and $b = 10$.



(a) Stable and unstable equilibrium points

(b) Hysteretic input/output map $y = G^{\dagger}(r)$

(c) Input/output behavior

Figure 2.19: System with positive feedback and saturation. (a) For a fixed reference value r, the intersections with the curve $r = G(y)$ corresponds to equilibrium points for the system. Equilibrium points at selected values of r are shown by circles (note that for some reference values there are multiple equilibrium points). Arrows indicate the sign of the derivative of y away from the equilibrium points, with the solid portions of $r = G(y)$ representing stable equilibria and dashed portions representing unstable equilibria. (b) The hysteretic input/output map given by the $y = G^{\dagger}(r)$, showing that some values of r have single equilibrium points while others have two possible (stable) steady state output values. (c) Simulation of the system dynamics showing the reference r (dashed curve) and the output y (solid curve).

The system is described by the differential equation

$$\frac{dx}{dt} = -ax + b(r+y) = b(r - G(y)), \qquad G(y) = \frac{aF^{-1}(y)}{b} - y = \frac{ay}{b(1-|y|)} - y.$$

Rewriting the dynamics in terms of the variable $y = F(x)$, we get the following relation between the input r and the output y:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dF^{-1}(y)}{dy} \cdot b(r - G(y)). \tag{2.43}$$

It can be shown that $dF^{-1}(y)/dy$ is everywhere nonzero and so the equilibria for a constant input r are given by the solutions of $r = G(y)$. The graph of the function G is shown in Figure 2.19a for $a = 1$ and $b = 4$. The function $G(y)$ has a local maximum $r_{\max} = 1 + a/b - 2\sqrt{a/b} = 0.25$ at $y = -1/\sqrt{1+a/b} = -0.5$ and a local minimum $r_{\min} = -0.25$ at $y = 0.5$. The set of possible equilibria as a function of r is shown in Figure 2.19b. There is one unique equilibria if $|r| > 0.25$, two equilibria if $|r| = 0.25$ and three equilibria if $|r| < 0.25$.

The differential equation (2.43) is of first order and the equilibrium is stable if $g'(y_0)$ is positive and unstable if $g'(y_0)$ is negative. Stable equilibria are shown in solid lines and unstable equilibria by dashed lines in Figure 2.19a. The differential equation thus has two stable equilibria when $r_{\min} < r < r_{\max}$ and one stable equilibrium when $|r| \geq r_{\max}$.

To understand the behavior of the system, we will explore what happens when the reference is changed. If the reference r is zero there are two stable equilibria, as can be seen in Figure 2.19a by looking at the horizontal line at $r = 0$ (labeled C). We assume that the system is in the stable left equilibrium, where y is negative. If the reference is increased, the equilibrium moves slightly to the right. When the reference reaches the value 0.25, which corresponds an unstable equilibrium, the solution moves towards the right stable equilibrium point, where y is positive, as indicated by the line marked B in Figure 2.19a. If the value of r is increased further, the output y also increases. The static input/output relation is thus given by the "inverse function" $y = G^{\dagger}(r)$, which gives the value(s) of the stable output values as a function of r. The system has hysteretic behavior as shown in Figure 2.19b, where the dashed line indicates the switches between the branches of the solution curves, and they occur at $r = \pm r_{\max} = \pm 0.25$.

The temporal behavior of the system is illustrated by the simulations in Figure 2.19c, where the input r is dashed and the output y is solid. The shapes of the signals depend on the parameters; the values $a = 5$, $b = 50$ were used in the figure. The hysteresis width is $2r_{\max}$ and the parameter a gives the sharpness of the corners of the output. The circuit shown in the Figure 2.18 is commonly used as a trigger to detect changes in a signal (known as a Schmitt trigger). It is also used as a memory element in solid state memories, illustrating that feedback can be used to obtain discrete behavior.

