Feedback Systems

An Introduction for Scientists and Engineers

SECOND EDITION

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Chapter Two
Feedback Principles

Feedback - it is the fundamental principle that underlies all self-regulating systems, not only machines but also the processes of life and the tides of human affairs.

A. Tustin, "Feedback", *Scientific American*, 1952 [Tus52].

This chapter presents some examples that illustrate fundamental properties of feedback: disturbance attenuation, command signal following, robustness and shaping of behavior. Simple methods for analysis and design of low-order systems are introduced. After reading this chapter, readers should have some insight into the power of feedback, they should know about transfer functions and block diagrams, and they should be able to design simple feedback systems. The basic concepts described in this chapter are explained in more detail in the remainder of the text, and this chapter can be skipped for readers who prefer to move directly to the more detailed analysis and design techniques.

2.1 Mathematical Models

The fundamental properties of feedback will be illustrated using a collection of examples. We need a modest set of concepts and tools to analyze simple feedback systems: linear differential equations, transfer functions, block diagrams and block diagram algebra. In addition we need a simulation tool. In this section we will introduce some of these tools.

Linear Differential Equations and Transfer Functions

In many practical situations, the input/output behavior of a system can be modeled by a linear differential equation of the form

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_n u, \tag{2.1}$$

where u is the input, y is the output and the coefficients a_k and b_k are real numbers. The differential equation (2.1) is characterized by two polynomials

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n, \qquad b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n, \tag{2.2}$$

where $a(s)$ is the *characteristic polynomial* of the differential equation (2.1). We assume that the polynomials $a(s)$ and $b(s)$ do not have common roots.

Equation (2.2) represents a *time invariant* system because if the pair $u(t), y(t)$ satisfies the equation so does $u(t+\tau), y(t+\tau)$. The equation is also *linear* because

if $u_1(t), y_1(t)$ and $u_2(t), y_2(t)$ satisfy the equation so does $\alpha u_1(t)+\beta u_2(t), \alpha y_1(t)+\beta y_2(t)$, where α and β are real numbers. Systems that are linear and time invariant are often called *LTI systems*. We can visualize these systems as being characterized by a huge table of corresponding input/output pairs. An interesting property of an LTI system is that they can be characterized by a single carefully chosen pair, for example the response of the system to a step input.

The solution to equation (2.1) is the sum of two terms: the general solution to the *homogeneous equation*, which does not depend on the input, and a *particular solution*, which depends on the input. The homogeneous equation associated with equation (2.1) is

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = 0. \tag{2.3}$$

Letting s_k represent the roots of the *characteristic equation* $a(s) = 0$, the solution to equation (2.3) is

$$y(t) = \sum_{k=1}^{n} C_k e^{s_k t} \tag{2.4}$$

if the characteristic does not have multiple roots s_k. The parameters $C_1, \ldots, C_n$ are constants that can be determined from the initial conditions at $t = 0$.

Since the coefficients a_k are real, the roots of the characteristic equation are either real-valued or occur in complex conjugate pairs. A real root s_k of the characteristic equation corresponds to the exponential function $e^{s_k t}$. This function decreases over time if s_k is negative, is constant if $s_k = 0$ and increases if s_k is positive, as shown in the top row of Figure 2.1. For real roots s_k the parameter $T = 1/s_k$ is called the *time constant*, because it tells how the signal decays.

A complex root $s_k = \sigma \pm i\omega$ corresponds to the time functions

$$e^{\sigma t} \sin(\omega t), \qquad e^{\sigma t} \cos(\omega t),$$

which have oscillatory behavior, as illustrated in the bottom row of Figure 2.1. The sine terms are shown as solid line; they have zero crossings with the spacing π/ω. The dashed lines show the envelopes, which correspond to the exponential function $\pm e^{\sigma t}$.

When the characteristic equation (2.4) has multiple roots, the solutions to the homogeneous equation (2.3) take the form

$$y(t) = \sum_{k=1}^{m} C_k(t) e^{s_k t}, \tag{2.5}$$

where $C_k(t)$ is a polynomial with degree less than the multiplicity of the root s_k. The solution (2.5) has $\sum_{k=1}^{m} (\deg C_k + 1) = n$ free parameters.

Having explored the solution to the homogeneous equation, we now turn to the input-dependent part of the solution. The solution to equation (2.1) for an exponential input is of particular interest, as will be shown in the following. We set $u(t) = e^{st}$, where $s \neq a_k$ is a complex number, and investigate if there is a unique particular solution of the form $y(t) = G(s)e^{st}$. Assuming this to be the case, we

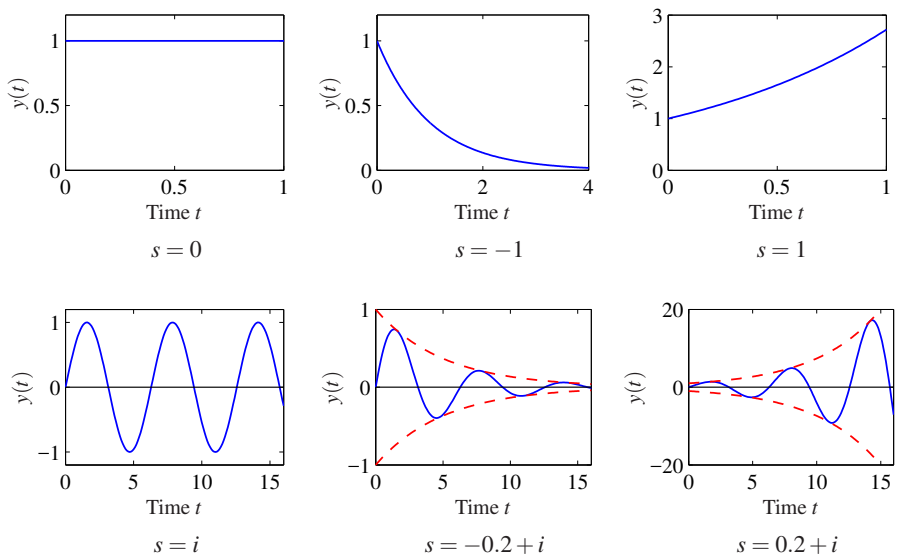


Figure 2.1: Examples of exponential signals. The top row corresponds to exponential signals with a real exponent, and the bottom row corresponds to those with complex exponents. The dashed line in the last two cases denotes the bounding envelope for the oscillatory signals. In each case, if the real part of the exponent is negative then the signal decays, while if the real part is positive then it grows.

find

$$\begin{aligned} \frac{du}{dt} &= se^{st}, & \frac{d^2u}{dt^2} &= s^2e^{st}, & \cdots \quad \frac{d^nu}{dt^n} &= s^ne^{st} \\ \frac{dy}{dt} &= sG(s)e^{st}, & \frac{d^2y}{dt^2} &= s^2G(s)e^{st}, & \cdots \quad \frac{d^ny}{dt^n} &= s^nG(s)e^{st}. \end{aligned} \tag{2.6}$$

Inserting these expressions into the differential equation (2.1) gives

$$(s^n + a_1s^{n-1} + \cdots + a_n)G(s)e^{st} = (b_1s^{n-1} + b_2s^{n-2} + \cdots + b_n)e^{st}$$

and hence

$$G(s) = \frac{b_1s^{n-1} + b_2s^{n-2} + \cdots + b_n}{s^n + a_1s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)}. \tag{2.7}$$

This function is called the *transfer function* of the system. It describes a particular solution to the differential equation for the input e^{st}. Summarizing, we find that a solution to the differential equation (2.1) for the exponential input $u(t) = e^{st}$ is

$$y(t) = \sum_{k=1}^{m} C_k(t)e^{s_kt} + G(s)e^{st}. \tag{2.8}$$

The relation between the transfer function (2.7) and the differential equation (2.1) is clear: the transfer function (2.7) can be obtained by inspection from the differential equation (2.1), and conversely the differential equation can be obtained

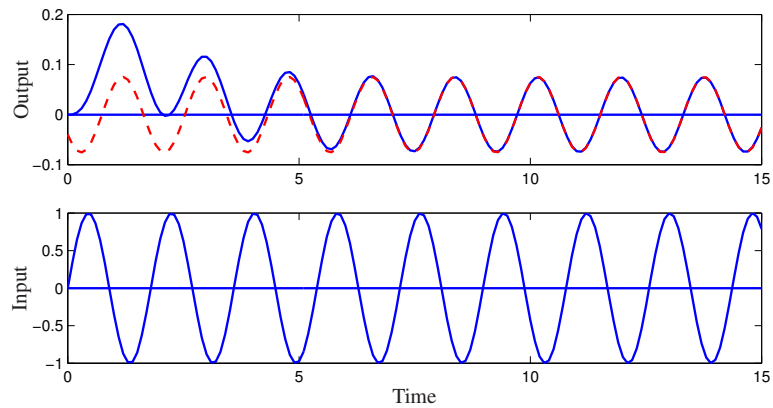


Figure 2.2: Two responses of a linear time-invariant system to a sinusoidal input. The dashed line shows the output when the initial conditions are chosen so that the output is purely sinusoidal. The full lines show the response response for the initial conditions $y(0) = 0$ and $y'(0) = 0$. The transfer function $G(s) = 1/(s+1)^2$.

from the transfer function. The transfer function can thus be regarded as a shorthand notation for the differential equation (2.1). It is a complete characterization of the differential equation even if it was derived as the response to a specific input $u(t) = e^{st}$. The input and the initial conditions must *both* be given to obtain the full solution of the differential equation, also refered to as the *response* of the system.

The particular solution for a constant input $u(t) = 1 = e^{0 \cdot t}$ is $y(t) = G(0) = b_n/a_n$. The quantity $G(0)$ is called the *zero frequency gain*, the *static gain* or sometimes the *steady state gain*.

To deal with oscillatory signals, like those shown in the bottom row of Figure 2.1, we allow s to be a complex number. The transfer function is then a function $G : \mathbb{C} \to \mathbb{C}$ that maps complex numbers to complex numbers. We let arg represent the argument (phase, angle) of a complex number and $|\cdot|$ the magnitude, and note that the complex response to an input $u = e^{i\omega t} = \cos \omega t + i \sin \omega t$ is given by $G(i\omega t)$. Using just the imaginary parts of the signals, it follows that the particular solution for the input $u = \sin(\omega t) = \operatorname{Im} e^{i\omega t}$ is

$$\begin{aligned} y(t) &= \operatorname{Im}\left(G(i\omega)\, e^{i\omega t}\right) = \operatorname{Im}\left(|G(i\omega)|\, e^{i \arg G(i\omega)}\, e^{i\omega t}\right) \\ &= |G(i\omega)| \operatorname{Im} e^{i(\arg G(i\omega) + \omega t)} = |G(i\omega)| \sin(\omega t + \arg G(i\omega)). \end{aligned}$$

The input is thus amplified by $|G(i\omega)|$ and the phase shift between input and output is $\arg G(i\omega)$. The functions $G(i\omega)$, $|G(i\omega)|$ and $\arg G(i\omega)$ are called the *frequency response*, *gain* and *phase*. Gain and phase are also called *magnitude* and *angle*.

