Feedback Systems

An Introduction for Scientists and Engineers

SECOND EDITION

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Chapter Three
Feedback Principles

Feedback - it is the fundamental principle that underlies all self-regulating systems, not only machines but also the processes of life and the tides of human affairs.

A. Tustin, "Feedback", *Scientific American*, 1952, [Tus52].

This chapter presents examples that illustrate fundamental properties of feedback: disturbance attenuation, command signal following, robustness and shaping of behavior. Simple methods for analysis and design of low order systems are introduced. After reading this chapter, readers should have some insight into the power of feedback, they should know about transfer functions and block diagrams and be able to design simple feedback systems.

3.1 Mathematical Models

The fundamental properties of feedback will be illustrated using a collection of examples. We need a modest set of concepts and tools to analyze simple feedback systems: linear differential equations, transfer functions, block diagrams and block diagram algebra. In addition we need a simulation tool. In this section we will introduce some of these tools, refining them in further chapters.

Linear Differential Equations and Transfer Functions

In many practical situations, the input/output behavior of a system can be modeled by a linear differential equation of the form

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_n u. \tag{3.1}$$

where u is the input, y is the output and the coefficients a_k and b_k are real numbers. The model (3.1) is more general than the model given by equation (2.7) in Section 2.2 because the right hand side has terms with derivatives of the input u. The differential equation (3.1) is characterized by two polynomials

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n, \qquad b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n, \tag{3.2}$$

where $a(s)$ is the *characteristic polynomial* of the differential equation (3.1).

The solution to equation (3.1) is the sum of two terms: the *general solution to the homogeneous equation*, which does not depend on the input, and a *particular solution*, which depends on the input. The *homogeneous equation* associated with

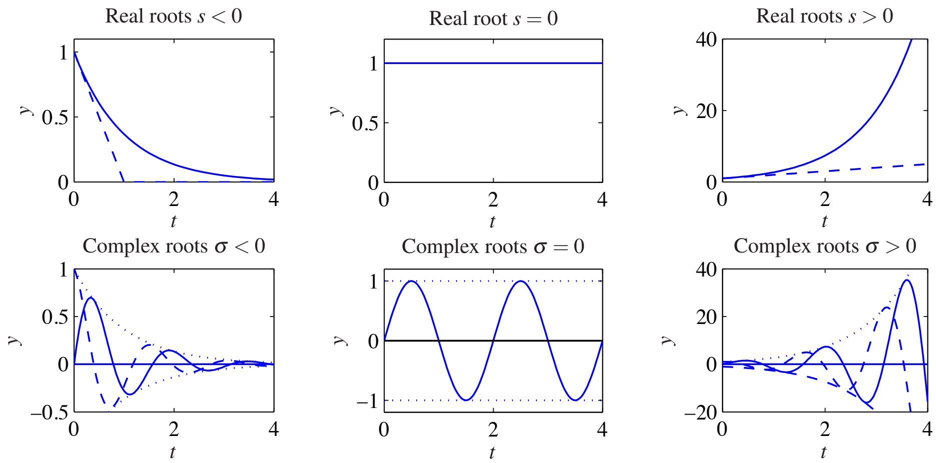


Figure 3.1: The exponential function $y(t) = e^{st}$. The top row shows the function for real s, the bottom row shows the function for complex $s = \sigma + i\omega$. The left column shows $\operatorname{Re} s < 0$, the center column $\operatorname{Re} s = 0$ and the right column $\operatorname{Re} s > 0$.

equation (3.1) is

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = 0. \tag{3.3}$$

The solution to (3.3) is

$$y(t) = \sum_{k=1}^{n} C_k e^{s_k t}, \tag{3.4}$$

if the *characteristic equation* $a(s) = 0$ does not have multiple roots s_k. The parameters $C_1, \ldots, C_n$ are constants, which can be determined from the initial conditions.

Since the coefficients a_k are real, the roots of the characteristic equation are real or complex conjugated pairs. A real root s_k of the characteristic equation corresponds to the exponential function $e^{s_k t}$. This function decreases over time if s_k is negative, it is constant if $s_k = 0$, and it increases if s_k is positive, as shown in the top row of Figure 3.1. For real roots s_k the parameter $T = 1/s_k$ is the *time constant*.