2.7 Further Reading

The books by Bennett [Ben79, Ben93] and Mindel [Min02, Min08] give interesting perspective on the development of control. Much of the material touched upon in this chapter is classical control; see [CM51], [JNP47], and [Tru55]. A more thorough introduction to the principles of feedback with minimal mathematical prerequisites is available in the textbook *Feedback Control for Everyone* [AM10]. The notion of controllers with two degrees of freedom was introduced by Horowitz [Hor63].

The analysis introduced here will be elaborated in the rest of the book. Transfer functions and other descriptions of dynamics are discussed in Chapters 3 and 9, methods to investigate stability in Chapters 5 and 10. The simple method to find parameters of controllers based on matching of coefficients of the closed loop characteristic polynomial is developed further in Chapters 7, 8, and 13. Feedforward control is discussed in Sections 8.5 and 12.4.

Exercises

2.1 Consider the system in Figure 2.1, where $F(w) = \text{sat}(w)$ with a negative sign in the feedback. Assume that $r = 0$ and $v = 1$. Sketch the input/output relation for $k = -3, -2, -1, 0, 1, 2$.

2.2 Let $y \in \mathbb{R}$ and $u \in \mathbb{R}$. Solve the differential equations

$$\frac{dy}{dt} + ay = bu, \qquad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\frac{du}{dt} + u,$$

for $t > 0$. Determine the responses to a unit step $u(t) = 1$ and the exponential signal $u(t) = e^{st}$ when the initial condition is zero. Derive the transfer functions of the systems.

2.3 Let $y_0(t)$ be the response of a system with the transfer function $G_0(s)$ to a given input. The transfer function $G(s) = (1 + sT)G_0(s)$ has the same zero frequency gain but it has an additional zero at $z = -1/T$. Let $y(t)$ be the response of the system with the transfer function $G(s)$ and show that

$$y(t) = y_0(t) + T\frac{dy_0}{dt}, \tag{2.44}$$

Next consider the system with the transfer function

$$G(s) = \frac{s+a}{a(s^2+2s+1)},$$

which has unit zero-frequency-gain ($G(0) = 1$). Use the result in equation (2.44) to explore the effect of the zero $s = -1/T$ on the step response of a system

2.4 Consider a closed loop system with process dynamics and a PI controller modeled by

$$\frac{dy}{dt} + ay = bu, \qquad u = k_\mathrm{p}(r-y) + k_\mathrm{i}\int_0^t (r(\tau) - y(\tau))d\tau,$$

where r is the reference, u the control variable, and y the process output. a) Derive a differential equation relating the output y to the difference by direct manipulation of the equations. b) Draw a block diagram of the system. c) Derive the transfer functions of the process and the controller. d) Compute the transfer function from reference r to output y of the closed loop system. Make the derivations both by direct manipulation of the system equations and by polynomial algebra. Compare the results with a direct determination of the transfer functions by inspection of the block diagram.

2.5 The dynamics of the pupillary reflex is approximated by a linear system with the transfer function

$$P(s) = \frac{0.2(1-0.1s)}{(1+0.1s)^3}.$$

Assume that the nerve system that controls the pupil opening is modeled as a proportional controller with the gain k. Use Routh–Hurwitz criterion to determine the largest gain that gives a stable closed loop system.

2.6 A simple model for the relation between speed v and throttle u for a car is given by the transfer function

$$G_{vu} = \frac{b}{s+a}$$

where $b = 1\ \mathrm{m/s^2}$ and $a = 0.025\ \mathrm{rad/s}$. The control signal is normalized to the range $0 \le u \le 1$. Design a PI controller for the system that gives a closed loop system with the characteristic polynomial

$$a_{\mathrm{cl}}(s) = s^2 + 2\zeta\omega_{\mathrm{c}}s + \omega_{\mathrm{c}}^2.$$

What are the consequences of choosing different values of the design parameters ζ and ω_{c}? Use your judgment to find suitable values. Hint: Investigate maximum acceleration and maximum velocity for step changes in the velocity reference.