The full response to a sine function is the sum of a particular solution and a solution to the homogeneous equation that is determined by the initial conditions, as given in equation (2.8). An illustration is given in Figure 2.2. The dashed line, which is a pure sine wave, is the solution obtained when all C_k in equation (2.8) are zero. The full line shows the response obtained when the C_k are chosen so that

$y(0)$ and its derivatives $y^{(k)}(0)$, $k = 1,\ldots,n-1$ are all zero. Since all roots of the characteristic equation have negative real parts, the solution to the homogeneous equation (2.5) goes to zero as $t \to \infty$ and the general solution converges to the particular solution.

The transfer function has many physical interpretations that can be exploited for analysis and design. The roots s_k of the characteristic equation $a(s) = 0$ are called *poles* of the transfer function: the transfer function is infinite for $s = s_k$. The poles s_k appear as exponents in the general solution to the homogeneous equation, as seen in equations (2.4) and (2.5). Systems with poles that are "lightly damped" ($\mathrm{Re}(s_k)$ is negative but small) can exhibit resonances when a sinusoidal input is applied whose frequency is near the imaginary part of s_k.

The roots s_j of the polynomial $b(s)$ are called *zeros* of the transfer function. The reason is that if $b(s_j) = 0$ it follows that $G(s_j) = 0$, and the particular solution for the input $e^{s_k t}$ is then zero. A system theoretic interpretation is that the transmission of the exponential signal $e^{s_j t}$ is blocked by the zero $s = s_j$, which is therefore also called a *transmission zero*.

The transfer function can also convey a great deal of intuition: the approximations of $G(s)$ for small and large s capture the propagation of slow and fast signals respectively. Consider for example a spring-mass system with input u (force) and output q (position), whose dynamics satisfy the second-order differential equation

$$m\ddot{q} + c\dot{x} + ky = u.$$

The system has the transfer function

$$G(s) = \frac{1}{ms^2 + cs + k}.$$

For small s we have $G(s) \approx 1/k$. The corresponding input/output relation is $q = (1/k)u$, which implies that for low frequency inputs, the system behaves like a spring driven by a force. For large s we have $G(s) \approx 1/(ms^2)$. The corresponding differential equation is $m\ddot{q} = u$ and thus high frequency inputs the system thus behaves like mass driven by a force (a double integrator).

The frequency response of a stable system can be determined experimentally by exploring the steady state response of a system to sinusoidal signals. This is an alternative or a complement to physical modeling. A more elaborate treatment of transfer functions and the frequency response will be given in Chapter 9.

Stability: The Routh-Hurwitz Criterion

When using feedback there is always the danger that the system may become unstable, and it is therefore important to have a stability criterion. The differential equation (2.1) is called *stable* if all solutions of the homogeneous equation (2.3) go to zero for any initial condition. It follows from equation (2.5) that this requires that all the roots of the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

have negative real parts.

It can often be difficult to analytically compute the roots of a high-order polynomial. The *Routh-Hurwitz criterion* is a stability criterion that does not require explicit calculation of the roots, because it gives conditions in terms of the coefficients of the characteristic polynomial.

We illustrate the Routh-Hurwitz criterion by describing it for low-order differential equations. A first-order differential equation is stable when the coefficient a_1 of the characteristic polynomial is positive, since the root of the characteristic polynomial will be $s = -a_1 < 0$. A second-order polynomial has the roots

$$s = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_2}\right),$$

and it is easy to verify that the real parts of the roots are both negative if and only if $a_1 > 0$ and $a_2 > 0$. A third order differential equation is more complicated, but the roots can be shown to have negative real parts if and only if

$$a_1, a_2, a_3 > 0, \quad \text{and} \quad a_1 a_2 > a_3. \tag{2.9}$$

The corresponding conditions for a fourth order differential equation are

$$a_1, a_2, a_3, a_4 > 0, \qquad a_1 a_2 > a_3, \quad \text{and} \quad a_1 a_2 a_3 > a_1^2 a_4 + a_3^2. \tag{2.10}$$

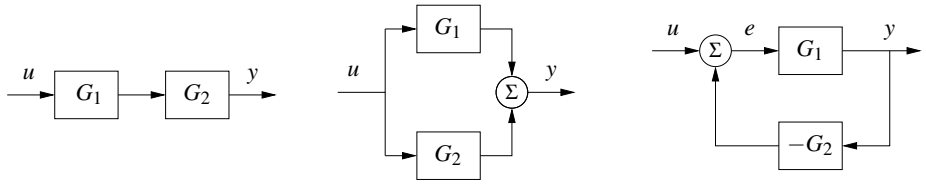
The Routh-Hurwitz criterion [Gan60] gives similar conditions for arbitrarily high order polynomials. Stability of a linear differential equation can thus be investigated just by analyzing the signs of various combinations of the coefficients of the characteristic polynomial.

Block Diagrams and Transfer Functions

As we saw already in Chapter 1, control systems are often described using block diagrams, such as the ones shown in Figures 1.1 and 1.3. If the behavior of the blocks are represented by transfer functions, the transfer function of a system can be obtained simply by algebraic manipulations. It follows from equation (2.8) that the transfer function can be derived from the particular solution for the input e^{st}. To derive the transfer function for a system composed of several blocks we assume that the input signal is an exponential $u(t) = e^{st}$, and compute the corresponding particular solutions for all blocks.

Consider for example the system in Figure 2.3a, which is a series connection of two systems with the transfer functions $G_1(s)$ and $G_2(s)$. Let the input of the system be $u(t) = e^{st}$ The output of the first block is then $y_1(t) = G_1(s)e^{st}$, which is also an exponential, and the output of the second system is $y(t) = G_2(s)G_1(s)u(t)$. The transfer function of the system is thus $G_{yu}(s) = G_2(s)G_1(s)$, where we use the convention that the right subscript is the input and the left subscript is the output, so that $y = G_{yu}u$.

Next we will will consider parallel connections of systems as shown in Figure 2.3b. Assuming that the input is $u(t) = e^{st}$, the exponential outputs of the blocks



(a) $G_{yu}(s) = G_2(s)G_1(s)$ (b) $G_{yu}(s) = G_1(s) + G_2(s)$ (c) $G_{yu}(s) = \dfrac{G_1(s)}{1+G_1(s)G_2(s)}$

Figure 2.3: Interconnections of linear systems. Series (a), parallel (b) and feedback (c) connections are shown. The transfer functions for the composite systems can be derived by algebraic manipulations assuming exponential functions for all signals.

are $y_1(t) = G_1(s)e^{st}$ and $y_2(t) = G_2(s)e^{st}$. The output of the system is then

$$y(t) = \big(G_1(s)e^{st} + G_2(s)e^{st}\big) = \big(G_1(s) + G_2(s)\big)\, e^{st},$$

and the transfer function of a parallel connection of systems with the transfer functions $G_(s)$ and $G_2(s)$ is thus $G_{yu}(s) = G_1(s) + G_2(s)$.

Finally we will consider the feedback connection shown in Figure 2.3c. If the input $u(t) = e^{st}$ is an exponential we find

$$y(t) = G_1(s)e(t) = G_1(s)\big(u(t) - G_2(s)y(t)\big) = G_1(s)\big(e^{st} - G_2(s)y(t)\big).$$

Solving for $y(t)$ gives

$$y(t) = \frac{G_1(s)}{1+G_1(s)G_2(s)}\, e^{st} = G_{yu}(s)\, e^{st}.$$

The transfer function of a feedback connection of systems with the transfer functions $G_1(s)$ and $G_2(s)$ is thus

$$G_{yu}(s) = \frac{G_1(s)}{G_1(s) + G_2(s)}. \tag{2.11}$$

By using polynomials and transfer functions the relations between signals in a feedback system can thus be obtained by algebra. With some practice the transfer functions can often be obtained by inspection, as we explore in more detail in Chapter 9.

2.2 Using Feedback to Improve Disturbance Attenuation

Reducing the effects of disturbances is a primary use of feedback. It was used by James Watt to make steam engines run at constant speed in spite of varying load and by electrical engineers to make generators driven by water turbines deliver electricity with constant frequency and voltage. Feedback is commonly used to alleviate effects of disturbances in the process industry, for machine tools and for engine and cruise control in cars. The human body exploits feedback to keep body

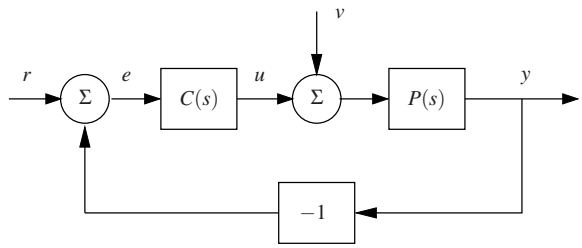


Figure 2.4: Block diagram of simple feedback system. The controller transfer function is $C(s)$ and the process transfer function is $P(s)$. The process output is y, the external signals are the reference r and the load disturbance v.

temperature, blood pressure and other important variables constant. For example the pupillary reflex guarantees that the light intensity of the retina is reasonably constant in spite of large variations in the ambient light intensity. Keeping variables close to a desired, constant reference value in spite of disturbances is called a *regulation problem*.

To discuss disturbance attenuation we consider the system in Figure 2.4. Since we will focus on the effects of a load disturbance v we will assume for now that the reference r is zero. To derive the transfer functions from the disturbance input v to the process output y, which we write as G_{yv}, we assume that the disturbance is an exponential function $v = e^{st}$. Applying block diagram algebra to Figure 2.4 gives

$$y(t) = P(s)\Big(e^{st} - C(s)P(s)y(t)\Big), \qquad y(t) = \frac{P(s)}{1+P(s)C(s)}\,e^{st}.$$

The transfer function relating the output y to the load disturbance v is thus

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)} \tag{2.12}$$

To explore the use of feedback to improve disturbance attentuation, we will focus on a simple process modeled by the first order differential equation

$$\frac{dy}{dt} + ay = bu, \qquad a > 0, \quad b > 0.$$

The corresponding transfer function is

$$P(s) = \frac{b}{s+a}. \tag{2.13}$$

This model is a reasonable approximation for a physical process if the storage of mass, momentum or energy can be captured by a single state variable. Typical examples are the velocity of a car on a road, the angular velocity of a rotating system and the fluid level of a tank.