A complex root $s_k = \sigma \pm i\omega$ corresponds to the time functions

$$e^{\sigma t} \sin(\omega t), \qquad e^{\sigma t} \cos(\omega t),$$

which have oscillatory behavior, as illustrated in the bottom row of Figure 3.1. The sine terms are shown in full lines and the cosine terms in dashed lines, they have zero crossings with the spacing π/ω. The dotted lines shows the envelopes, which correspond to the exponential function $\pm e^{\sigma t}$.

When the characteristic equation (3.4) has multiple roots, the solutions to the homogeneous equation (3.3) are

$$y(t)=\sum_{k=1}^{m} C_k(t)e^{s_k t}, \tag{3.5}$$

where $C_k(t)$ is a polynomial with degree less than the multiplicity of the root s_k. The solution (3.5) has $\sum_{k=1}^{m}(\deg C_k+1)=n$ free parameters.

Having explored the solution to the homogeneous equation, we now turn to the input-dependent part of the solution. The solution to equation (3.1) for an exponential input is of particular interest. We set $u(t)=e^{st}$ and investigate if there is a unique particular solution of the form $y(t)=G(s)e^{st}$. Assuming this to be the case, we find

$$\begin{aligned} &\frac{du}{dt}=se^{st}, \quad &&\frac{d^2u}{dt^2}=s^2e^{st}, \quad &&\cdots \quad &&\frac{d^nu}{dt^n}=s^ne^{st}\\ &\frac{dy}{dt}=sG(s)e^{st}, \quad &&\frac{d^2y}{dt^2}=s^2G(s)e^{st}, \quad &&\cdots \quad &&\frac{d^ny}{dt^n}=s^nG(s)e^{st}. \end{aligned} \tag{3.6}$$

Inserting these expression in the differential equation (3.1) gives

$$(s^n+a_1s^{n-1}+\cdots+a_n)G(s)e^{st}=(b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n)e^{st},$$

and hence

$$G(s)=\frac{b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n}{s^n+a_1s^{n-1}+\cdots+a_n}=\frac{b(s)}{a(s)}. \tag{3.7}$$

This function is called the *transfer function* of the system. It describes a particular solution to the differential equation for the input e^{st} and it is a convenient way to characterize the system described by the differential equation.

To further show the relation between the transfer function and the differential equation, introduce the differential operator $p=\frac{d}{dt}$ and the notation $p^k=\frac{d^k}{dt^k}$. The differential equation (3.1) can be written as

$$p^ny+a_1p^{n-1}y+\cdots a_ny=b_1p^{n-1}u+b_2p^{n-1}u+\cdots+b_nu,$$

or

$$(p^n+a_1p^{n-1}+\cdots+a_n)y=(b_1p^{n-1}+b_2p^{n-2}+\cdots+b_n)u.$$

The relation between the transfer function (3.7) and the differential equation (3.1) is clear: the transfer function (3.7) can be obtained by inspection from the differential equation (3.1), and conversely the differential equation can be obtained from the transfer function. The transfer function can thus be regarded as a shorthand notation for the differential equation (3.1).

To deal with oscillatory signals, like those shown in Figure 3.1, it is convenient to allow s to be a complex number. The transfer function is a function $G:\mathbb{C}\to\mathbb{C}$ that maps complex numbers to complex numbers. The roots of the characteristic equation $a(s)=0$ are called *poles* of the transfer function. A pole s_k appears as exponent in the general solution to the homogeneous equation (3.4). The roots of the polynomial $b(s)$ are called *zeros* of the transfer function. The reason is that if $b(s_k)=0$ it follows that $G(s_k)=0$, and the particular solution for the input e^{s_kt} is zero. A system theoretic interpretation is that the transmission of the exponential signal $e^{s_k t}$ is blocked by the zero $s = s_k$.