2.7 Consider the feedback system in Figure 2.7. Let the disturbance $v = 0$, $P(s) = 1$ and $C(s) = k_{\mathrm{i}}/s$. Determine the transfer function G_{yr} from reference r to output y. Also determine how much G_{yr} is changed when the process gain changes by 10%.

2.8 The calculations in Section 2.3 can be interpreted as a design method for a PI controller for a first-order system. A similar calculation can be made for PID control of the second order system. Let the transfer functions of the process and the controller be

$$P(s) = \frac{b}{s^2 + a_1 s + a_2}, \qquad C(s) = k_{\mathrm{p}} + \frac{k_{\mathrm{i}}}{s} + k_{\mathrm{d}}s.$$

Show that the controller parameters

$$k_{\mathrm{p}} = \frac{(1+2\alpha\zeta)\omega_{\mathrm{c}}^2 - a_2}{b}, \qquad k_{\mathrm{i}} = \frac{\alpha\omega_{\mathrm{c}}^3}{b}, \qquad k_{\mathrm{d}} = \frac{(\alpha+2\zeta)\omega_{\mathrm{c}} - a_1}{b}.$$

give a closed loop system with the characteristic polynomial

$$(s^2 + 2\zeta\omega_{\mathrm{c}}s + \omega_{\mathrm{c}}^2)(s + \alpha\omega_{\mathrm{c}}).$$

2.9 Consider an open loop system with the nonlinear input-output relation $y = F(u)$. Assume that the system is closed with the proportional controller $u = k(r - y)$. Show that the input-output relation of the closed loop system is

$$y + \frac{1}{k}F^{-1}(y) = r.$$

Estimate the largest deviation from ideal linear response $y = r$. Illustrate by plotting the input output responses for a) $F(u) = \sqrt{u}$ and b) $F(u) = u^2$ with $0 \le u \le 1$ and $k = 5, 10$ and 100.

2.10 The effect of distortion in an amplifier can be illustrated by the following MATLAB script:

```
load handel            % Load Handel's Messiah
sound(y, Fs); pause    % Play the original music through speaker

% Music filtered through two cascaded open loop amplifiers
y1 = anm_ol(y, 1); y2 = amp_ol(y1, 1);
sound(y2, Fs); pause

% Music filtered through cascaded amplifiers with feedback k=100
y3 = amp_cl(y, 1, 100); y4 = amp_cl(y3, 1, 100);
sound(y4, Fs); pause
```

where the functions representing the open and closed loop amplifiers are:

```
% Nonlinear static amplifier
function y = amp_ol(x, a)
  z = (x + 1)/2;
  y = 2 * (z + a * z.*(1 - z) - 0.5);
end

% Nonlinear amplifier with negative feedback
function y = amp_cl(x, a, k)
  y = x - (1/k) * (0.5 + x + a * (1 - x.^2)/2);
end
```

The script operates as follows: A file with Handel's Messiah is first loaded as y and played. The music is then sent through two amplifiers with the nonlinearity `amp_ol` and played again. Finally, the music is sent through the same amplifiers with feedback $k = 10$ `amp_cl` and played. Listen to the music when you run the script and explain the action of the filters on the music.

2.11 Consider a queuing system modeled by

$$\frac{dx}{dt} = \lambda - \mu_{\text{max}} \frac{x}{x+1}.$$

The model is nonlinear and the dynamics of the system changes significantly with the queuing length; see Example 3.14. Investigate the situation when a PI controller is used for admission control. The arrival intensity λ is then given by

$$\lambda = k_{\text{p}}(r - x) + k_{\text{i}} \int^{t} (r(t) - x(t))dt.$$

The controller parameters are determined from the approximate model

$$\frac{dx}{dt} = \lambda.$$

Find controller parameters that give the closed loop characteristic polynomial $s^2 + 2s + 1$ for the approximate model. Investigate the behavior of the control strategy for the nonlinear model by simulation for the input $r = 5 + 4\sin(0.1t)$.