Proportional Control

We will first investigate the case of proportional control, when the control signal is proportional to the output error: $u = k_p e$, as introduced already in Section 1.6. The controller transfer function is then $C(s) = k_p$. The process transfer function is given by equation (2.13) and the effect of the disturbance on the output is then described by the transfer function (2.12):

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{\frac{b}{s+a}}{1+\frac{bk_p}{s+a}} = \frac{b}{s+(a+bk_p)}.$$

The relation between the disturbance v and the output y is thus given by the differential equation

$$\frac{dy}{dt} + (a+bk_p)y = bv.$$

The closed loop system is stable if $a+bk_p > 0$. A constant disturbance $v = v_0$ then gives an output that exponentially approaches the value

$$y_0 = G_{yv}(0) = \frac{b}{a+bk_p} v_0$$

with the time constant $T = 1/(a+bk_p)$. Without feedback, $k_p = 0$ and for a constant disturbance v_0, the output will instead approach bv_0/a. The effect of the disturbance is thus reduced if $k_p > 0$.

We have thus shown that a constant disturbance gives an error that can be reduced by feedback using a proportional controller. The error decreases with increasing controller gain. Figure 2.5a shows the responses for a few values of the controller gain k_p.

Proportional-Integral (PI) Control

The PI controller, introduced in Section 1.6, is described by

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)d\tau. \tag{2.14}$$

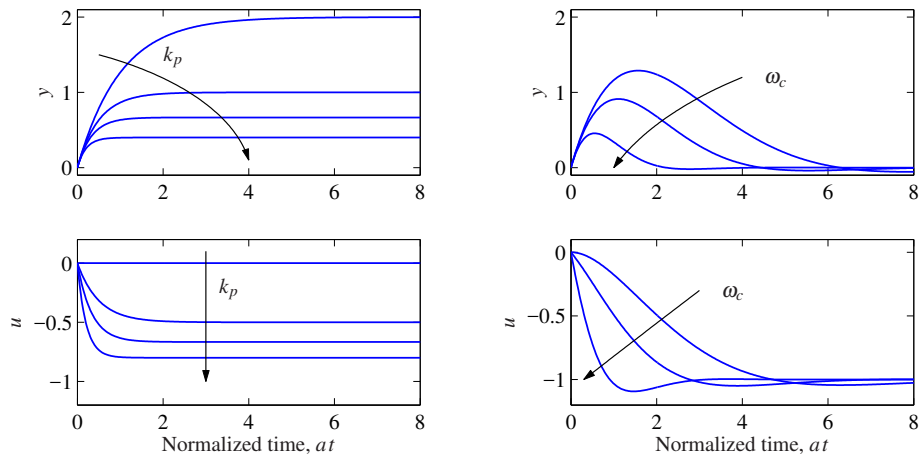
To determine the transfer function of the controller we differentiate to obtain

$$\frac{du}{dt} = k_p \frac{de}{dt} + k_i e,$$

and we find that the transfer function is $C(s) = k_p + k_i/s$. To investigate the effect of the disturbance v on the output we use the block diagram in Figure 2.4 and the transfer function from v to y is

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{bs}{s^2+(a+bk_p)s+bk_i}. \tag{2.15}$$

Using the relationship between transfer functions and differential equations given by equations (2.1) and (2.7), it follows that the relation between the load distur-



(a) Proportional control (b) Proportional-Integral (PI) control

Figure 2.5: Step responses for a first-order, closed loop system with proportional control (a) and PI control (b). The process transfer function is $P = 2/(s+1)$. The controller gains for proportional control are $k_p = 0$, 0.5, 1 and 2. The PI controller is designed using equation (2.19) with $\zeta_c = 0.707$ and $\omega_c = 0.707$, 1 and 2, which gives the controller parameters $k_p = 0$, 0.207, 0.914 and $k_i = 0.25$, 0.50 and 2.

bance and the output is given by the differential equation

$$\frac{d^2y}{dt^2} + (a + bk_p)\frac{dy}{dt} + bk_i y = b\frac{dv}{dt}. \tag{2.16}$$

Notice that since the disturbance enters as a derivative on the right hand side, a constant disturbance gives no steady state error. The same conclusion can be drawn from the observation that $G_{yv}(0) = 0$. This is consistent with the discussion of integral action and steady state error in Section 1.6.

To find suitable values of the controller parameters k_p and k_i, we consider the characteristic polynomial of the differential equation (2.16),

$$a_{cl}(s) = s^2 + (a + bk_p)s + bk_i. \tag{2.17}$$

We can assign arbitrary roots to the characteristic polynomial by choosing the controller gains k_p and k_i. The most common case is that we assign complex roots that give the characteristic polynomial

$$(s + \sigma_d + i\,\omega_d)(s + \sigma_d - i\,\omega_d) = s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2. \tag{2.18}$$

This polynomial has roots at $s = -\sigma_d \pm i\,\omega_d$. The general solution to the homogeneous equation is then a linear combination of the terms

$$e^{-\sigma_d t}\sin(\omega_d t), \qquad e^{-\sigma_d t}\cos(\omega_d t),$$

which are damped sine and cosine functions, as shown in the lower middle plot in Figure 2.1. The coefficient σ_d determines the decay rate and the parameter ω_d gives the frequency of the decaying oscillation. Identifying coefficients of equal

powers of s in the polynomials (2.17) and (2.18) gives

$$k_p = \frac{2\sigma_d - a}{b}, \qquad k_i = \frac{\sigma_d^2 + \omega_d^2}{b}. \tag{2.19}$$

Instead of parameterizing the closed loop system in terms of σ_d and ω_d it is common practice to use the *undamped natural frequency* $\omega_c = \sqrt{\sigma_d^2 + \omega_d^2}$ and the *damping ratio* $\zeta_c = \sigma_d/\omega_c$. The closed loop characteristic polynomial is then

$$a_{cl}(s) = s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2 = s^2 + 2\zeta_c\omega_c s + \omega_c^2.$$

This parameterization has the advantage that ζ_c, which is in the range $[-1,1]$, determines the shape of the response and ω_c gives the response speed.

Figure 2.5b shows the output y and the control signal u for $\zeta_c = 1/\sqrt{2} = 0.707$ and different values of the design parameter ω_c. Proportional control gives a steady-state error that decreases with increasing controller gain k_p. With PI control the steady-state error is zero. Both the decay rate and the peak error decrease when the design parameter ω_c is increased. Larger controller gains give smaller errors and control signals that react faster to the disturbance.

With the controller parameters (2.19), the transfer function (2.15) from disturbance v to process output y becomes

$$G_{yv}(s) = \frac{P(s)}{1 + P(s)C(s)} = \frac{bs}{s^2 + 2\zeta_c\omega_c s + \omega_c^2}.$$

For efficient attenuation of disturbances, it is desirable that $|G_{yv}(i\omega)|$ is small for all ω. For small values of ω we have $|G_{yv}(i\omega)| \approx b\omega/\omega_c$, while for large ω we have $|G_{yv}(i\omega)| \approx b/\omega$. The largest value of $|G_{yv}(i\omega)|$ is $b/(2\zeta_c\omega_c)$ for $\omega = \omega_c$. It thus follows that a large value of ω_c gives good load disturbance attenuation.

In summary, we find that the analysis gives a simple way to find the parameters of PI controllers for processes whose dynamics can be approximated by a first-order system. The technique can be generalized to more complicated systems but the controller will be more complex. To achieve the benefits of large control gains the model must be accurate over wide frequency ranges, as will be discussed next.

Unmodeled Dynamics

The analysis we have made so far indicates that there are no limits to the performance that can be achieved. Figure 2.5b shows that arbitrarily fast response can be obtained simply by making ω_c sufficiently large. In reality there are of course limitations on what is achievable. One reason is that the controller gains increase with ω_c: the proportional gain is $k_p = (2\zeta_c\omega_c - a)/b$ and the integral gain is $k_i = \omega_c^2/b$. A large value of ω_c thus gives large controller gains and the control signal may saturate. Another reason is that the model (2.13) is a simplification: it is only valid in a given frequency range. If the model is instead

$$P(s) = \frac{b}{(s+a)(1+sT)}, \tag{2.20}$$

where the term $1+sT$ represents the dynamics of sensors, actuators or other dynamics that were neglected when deriving equation (2.13)—so-called *unmodeled dynamics*—the closed loop characteristic polynomial for the closed loop system becomes

$$a_{cl} = s(s+a)(1+sT)+k_p s+k_i = s^3T+s^2(1+aT)+2\zeta_c\omega_c s+\omega_c^2.$$

It follows from the Routh-Hurwitz criterion (2.9) that the closed loop system is stable if $\omega_c^2 T < 2\zeta_c\omega_c(1+aT)$ or if

$$\omega_c T < 2\zeta_c(1+aT).$$

The frequency ω_c and the achievable response time are thus limited by the unmodeled dynamics represented by T, which typically is smaller than the time constant $1/a$ of the process. When models are developed for control it is therefore important to also consider the unmodeled dynamics.

The fact that unmodeled dynamics limit the performance of a feedback system is an important property and must be considered during the system design. It is common to use simplified models when designing components of complex systems and if the unmodeled dynamics of those components (or the other subsystems they interact with) are not properly taken into account, the implementation of the system can display poor behavior (of which instability is one extreme example). As we shall see in later chapters, it is the ability to reason about the effects of uncertainty that makes control theory a particularly powerful mathematical tool for systems design.

2.3 Using Feedback to Follow Command Signals

Another major application of feedback is to make a system output follow a reference value, which is called the *servo problem*. Cruise control, steering of a car, tracking a satellite with an antenna or a star with a telescope are some examples. Other examples are high performance audio amplifiers, machine tools and industrial robots.