The particular solution for a constant input $u(t) = 1$ is $y(t) = G(0) = b_n/a_n$. The quantity $G(0)$ is called the *zero frequency gain* or the *static gain*. The particular solution for the input $u = \cos(\omega t) = \operatorname{Re} e^{i\omega t}$ is

$$\begin{aligned} y(t) &= \operatorname{Re}\left(G(i\omega)\, e^{i\omega t}\right) = \operatorname{Re}\left(|G(i\omega)|\, e^{i\arg G(i\omega)}\, e^{i\omega t}\right) \\ &= |G(i\omega)| \operatorname{Re} e^{i(\arg G(i\omega)+\omega t)} = |G(i\omega)| \cos(\omega t + \arg G(i\omega)). \end{aligned}$$

The input is thus amplified by $|G(i\omega)|$ and the phase shift between input and output is $\arg G(i\omega)$, where arg denotes the angle of a complex variable. The functions $G(i\omega)$, $|G(i\omega)|$ and $\arg G(i\omega)$ are called the *frequency response*, *gain* and *phase*. The gain and the phase are also called *magnitude* and *angle*.

The actual response to a sine or a cosine function is the sum of a particular solution and the general solution to the homogeneous equation (3.4) or (3.5). The coefficients in the general solution can be determined from the initial conditions. If all roots of the characteristic equation have negative real parts, all solutions to the homogeneous equation go to zero and the general solution converges to the particular solution as time increases.

The transfer function $G(s)$ is a useful representation of the differential equation (3.1) and of the system modeled by the differential equation. The transfer function has many physical interpretations that can be exploited for analysis and design. The transfer function makes it possible to apply algebra to determine relations between signals in a complex system. The transfer function can also convey intuition: $G(0)$ is the steady state gain for constant inputs and the frequency response $G(i\omega)$ captures the steady state response to sinusoidal functions. The frequency response of a stable system can be determined experimentally by exploring the response of a system to sinusoidal signals. The approximations of $G(s)$ for small and large s captures the propagation of slow and fast signals respectively. Consider for example the spring-mass system in equation (2.15), with input u and output q, which has the transfer function

$$G(s) = \frac{1}{ms^2 + cs + k.}$$

For small s we have $G(s) \approx 1/k$. The corresponding input/output relation is $q = (1/k)u$ which implies that for low frequency inputs, the system behaves like a spring driven by a force. For large s we have $G(s) \approx 1/(ms^2)$. The corresponding differential equation is $m\ddot{q} = u$, and for high frequency inputs the system behaves like mass driven by a force (a double integrator). A more eleborate treatment of transfer functions and the frequency response will be given in later chapters, particularly in Chapter 8.

Stability: The Routh-Hurwitz Criterion

When using feedback there is always the danger that the system may become unstable. It is therefore important to have a stability criterion. The differential equa-

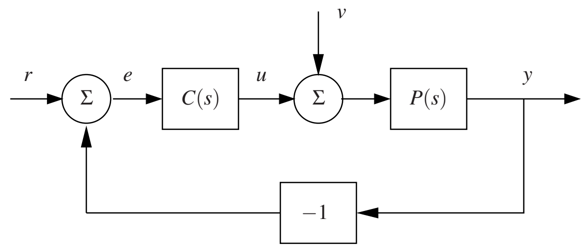


Figure 3.2: Block diagram of simple feedback system. The controller transfer function is $C(s)$ and the process transfer function is $P(s)$. The process output is y, the external signals are the reference r and the load disturbance v.

tion (3.1) is called *stable* if all solutions of the homogeneous equation (3.3) go to zero for any initial condition. It follows from equation (3.5) that this requires that all the roots of the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n = 0,$$

have negative real parts. The *Routh-Hurwitz criterion* is a stability criterion that does not require calculation of the roots, because it gives conditions in terms of the coefficients of the characteristic polynomial..