To illustrate command signal following we will consider the system in Figure 2.4 where the process is a first-order system and the controller is a PI controller. The transfer functions of the process and the controller are

$$P(s) = \frac{b}{s+a}, \qquad C(s) = \frac{k_p s+k_i}{s}. \tag{2.21}$$

Since we will focus on following the command signal r, we will neglect the load disturbance and set $v=0$. Applying block diagram algebra to the system in Figure 2.4, we find that the transfer function from the command signal r to the output y is

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{bk_p s+bk_i}{s^2+(a+bk_p)s+bk_i}. \tag{2.22}$$

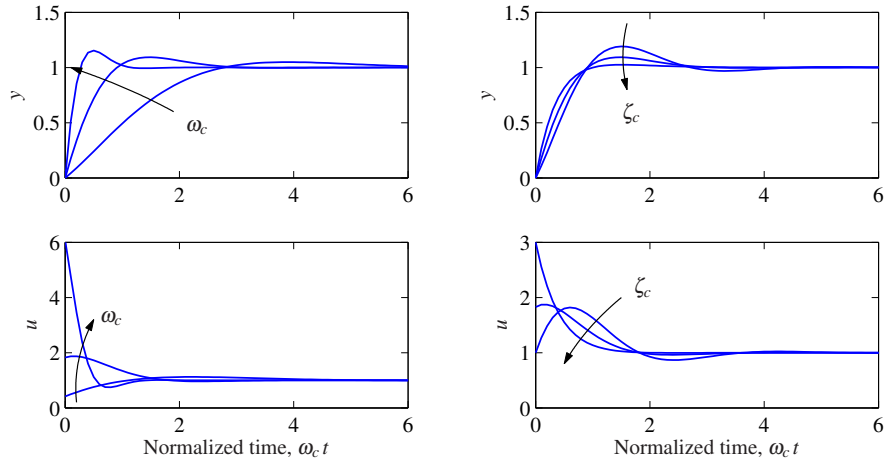


Figure 2.6: Responses to a unit step change in the command signal for different values of the design parameters ω_c and ζ_c. The left figure shows responses for fixed $\zeta_c = 0.707$ and $\omega_c = 1$, 2 and 5. The right figure shows responses for $\omega_c = 2$ and $\zeta_c = 0.5$, 0.707, and 1. The process parameters are $a = b = 1$. The initial value of the control signal is k_p.

Since $G_{yr}(0) = 1$ it follows that $r = y$ when r and y are constant, independent of the values of the parameters a and b, as long as the closed loop is stable. The steady state output is thus equal to the reference, a consequence of the integral action in the controller.

To determine suitable values of the controller parameters k_p and k_i, we proceed as in Section 2.2 by choosing controller parameters that make the closed-loop characteristic polynomial

$$a_{cl}(s) = s^2 + (a + bk_p)s + bk_i \tag{2.23}$$

equal to $s^2 + 2\zeta_c\omega_c s + \omega_c^2$ with $\zeta_c > 0$ and $\omega_c > 0$. Identifying coefficients of equal powers of s in these polynomials gives

$$k_p = \frac{2\zeta_c\omega_c - a}{b}, \qquad k_i = \frac{\omega_c^2}{b}, \tag{2.24}$$

which is equivalent to equation (2.19). Notice that integral gain increases with the square of ω_c. Figure 2.6 shows the output signal y and the control signal u for different values of the design parameters ζ_c and ω_c. The response time decreases with increasing ω_c and the initial value of the control signal also increases because it takes more effort to move rapidly. The overshoot decreases with increasing ζ_c. For $\omega_c = 2$, the design choice $\zeta_c = 1$ gives a short settling time and a response without overshoot.

It is desirable that the output y will track the reference signal r for time-varying references. This means that we would like the transfer function $G_{yr}(s)$ to be close to 1 for large frequency ranges. With the controller parameters (2.24), it follows

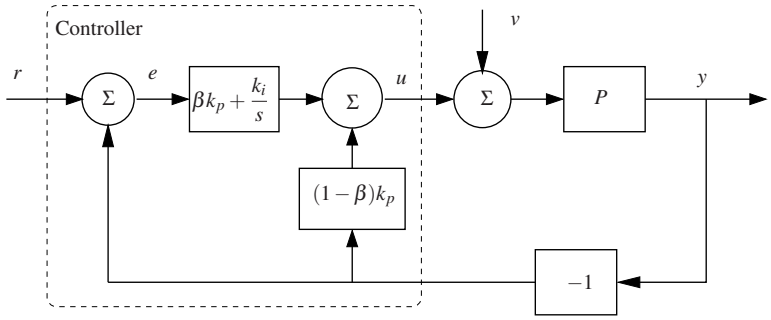


Figure 2.7: Block diagram of a closed-loop system with a PI controller having an architecture with two degrees of freedom.

from equation (2.22) that

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{(2\zeta_c\omega_c - a)s + \omega_c^2}{s^2 + 2\zeta_c\omega_c s + \omega_c^2}.$$

Since $G_{yr}(0) = 1$, tracking of constant inputs is perfect. In addition, if $s = i\omega$ is smaller in magnitude than ω_c, then we see that $G_{yr}(s)$ will be very close to one. The frequency ω_c thus determines the upper bound of the frequency of input signals that can be tracked with small error, and this bound is referred to as the *bandwidth* of the closed loop system. The frequency response of G_{yr} thus provides a quantitative representation of the tracking abilities.

Controllers with Two Degrees of Freedom

The control law in Figure 2.4 has *error feedback* because the control signal u is generated from the error $e = r - y$. With proportional control, a step in the reference signal r gives an immediate step change in the control signal u. This rapid reaction can be advantageous, but it may give large overshoot, which can be avoided by a replacing the PI controller in equation (2.14) with a controller of the form

$$u(t) = k_p\big(\beta r(t) - y(t)\big) + k_i \int_0^t (r(\tau) - y(\tau))d\tau. \tag{2.25}$$

In this modified PI algorithm, the proportional action only acts on the fraction β of the reference signal. The signal transmissions from reference r to u and from output y to u can be represented by the transfer functions

$$C_{ur}(s) = \beta k_p + \frac{k_i}{s}, \qquad C_{uy}(s) = k_p + \frac{k_i}{s} = C(s). \tag{2.26}$$

The controller (2.25) is called a controller with *two degrees of freedom* since the transfer functions $C_{ur}(s)$ and $C_{uy}(s)$ are different.

A block diagram of a closed loop system with a PI controller having two degrees of freedom is shown in Figure 2.7. Let the process transfer function be $P(s) = b/(s+a)$. The transfer functions from reference r and disturbance v to

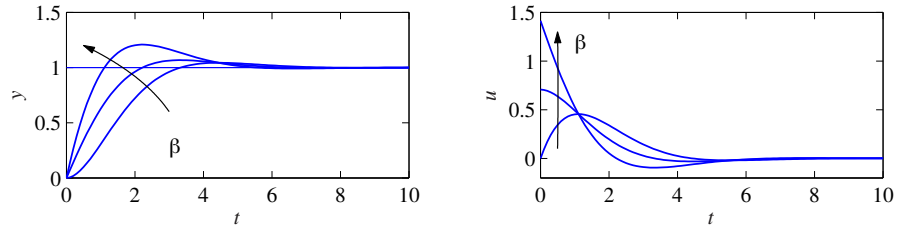


Figure 2.8: Response to a step change in the command signal for a system with a PI controller having two degrees of freedom. The process transfer function is $P(s) = 1/s$ and the controller gains are $k_p = 1.414$, $k_i = 1$ and $\beta = 0$, 0.5 and 1.

output y are

$$G_{yr}(s) = \frac{b\beta k_p s + bk_i}{s^2 + (a + bk_p)s + bk_i}, \qquad G_{yv}(s) = \frac{bs}{s^2 + (a + bk_p)s + bk_i}. \tag{2.27}$$

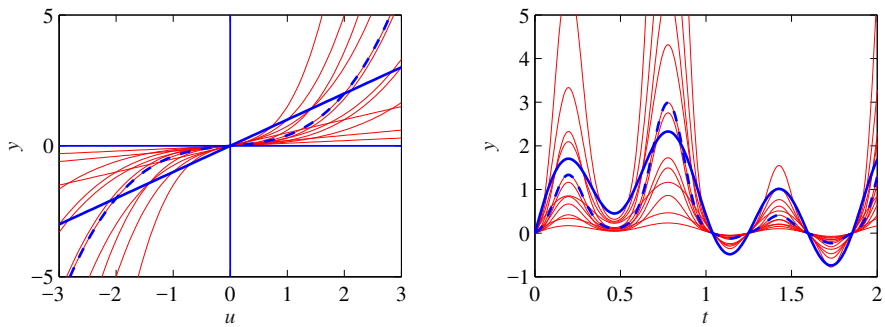
Comparing with the corresponding transfer function for a controller with error feedback in equations (2.15) and (2.22) we find that the responses to the load disturbances is the same but the response to reference signals is different.

A simulation of the closed loop system for $a = 0$ and $b = 1$ is shown in Figure 2.8. The figure shows that the parameter β has a significant effect on the responses. Comparing the system with error feedback ($\beta = 1$) to the system with smaller values of β we find that using a system with two degrees of freedom gives less overshoot and gentler control actions.

The example shows that command signal response can be improved by using a controller architecture having two degrees of freedom. In Section 2.6 we will further show that the responses to command signals and disturbances can be completely separated by using a more general system architecture. To use a system with two degrees of freedom both the reference signal r and the output signal y must be measured. There are situations where only the error signal $e = r - y$ can be measured; typical examples are DVD players, optical memories and atomic force microscopes.

2.4 Using Feedback to Provide Robustness

Feedback can be used to make good systems from poor components. Black's invention of the feedback amplifier for the telephone network is an early example [Bla77]. The vacuum tube, which was nonlinear and time varying, was the primary amplifier at the time, and Black's idea was to close a loop with negative feedback around the tube amplifier. In this way he could obtain a closed loop system with a linear input/output relation having constant gain. The general recipe is to localize the nonlinearities and the source of process variations, and to close feedback loops around them. We illustrate the approach with a simple model of an electronic amplifier.



(a) Input/output relationship (b) Output response

Figure 2.9: Responses of a static nonlinear system. The left figure shows the input/output relation of the open-loop system and the right figure shows responses to the input signal (2.29). The ideal response is shown solid bold lines. The nominal response of the nonlinear system is shown using dashed bold lines and the responses for different parameter values are shown using thin lines.

Reducing Effects of Parameter Variations and Nonlinearities

Consider an amplifier with a static, nonlinear input/output relation with considerable variability, as illustrated in Figure 2.9a. The nominal input/output characteristics is shown as a dashed bold line and examples of variations as thin lines. The nonlinearity in the figure is given by

$$y = f(u) = \alpha(u + \beta u^3), \qquad -3 \le u \le 3. \tag{2.28}$$

The nominal values corresponding to the dashed line are $\alpha = 0.2$ and $\beta = 1$. The variations of the parameters α and β are in the ranges $0.1 \le \alpha \le 0.5$, $0 \le \beta \le 2$. The responses of the system to the input $u = r$ with

$$r(t) = \sin(t) + \sin(\pi t) + \sin(\pi^2 t). \tag{2.29}$$

are shown in Figure 2.9b. The desired response $y = u$ is shown as a solid bold line and responses for a range of parameters are shown with thin lines. The nominal response of the nonlinear system is shown as a dashed bold line. It is distorted due to the nonlinearity. Notice in particular the heavy distortion both for small and large signal amplitudes.

The behavior of the system is clearly not satisfactory, but it can be improved significantly by introducing feedback. A block diagram of a system with a simple integral controller is shown in Figure 2.10. Figure 2.11 shows the behavior of the closed loop system with the same parameter variations as in Figure 2.9. The input/output plot in Figure 2.11a is a scatter plot of the inputs and the outputs of the feedback system. The input/output relation is practically linear and close to the desired response. There is some variability because of the dynamics introduced by the feedback. Figure 2.11b shows the responses to the reference signal; notice the dramatic improvement compared with Figure 2.9b. The tracking error is shown in

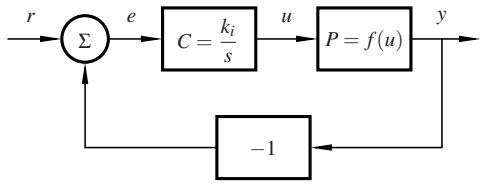


Figure 2.10: Block diagram of a nonlinear system with integral feedback.

Figure 2.11c.

Analysis

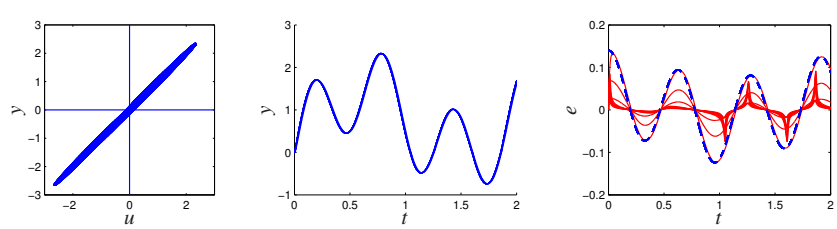
Analysis of the closed loop system is difficult because it is nonlinear. We can, however, obtain significant insight by using approximations. We first observe that the system is linear when $\beta = 0$. In other situations we can thus approximate the nonlinear function by a straight line around an operating point $u = u_0$. The slope of the nonlinear function at $u = u_0$ is $f'(u_0)$ and we will approximate the process with a linear system with the gain $f'(u_0)$. The transfer functions of the process and the controller are

$$P(s) = f'(u_0) = \alpha(1 + 3\beta u_0^2) = b, \qquad C(s) = \frac{k_i}{s}, \tag{2.30}$$

where u_0 denotes the operating condition. The process gain $b = \alpha(1 + 3\beta u_0^2)$ is in the range 0.1–27.5 depending on the values of α, β and u_0. It follows from equation (2.12) that the transfer functions relating the output y and the error e to the reference signal r are

$$G_{yr}(s) = \frac{bk_i}{s + bk_i}, \qquad G_{er}(s) = 1 - G_{yr} = \frac{s}{s + bk_i}. \tag{2.31}$$

The closed loop system is a first-order system with the pole $s = -bk_i$ and time constant $T_{cl} = 1/(bk_i)$. If the integral gain is chosen as $k_i = 1000$, the closed loop



(a) (b) (c)

Figure 2.11: Responses of the system with integral feedback ($k_i = 1000$). The left plot, is a scatter plot of inputs and outputs. The center plot shows the response of the closed loop system to the input signal r, and the right plot shows the control error. The parameter variations are the same as in Figure 2.9. Notice the dramatic improvement compared to Figure 2.9b. The dashed line in (c) corresponds to the approximate error given by equation (2.33).

pole ranges from 100 rad/s to 2.75×10^4 rad/s, which is fast compared to the high frequency component 9.86 rad/s of the input signal.

The error for the approximated system is described by the differential equation

$$\frac{de}{dt} = -bk_i e + \frac{dr}{dt}, \qquad \frac{dr}{dt} = \cos(t) + \pi\cos(\pi t) + \pi^2\cos(\pi^2 t). \tag{2.32}$$

The fast frequency component of the input (2.29) has the frequency $\pi^2 = 9.86$; it is slower than the process dynamics for all parameter variations. Neglecting the term de/dt in equation (2.32) gives

$$e \approx \frac{1}{bk_i}\frac{dr}{dt} \approx \frac{\pi^2}{bk_i}\cos(\pi^2 t). \tag{2.33}$$

An estimate of the largest error $e(t) \approx 0.1\cos(\pi^2 t)$ is obtained for the smallest value of $b = 0.1$. It is shown in dashed line in Figure 2.11c, and we see that it gives a good estimate of the maximum error across the uncertain parameter space.

This analysis is based on the assumption that the amplifier can be modeled by a constant gain. The closed loop system is however a dynamic system because the controller is an integrator. It follows from equation (2.31) that the closed loop dynamics has the time constant $T_{cl} = 1/(bk_i)$. If the amplifier has dynamics, its time constant must thus be small compared to T_{cl} in order to provide good tracking. It follows that the largest admissible integral gain k_i is determined by the unmodeled dynamics.

This example illustrates that feedback can be used to design an amplifier that has practically linear input/output relation even if the basic amplifier is nonlinear with strongly varying characteristics.

2.5 Positive Feedback

Most of this book is focused on negative feedback because of its amazingly good properties, which have been illustrated in the previous sections. In this section we will briefly discuss positive feedback, which has complementary properties. In spite of this, positive feedback has found good use in several contexts.

Systems with negative feedback can be well understood by linear analysis. To understand systems with positive feedback it is necessary to consider nonlinear effects, because without the nonlinearities the instability caused by positive feedback will grow without bound. The nonlinear elements can create interesting and useful effects by limiting the signals.

Positive feedback is common in many settings: encouraging a student or a coworker when they have performed well encourages them do to even better. In biology, it is standard to distinguish inhibitory connections (negative feedback) from excitatory feedback (positive feedback) as illustrated in Figure 2.12. Neurons use a combination of positive and negative feedback to generate spikes.

Positive feedback may cause instabilities. Exponential growth, where the rate

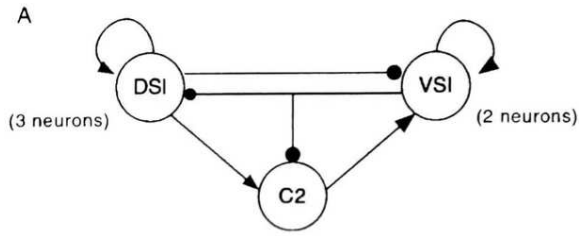


Figure 2.12: Schematic diagram of the neural network that controls swimming motions in the marine mollusk Tritonia, which has both positive and negative feedback [Wil99]. An exhibitory connection (positive feedback) is denoted with a line ending with an arrow, an inhibitory interaction (negative feedback) is denoted with an arrow ending with a circle. (Figure adapted from [Wil99].)

of change of a quantity x is proportional to x,

$$\frac{dx}{dt} = \alpha x,$$

is a typical example, which results in exponential growth $x(t) = e^{\alpha t}$. In nature, exponential growth of a species is limited by the finite amount of food. Another common example is when a microphone is placed close to a speaker in public address systems, resulting in a howling noise. Positive feedback can create stampedes in cattle heard, runs on banks and boom-bust behavior. In all these cases there is exponential growth that is finally limited by finite resources.

The notions of positive and negative feedback are clear if the feedback is static. If the feedback is dynamic its action can change from positive to negative depending on the frequency of the signals. Use of positive feedback will be illustrated by a few examples.

Hewlett's Oscillator

Since positive feedback may generate instability, it can be used to construct oscillators. To limit the exponential growth it is necessary to introduce some nonlinearity that limits the amplitude of the oscillation. An example is given in Figure 2.13, from William Hewlett's Masters thesis from Stanford University in 1938. Hewlett used two vacuum tubes with positive feedback and a nonlinear element in the form of a lamp to maintain constant amplitude of the oscillation. The positive feedback in the basic loop creates an oscillation. The resistance of the lamp decreases as the signal amplitude increases and the amplitude is limited. The oscillator HP200A, based on Hewlett's thesis, was the first product made by Hewlett-Packard, a company that Hewlett started with David Packard in 1939.

The Superregenerative Receiver

In the previous sections we have shown that negative feedback has some very useful properties. The negative feedback amplifier was an enabler for long-distance

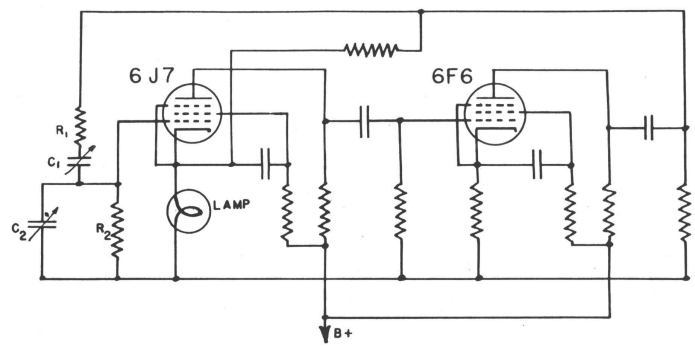


Figure 2.13: Circuit diagram of William Hewlett's oscillator that gives a stable oscillation by using positive feedback and a nonlinearity (lamp) that stabilized the amplitude of the oscillation, [].

telephony. A key idea was to design an amplifier with a very large open loop gain and to reduce the gain by negative feedback. The result was an amplifier that is robust and linear. Positive feedback has complementary properties: it is possible to create high gain but the closed loop system is sensitive to parameter variations.

To understand that positive feedback can generate high gains we consider an amplifier with gain A_{ol}. Referring back to our standard feedback loop in Figure 2.4, we can take the process dynamics to be A_{ol} and the controller dynamics to be k. Neglecting dynamics and closing the loop with *positive* feedback (instead of the negative feedback shown in the diagram), it can be shown that the closed loop system has gain

$$A_{cl} = \frac{A_{ol}}{1 - kA_{ol}}.$$

A very large closed loop gain A_{cl} can be obtained by selecting a feedback gain k that is just below the stability limit $1/A_{ol}$. Choosing $kA_{ol} = 0.99$ gives $A_{cl} = 100A_{ol}$. Armstrong constructed a "superregenerative" radio receiver with only one vacuum tube in 1914, when he was still undergraduate at Columbia University. The drawback by using positive feedback is that the system is highly sensitive and that the gain has to be adjusted carefully to avoid oscillations. It is still used in simple walkie-talkies, garage door openers and toys.

Implementation of Integral Action by Positive Feedback

Positive feedback was used in early controllers where integral action was provided by positive feedback around a system with first order dynamics, as shown in the block diagram of Figure 2.14. Intuitively the system can be explained as follows. Proportional feedback typically gives a steady state error. This can be overcome by adding a bias signal. In Figure 2.14 the bias is estimated by low pass filtering the control signal and adding it back into to the signal path. This serves to compensate for any error that is present.

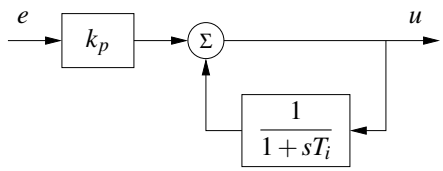


Figure 2.14: Implementation of integral action by *positive feedback* .

The circuit can be understood better by a little analysis. Using block diagram algebra we find that the transfer function of the system is

$$G_{ue} = \frac{k_p}{1 - \frac{1}{1+sT}} = k_p + \frac{k_p}{sT},$$

which is a transfer function of a PI controller. This way of implementing integral action is still used in many industrial regulators.

Positive Feedback Combined with Saturation

Systems with interesting and useful properties can be obtained by combining linear and nonlinear components with positive feedback. Consider the system in Figure 2.15, which consists of a linear block with first-order dynamics and a nonlinear block with positive feedback. Assume that the nonlinearity is

$$y = f(x) = \frac{x}{1+|x|}, \qquad x = f^{-1}(y) = \frac{y}{1-|y|}. \tag{2.34}$$

The system is described by the differential equation

$$\frac{dx}{dt} = -ax + b(r+y) = b(r - G(y)), \qquad G(y) = \frac{af^{-1}(y)}{b} - by. \tag{2.35}$$

The equilibria for a constant input r are given by

$$r = -y + \frac{a}{b} f^{-1}(y) = G(y) \qquad \text{where} \qquad \frac{dG(y)}{dy} = -1 + \frac{a}{b(1-|y|)^2}. \tag{2.36}$$

The graph of the function G is shown in Figure 2.16a for $a = 1$ and $b = 4$. The function has a local maximum $r_{max} = 1 + a/b - 2\sqrt{a/b} = 0.25$ at $y = -1/\sqrt{1+a/b} = -0.5$ and a local minimum $r_{min} = -0.25$ at $y = 0.5$. There is one unique equilibria if $|r| > 0.5$, two equilibria if $|r| = 0.5$ and three equilibria if $|r| < 0.5$. The set of possible equilibria as a function of r is shown in Figure 2.16b.

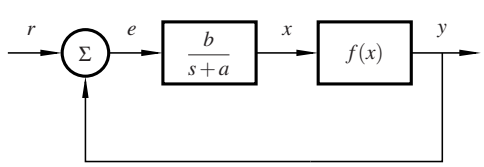
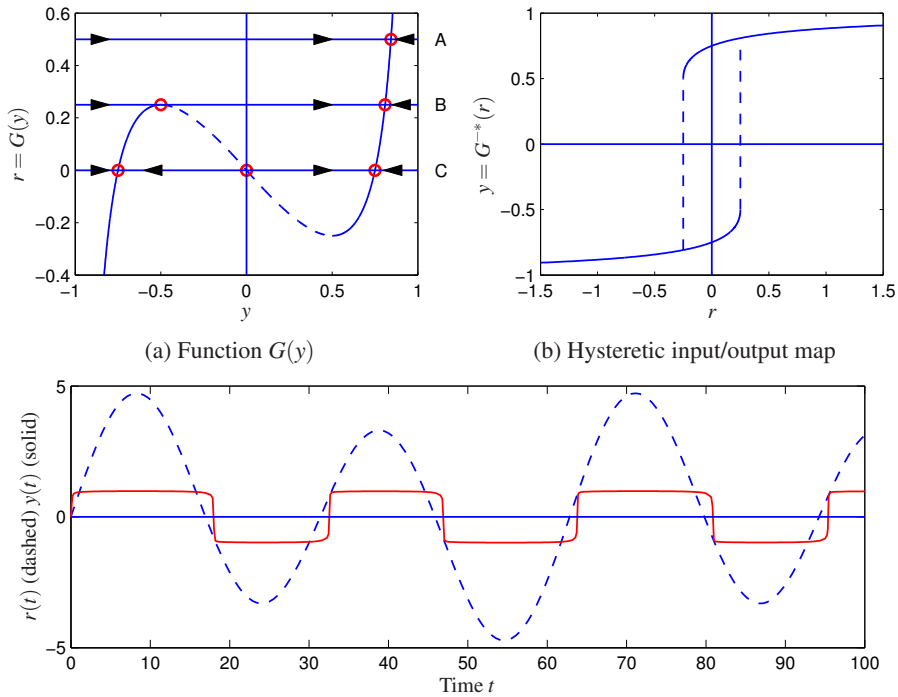


Figure 2.15: Block diagram of system with positive feedback and saturation. The parameters are $a = 1$ and $b = 10$.



(a) Function $G(y)$

(b) Hysteretic input/output map

(c) Input/output behavior

Figure 2.16: System with positive feedback and saturation, (a) show the function $G(y)$ used in the analysis, (b) shows the hysteretic input/output map and (c) shows a simulation of the system. The input r is the dashed curve and the output y is the solid line.

The differential equation (2.36) is of first order and the state $x \in \mathbb{R}$ is a value on the real line, so its behavior can be understood simply by investigating the sign of the derivative dx/dt in the intervals between the equilibria. Since x is a monotone function of y it follows follows from equation (2.35) that dx/dt is positive to the left of an equilibrium point where $G'(y)$ is positive, and negative to the right of this equilibrium point. The stable equilibria correspond to the values of y where the slope of $G(y)$ is positive, which is shown by solid lines in Figure 2.16b. The differential equation thus has two stable equilibria when $r_{min} < r < r_{max}$ and one stable equilibrium when $|r| \geq r_{max}$.

To understand the behavior of the system, we will explore what happens when the reference is changed. If the reference r is zero there are two stable equilibria: we assume that the system is in the stable left equilibrium, where y is negative. If the reference is increased slowly, the equilibrium moves slightly to the right. When the reference reaches the value 0.25, which corresponds an unstable equilibrium, the solution quickly moves to the stable equlibrium, where y is positive, as indicated by the line marked B in Figure 2.16a. If the value of r is increased further, the output y also increases. The input/output relation of the system has

the hysteretic behavior shown in Figure 2.16b, where the dashed line indicate the switches, which occur for $r = \pm r_{max} = \pm(1 + a/b - 2\sqrt{a/b})$.

The temporal behavior of the system is illustrated by the simulations in Figure 2.16c, where the input r is dashed and the output y is solid. The shapes of the signals depend on the parameters; the values $a = 5$, $b = 50$ were used in the figure. The hysteresis width is $2r_{max}$ and the parameter a gives the sharpness of the corners of the output.

The circuit shown in the Figure 2.15 is commonly used as a trigger to detect changes in a signal (known as a Schmitt trigger). It is also used as a memory element in solid state memories. It illustrates that feedback can be used to obtain discrete behavior.

2.6 Feedback and Feedforward

Feedback and feedforward have complementary properties as was discussed in Section 1.5. Feedback only acts when there are deviations between the actual and the desired behaviors, feedforward acts on planned behavior. Feedback and feedforward can be combined to improve the response to command signals and to reduce the effect of disturbances that can be measured. In this section we will discuss feedforward control and how it can be combined with feedback.

Feedforward and System Inversion

To explore feedforward control we will first investigate command signal following. Consider the system modeled by the differential equation (2.1):

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u.$$

Assume that we want to find a control signal $u(t)$ that gives the response $y_{\mathrm{m}}(t)$, a desired output. It follows from equation (2.1) that the desired control signal satisfies

$$b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u = \frac{d^n y_{\mathrm{m}}}{dt^n} + a_1 \frac{d^{n-1} y_{\mathrm{m}}}{dt^{n-1}} + \ldots + a_n y_{\mathrm{m}}, \tag{2.37}$$

where we consider y_{m} as the input and u as the output. Equation (2.37) is called the *inverse* of equation (2.1) because it is obtained by exchanging inputs and outputs. If the transfer function of the original system is P, the transfer function of the inverse system is simply $1/P$.

More generally, consider a system with the transfer function P and assume that we want to find a feedforward controller so that the response to command signals is given by the transfer function F_{m}, as shown in Figure 2.17. The feedforward transfer function is then

$$F_{\mathrm{u}} = P^{-1} F_{\mathrm{m}} = \frac{F_{\mathrm{m}}}{P}. \tag{2.38}$$

Design of a feedforward compensator is thus closely related to system inversion.

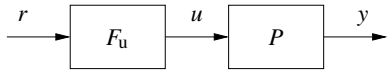


Figure 2.17: Block diagram of a process with pure feedforward compensation. The process transfer function is P, the desired response is characterized by the transfer function F_m the feedforward compensator has the transfer function F_u and the combined system has transfer function $F_m = PF_u$.

Feedforward control has the advantage that it cannot create feedback instabilities, but it also has some disadvantages. If the process transfer function has zeros in the right half plane, the feedforward transfer function given by equation (2.38) will be unstable and the computed response y_m may be unbounded. If the process has time delays, system inversion requires an ideal predictor. The transfer function $F_u(s)$ can also generate very large input signals. Assume for example that

$$P(s) = \frac{0.5}{(s+1)^2}, \quad F_m(s) = \frac{1}{s+1}.$$

It then follows from equation (2.38) that $F_r(s) = 2(s+1)$. To generate the feedforward signal it is thus necessary to differentiate the command signal, which can result in very large control signals if the command signal changes rapidly. In this particular case we must require that the rate of change of the control signal is not too large.

Another drawback of feedforward control is that good process models are required. To understand the effects of errors in the process model we assume that a feedforward F_r is designed for a nominal process model P. It follows from equation (2.38) that the transfer function from reference to process output is $F_m = F_u P$. To investigate what happens when there are small errors dP in the process model, we differentiate F_m with respect to P, which gives

$$\frac{dF_m}{dP} = F_u = \frac{PF_u}{P} = \frac{F_m}{P} \quad \Longrightarrow \quad \frac{dF_m}{F_m} = \frac{dP}{P}. \tag{2.39}$$

The relative error in F_m is thus equal to the relative error in P, which is an analytic justification of the entry in Table 1.1 in Section 1.5 that says that feedforward is sensitive to modeling errors.

Combining Feedforward with Feedback

It seems natural to combine feedforward with feedback. An architecture of such a controller is shown in the block diagram of Figure 2.18. The controller has three blocks: the transfer function F_m that describes the desired response, the feedback transfer function C and the feedforward transfer function F_u.

The controller architecture in Figure 2.18 is highly intuitive. The feedforward signal u_{ff} generates the ideal output $y = y_m$. If there are no disturbances and no modeling errors the feedback error e is then zero and the feedback signal u_{fb} is also zero. All control is thus handled by the feedforward action. If there are disturbances and modeling errors, the error e will not be zero and the feedback controller

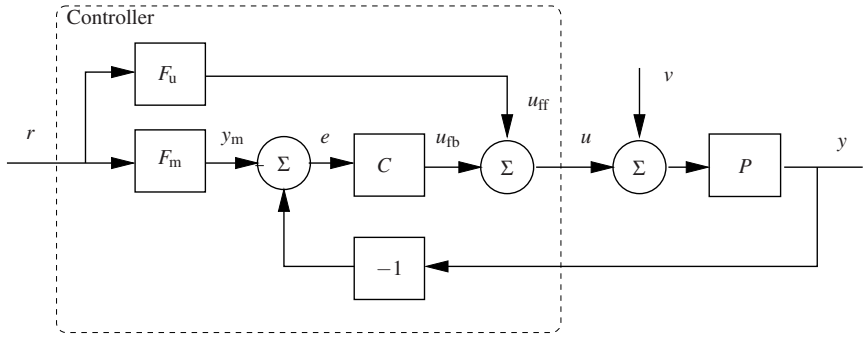


Figure 2.18: Block diagram of a closed loop system where the controller has an architecture with two degrees of freedom. The desired response y_m and the feedforward signal u_{ff} are generated from the reference r. The feedback controller C acts on the control error $e = y_{\mathrm{m}} - y$ and generates the feedback control signal u_{fb}.

C will make appropriate corrections.

The controller architecture in Figure 2.18 is a generalization of the controller with two degrees of freedom introduced in Section 2.3 (see Figure 2.7). A nice property is that it separates command signal following, robustness and disturbance attenuation. Command signal following is dealt with by design of the transfer function F_{m}, which gives the desired response, and the feedforward transfer function F_{u}, which generates the nominal input that is required. Robustness and disturbance attenuation is dealt with by design of the feedback transfer function C.

The transfer function from r to y for the system in Figure 2.18 is

$$G_{yr} = \frac{P(F_{\mathrm{u}} + CF_{\mathrm{m}})}{1+PC} = F_{\mathrm{m}} + \frac{PF_{\mathrm{u}} - F_{\mathrm{m}}}{1+PC}. \tag{2.40}$$

The transfer function G_{yr} from reference r to output y equals F_{m} if the feedforward transfer function F_{u} is chosen to be $F_{\mathrm{u}} = P^{-1}F_{\mathrm{m}}$, which is the same as the condition (2.38) for pure feedforward in an open loop system.

The transfer functions relating the output y and the feedback signal u to the disturbances v are

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)}, \qquad G_{uv}(s) = -\frac{C(s)}{1+P(s)C(s)}, \tag{2.41}$$

which do not depend on the transfer functions F_{m} and F_{u}. The controller architecture in Figure 2.18 thus admits a decoupling of the response to command signals and the response to disturbances. The feedback controller $C(s)$ provides robustness to process variations and attenuation of load disturbances. The transfer functions F_{m} and F_{u} determine the desired response to command signals.

To find the effects of small modeling errors on G_{yr} we differentiate equation (2.40) with respect to P, which gives

$$\frac{dG_{yr}}{dP} = \frac{G_{yr}}{P} - \frac{G_{yr}C}{1+PC}$$

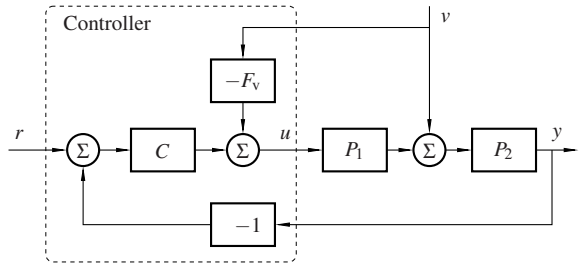


Figure 2.19: Block diagram of a system with a control architecture that combines feedback with feedforward from a disturbance v that can be measured.

Rearranging the equation to compute the relative error gives

$$\frac{dG_{yr}}{G_{yr}} = \frac{dP}{P} - \frac{CdP}{1+PC} = \frac{1}{1+PC}\frac{dP}{P} = S\frac{dP}{P}. \tag{2.42}$$

Comparing this with equation (2.39), we find that the relative errors in the process are reduced by the factor $S = 1/(1+PC)$. Combining feedback and feedforward thus has significant advantages. The function S, called the *sensitivity function*, will be discussed in Section 12.2.

Using Feedforward to Attenuate Measured Disturbances

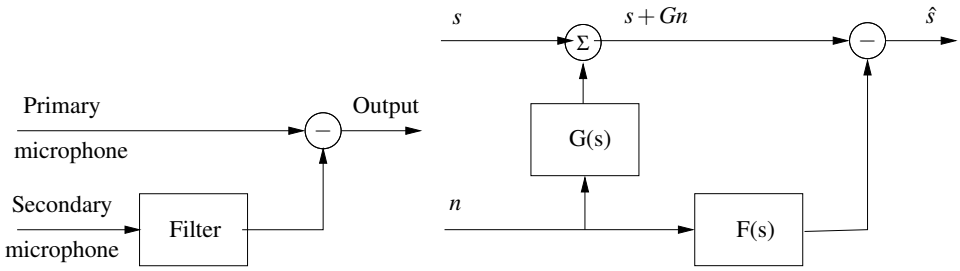
Feedforward can also be used to mitigate the effect of disturbances that can be measured. Such a scheme is shown in Figure 2.19. The process transfer function P is composed of two factors, $P = P_1P_2$. A measured disturbance v enters at the input of process section P_2. The measured disturbance is fed to the process input via the feedforward transfer function F_v.

The transfer function from the disturbance v to process output y is

$$G_{yv} = \frac{P_2(1-P_1F_v)}{1+PC} \tag{2.43}$$

This equation shows that there are two ways of reducing the disturbance. The transfer function $1-P_1F_v$ can be made small by a proper choice of the feedforward transfer function F_v. In feedback compensation the effect of the disturbance is instead reduced by making the the loop transfer function PC large. Feedforward thus makes the error small by subtraction, while feedback instead makes the error small by dividing with $1+PC$. An immediate consequence is that feedforward is more sensitive than feedback since we are trying to match two terms. Feedback gives better robustness but there is a risk of instability. Feedback and feedforward are therefore complementary, and we again can see that it is useful to combine them.

Feedforward is most effective when the disturbance v enters early in the process. This occurs when most of the dynamics are in process section P_2. When $P_2 = P$, and therefore $P_1 = 1$, the feedforward compensator is simply a proportional controller.



(a) Schematic diagram (b) Block diagram

Figure 2.20: Schematic and block diagrams for noise cancellation.

Noise Cancellation

Noise cancellation is a common example of the use of feedforward to cancel effects of disturbances. Consider, for example, a pilot that has to communicate in a noisy cabin. The environmental noise will seriously deteriorate the communication because the pilots microphone will pick up ambient noise. The noise can be reduced significantly by using two microphones as illustrated in Figure 2.20a. The primary microphone is directed towards the pilot. It picks up the pilots voice and ambient noise. The second microphone is directed away from the pilot and it picks up the ambient noise. The effect of the noise can be reduced by filtering the signal from the secondary microphone and subtracting it from the signal from the primary microphone. A block diagram of the system is shown in Figure 2.20b. The transfer function $G(s)$ represents the dynamics of the acoustic transmission from the secondary microphone to the first microphone. The transfer function $F(s)$ is the transfer function of the filter. To cancel the effect of the noise the transfer function $F(s)$ should be close to $G(s)$. Since the noise transmission depends on the situation, for example how the pilot turns his head, it is common to let the filter $F(s)$ be adaptive so that it can adjust, as described later in Example 5.16. Noise cancellation has many applications: in headphones, to create noise-free spaces by active noise control, or to measure electrocardiograms and separate the heartbeats of mother and fetus.

2.7 Using Feedback to Shape Behavior

The regulation and servo problems discussed in Sections 2.2 and 2.3 are classical applications of feedback. In Section 2.4 it was shown that feedback can be used to shape linear input/output behavior from nonlinear components. Feedback can also be used to shape many other behaviors. In robotics feedback is used to obtain collision avoidance. In automotive control it is used to create behaviors that avoid locking the brakes and skidding. Bacteria use simple feedback mechanisms to search for areas where there is high concentration of food or light. The principle is to sense a variable and to make exploratory moves to see if the concentration increases. A similar mechanism can be used to avoid harmful substances. Opti-

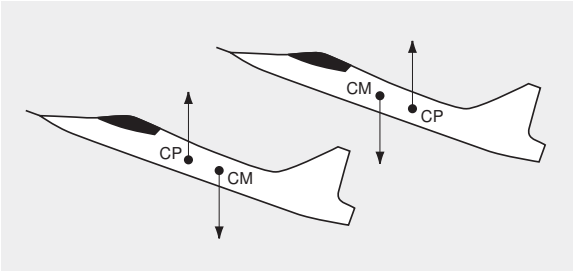


Figure 2.21: Schematic diagram of two aircraft. The aircraft at the top is stable because it has the center of pressure CP behind the center of mass CM. The aircraft at the bottom is unstable because the positions of center of mass and center of pressure are reversed.

mization is also used in computer systems to maintain high throughput of servers. In this section we elaborate on a few cases.

Stabilization

Stabilizing an unstable system is a typical example of how feedback can be used to change behavior. Many systems are naturally unstable. The ability to stand upright, walk and run has given humans many advantages but it requires stabilization. Stability and maneuverability are conflicting goals in vehicle design. The ship designer Minorsky realized that there was a trade-off between maneuverability and stability and he emphasized that a stable ship is difficult to steer. The Wright Flyer, which was maneuverable but unstable, inspired Sperry to design an autopilot. Feedback has been used extensively in aircraft, from simple systems for stability augmentation to systems that provide full autonomy.

Military airplanes gain significant competitive advantage by being made unstable. Schematic pictures of two airplanes are shown in Figure 2.21. The positions of the center of mass CM and the center of pressure CP are key elements. To be stable, the center of pressure must be behind of the center of mass. The center of pressure of an aircraft shifts backwards when a plane goes supersonic. If the plane is stable at subsonic speeds it becomes even more stable at supersonic speeds because of the long distance between CM and CP. Large forces and large control surfaces are then required to maneuver the airplane and the plane will be more sluggish. A more balanced design is obtained by placing the center of pressure in front of the center of mass at subsonic speeds. Such an airplane will have superior performance, but it is unstable at subsonic speeds, typically at takeoff and landing. The control system that stabilizes the aircraft in these operating conditions is mission-critical, with strong requirements on robustness and reliability.

The evolutionary biologist John Maynard Smith [Smi52a] has claimed that while early flying animals were inherently stable, they later developed unstable configurations when their sensory and nervous systems became more sophisticated and able to stabilize. The unstable configuration had significant advantages in maneuverability both for predator and prey.

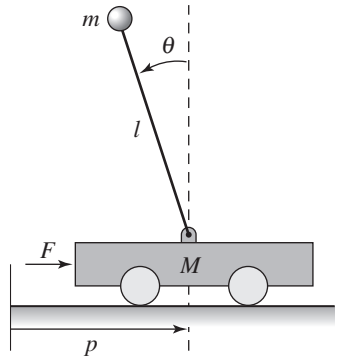


Figure 2.22: Inverted pendulum on a cart.

Keeping an inverted pendulum in the upright position is another example of stabilization. Consider the cart–pendulum system shown in Figure 2.22. Neglecting damping, assuming that the cart is much heavier than the pendulum and assuming that the tilt angle θ is small, the dynamics of the system can be approximated by the differential equation

$$J_t\ddot{\theta} - mgl\theta = u \tag{2.44}$$

(a more detailed derivation is given in Example 3.1). The transfer function of the open loop system is

$$G_{\theta u} = \frac{1}{J_t s^2 - mgl}, \qquad a_{cl}(s) = J_t s^2 - mgl.$$

The system is unstable because it has a pole $s = \sqrt{mgl/J_t} = \omega_0$ that is positive. It can be stabilized with a *proportional-derivative (PD) controller* that has the transfer function

$$C(s) = k_d s + k_p. \tag{2.45}$$

The closed-loop characteristic polynomial is

$$a_{cl}(s) = J_t s^2 + k_d s + (k_p - mgl),$$

and all of its roots are in the left half plane if $k_p > mgl$.

One way to find controller parameters is to choose the controller gains so that the characteristic polynomial has natural frequency ω_c and damping ratio ζ_c, hence

$$k_d = 2\zeta_c\omega_c J_t, \qquad k_p = J_t\omega_c^2 + mgl.$$

Choosing $\omega_c = \omega_0$ gives the closed loop poles $-\zeta_c\omega_0 \pm i\sqrt{1-\zeta_c^2}\omega_0$ which can be compared with the open loop poles at $\pm\omega_0$. The controller gains required are $k_p = 2mgl$ and $k_d = 2\zeta_c\sqrt{mglJ_t}$. The control law (2.45) stabilizes the pendulum but is does not stabilize the motion of the cart. To do this it is necessary to introduce feedback from cart position and cart velocity.

Figure 2.23: Haptic devices, the left figure shows the PHANTOM® and the right a system is developed by Quanser.

Impedance Control and Haptics

Changing behavior of a mechanical system is common in robotics and haptics. Position control is not sufficient when industrial robots are used for grinding, polishing and assembly. The robot can be brought into proximity with the workspace by position control, but to carry out the operations it is desirable to shape how the force depends on the distance between the tool and the workspace. A spring-like behavior is an example. The general problem is to create a behavior specified by a given differential equation between force and motion, a procedure called *impedance control*. Similar situations occur in teleoperation in hazardous environment or in telesurgery. In this situation the workpiece is operated remotely using a joystick. It is useful for the operator to have some indication of the forces between the tool and the workpiece. This can be accomplished by generating a force on the operator's joystick that mimics the force on the workpiece.

Figure 2.23 shows two haptic input devices. The systems are pen-like with levers or gimbals containing angle sensors and force actuation. By sensing position and orientation, and then generating a force depending on position and velocity, it is possible to create a behavior that simulates touching real or virtual objects. Forces that simulate friction and surface structure can also be generated.

We illustrate the principle by considering a joystick having a low friction joint. Let J be the moment of inertia, and let the actuation torque and the external torque from the operator be T_a and T, respectively. The equation of motion is

$$J\frac{d^2\theta}{dt^2} = T + T_a.$$

By measuring the angle θ and its first two derivatives we can create the feedback

$$T_a = k_p(\theta_r - \theta) - k_d\frac{d\theta}{dt} - k_a\frac{d^2\theta}{dt^2}.$$

The closed loop system is then

$$(J + k_a)\frac{d^2\theta}{dt^2} + k_d\frac{d\theta}{dt} + k_p(\theta - \theta_r) = T.$$

The feedback has thus provided virtual inertia $J + k_a$, virtual damping k_d and virtual spring action k_p. If no torque T is applied, the joystick will assume the orientation given by the reference signal θ_r. If a the user applies a torque, the joystick will behave like a damped spring–mass system.

2.8 Further Reading

The books by Bennett [Ben79, Ben93] and Mindel [Min02, Min08] give interesting perspective on the development of control. Much of the material touched upon in this chapter is classical control; see [CM51], [JNP47] and [Tru55]. The notion of controllers with two degrees of freedom was introduced by Horowitz [Hor63]. The analysis will be elaborated in the rest of the book. Transfer functions and other descriptions of dynamics are discussed in Chapters 6 and 9, methods to investigate stability in Chapter 10. The simple method to find parameters of controllers based on matching of coefficients of the closed loop characteristic polynomial is developed further in Chapters 7, 8 and 13. Feedforward control is discussed in Section 8.5 and Section 12.2.

Exercises

2.1 Let $y \in \mathbb{R}$ and $u \in \mathbb{R}$. Solve the differential equations

$$\frac{dy}{dt} + ay = bu, \qquad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\frac{du}{dt} + u,$$

for $t > 0$. Determine the responses to a unit step $u(t) = 1$ and the exponential signal $u(t) = e^{st}$ when the initial condition is zero. Derive the transfer functions of the systems.

2.2 Let $y_0(t)$ be the response of a system with the transfer function $G_0(s)$ to a given input. The transfer function $G(s) = (1 + sT)G_0(s)$ has the same zero frequency gain but it has an additional zero at $z = -1/T$. Let $y(t)$ be the response of the system with the transfer function $G(s)$ and show that

$$y(t) = y_0(t) + T\frac{dy_0}{dt}, \tag{2.46}$$

Next consider the system with the transfer function

$$G(s) = \frac{s + a}{a(s^2 + 2s + 1)},$$

which has unit zero-frequency-gain ($G(0) = 1$). Use the result in equation (2.46) to explore the effect of the zero $s = -1/T$ on the step response of a system

2.3 Consider a closed loop system with process dynamics and a PI controller modeled by

$$\frac{dy}{dt} + ay = bu, \qquad u = k_p(r-y) + k_i \int_0^t (r(\tau) - y(\tau))d\tau,$$

where r is the reference, u the control variable and y the process output. a) Derive a differential equation relating the output y to the difference by direct manipulation of the equations. b) Draw a block diagram of the system. c) Derive the transfer functions of the process and the controller. d) Compute the transfer function from reference r to output y of the closed loop system. Make the derivations both by direct manipulation of the system equations and by polynomial algebra. Compare the results with a direct determination of the transfer functions by inspection of the block diagram.

2.4 The dynamics of the pupillary reflex is approximated by a linear system with the transfer function

$$P(s) = \frac{0.2(1-0.1s)}{(1+0.1s)^3}.$$

Assume that the nerve system that controls the pupil opening is modeled as a proportional controller with the gain k. Use Routh-Hurwitz theorem to determine the largest gain that gives a stable closed loop system.

2.5 A simple model for the relation between speed v and throttle u for a car is given by the transfer function

$$G_{vu} = \frac{b}{s+a}$$

where $b = 1\ \mathrm{m/s^2}$ and $a = 0.025\ \mathrm{rad/s}$ (see Section 4.1). The control signal is normalized to the range $0 \le u \le 1$. Design a PI controller for the system that gives a closed loop system with the characteristic polynomial

$$a_{cl}(s) = s^2 + 2\zeta\omega_c s + \omega_c^2.$$

What are the consequences of choosing different values of the design parameters ζ and ω_c? Use your judgment to find suitable values. Hint: Investigate maximum acceleration and maximum velocity for step changes in the velocity reference.

2.6 Consider the feedback system in Figure 2.4. Let the disturbance $v = 0$, $P(s) = 1$ and $C(s) = k_i/s$. Determine the transfer function G_{yr} from reference r to output y. Also determine how much G_{yr} is changed when the process gain changes by 10%.

2.7 The calculations in Section 2.2 can be interpreted as a design method for a PI controller for a first-order system. A similar calculation can be made for PID control of the second order system. Let the transfer functions of the process and the controller be

$$P(s) = \frac{b}{s^2 + a_1 s + a_2}, \qquad C(s) = k_p + \frac{k_i}{s} + k_d s.$$

Show that the controller parameters

$$k_p = \frac{(1+2\alpha\zeta)\omega_c^2 - a_2}{b}, \qquad k_i = \frac{\alpha\omega_c^3}{b}, \qquad k_d = \frac{(\alpha+2\zeta)\omega_c - a_1}{b}.$$

give a closed loop system with the characteristic polynomial

$$(s^2 + 2\zeta\omega_c s + \omega_c^2)(s + \alpha\omega_c).$$

2.8 Consider an open loop system with the nonlinear input-output relation $y = f(u)$. Assume that the system is closed with the proportional controller $u = k(r - y)$. Show that the input-output relation of the closed loop system is

$$y + \frac{1}{k} f^{-1}(y) = r.$$

Estimate the largest deviation from ideal linear response $y = r$. Illustrate by plotting the input output responses for a) $f(u) = \sqrt{u}$ and b) $f(u) = u^2$ with $0 \le u \le 1$ and $k = 5, 10$ and 100.

2.9 Consider the system in Section 2.2 where the controller was designed to give a closed loop system characterized by $\omega_c = 1$ and $\zeta = 0.707$. The transfer functions of the process and the controller are

$$P(s) = \frac{2}{s+1}, \qquad C(s) = \frac{0.207s + 0.5}{s}.$$

The response of the closed loop system to step command signals has a settling time (time required to stay within 2% of the final value, see Figure 6.9) of $4/\zeta\omega_c \approx 5.66$. Assume that the attenuation of the load disturbances is satisfactory but that we want a closed loop system system that responds five times faster to command signals without overshoot. Determine the transfer functions of a controller with the architecture in Figure 2.18 that gives a response to command signals with a first-order dynamics. Simulate the system in the nominal case of a perfect model and explore the effects of modeling errors by simulation.

2.10 Consider a queuing system modeled by

$$\frac{dx}{dt} = \lambda - \mu_{\max}\frac{x}{x+1}.$$

The model is nonlinear and the dynamics of the system changes significantly with the queuing length; see Example 3.12. Investigate the situation when a PI controller is used for admission control. The arrival intensity λ is then given by

$$\lambda = k_p(r - x) + k_i \int^t (r(t) - x(t))dt.$$

The controller parameters are determined from the approximate model

$$\frac{dx}{dt} = \lambda.$$

Find controller parameters that give the closed loop characteristic polynomial $s^2 + 2s + 1$ for the approximate model. Investigate the behavior of the control strategy for the nonlinear model by simulation for the input $r = 5 + 4\sin(0.1t)$.