A first order differential equation is stable if the coefficient a_1 of the characteristic polynomial is positive, since the zero of the characteristic polynomial will be $s = -a_1 < 0$. A second order polynomial is stable if and only if the coefficients a_1 and a_2 are all positive. Since the roots are

$$s = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_2}\right),$$

it is easy to verify that the real parts are negative if and only if $a_1 > 0$ and $a_2 > 0$. A third order differential equation is more complicated, but the roots can be shown to have negative real parts if and only if

$$a_k > 0 \quad \forall k, \qquad a_1 a_2 > a_3. \tag{3.8}$$

The corresponding conditions for a fourth order differential equation are

$$a_k > 0 \quad \forall k, \qquad a_1 a_2 > a_3, \qquad a_1 a_2 a_3 > a_1^2 a_4 + a_3^2. \tag{3.9}$$

The Routh-Hurwitz criterion [Gan60] gives similar conditions for arbitrarily high order polynomials. Stability of a linear differential equation can thus be investigated just by analyzing the signs of various combinations of the coefficients of the characteristic polynomial.

Block Diagrams and Transfer Functions

Figure 3.2 shows a block diagram of a typical control system. If each block is modeled as a linear differential equation (3.1), we need to find the differential equation that relates the signals in the complete system. A block can be considered as a filter that generates the output from the input and the block is characterized by its transfer function, which is a nice shorthand notation for the differential equation describing the input/output relation.

Assume that the disturbance v is zero and that we want to find the differential equation that describes how the output y is influenced by the reference signal r. Let the transfer functions of the controller and the process be characterized by the polynomials $b_c(s)$, $a_c(s)$, $b_p(s)$ and $a_p(s)$, so that

$$C(s) = \frac{b_c(s)}{a_c(s)}, \qquad P(s) = \frac{b_p(s)}{a_p(s)}. \tag{3.10}$$

The corresponding differential equations are

$$a_c(p)\,u(t) = b_c(p)\,(r(t) - y(t)), \qquad a_p(p)\,y(t) = b_p(p)\,u(t),$$

recall that $p^k = \frac{d^k}{dt^k}$. Multiplying the first equation by $a_p(p)$ and the second with $a_c(p)$ we find that

$$a_c(p)\,a_p(p)\,y(t) = a_c(p)\,b_p(p)\,u(t) = b_p(p)\,b_c(p)\,(r(t) - y(t)).$$

Solving for $y(t)$ gives

$$\big(a_c(p)\,a_p(p) + b_p(p)\,b_c(p)\big)\,y(t) = b_p(p)\,b_c(p)\,r(t), \tag{3.11}$$

which is the differential equation that relates the output to the reference. We see that the polynomial notation makes it easy to manipulate differential equations. Forming linear combinations of differential equations and their derivatives corresponds to polynomial multiplication.

The differential equation (3.11) corresponds to the transfer function

$$G_{yr} = \frac{b_p(s)b_c(s)}{a_c(s)a_p(s) + b_p(s)b_c(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)}, \tag{3.12}$$

where we used the notation G_{yr} for the transfer function from r to y. Proceeding in the same way we obtain the following transfer functions

$$G_{ur} = \frac{C(s)}{1 + P(s)C(s)}, \qquad G_{yv} = \frac{P(s)}{1 + P(s)C(s)}, \qquad G_{uv} = \frac{-P(s)C(s)}{1 + P(s)C(s)}, \tag{3.13}$$

By using polynomials and transfer functions the relations between signals in a feedback system can be obtained by algebra. The transfer functions relating two signals can be obtained from the block diagram by inspection. The denominator is always $1 + P(s)C(s)$ and the numerator is a product of the transfer functions between the signals, for example, the transfer functions from disturbance v to control u in Figure 3.2 are $P(s)$, -1 and $C(s)$.

Transfer Functions and Laplace Transforms

We have defined the transfer function as a particular solution to a linear differential equation with the exponential input e^{st}. Transfer functions can also be conveniently defined using Laplace transforms. Let $u(t)$ be the input to the system (3.1) and let

$y(t)$ be the corresponding output when the initial conditions are zero. Furthermore let $U(s)$ and $Y(s)$ be the Laplace transforms of the input and the output

$$U(s) = \int_0^\infty e^{-st} u(t)dt, \qquad Y(s) = \int_0^\infty e^{-st} y(t)dt.$$

The transfer function of the system is then simply $G(s) = \frac{Y(s)}{U(s)}$.

3.2 Using Feedback to Improve Disturbance Attenuation

Reducing the effects of disturbances is a primary use of feedback. It was used by James Watt to make steam engines run at constant speed in spite of varying load and by electrical engineers to make generators driven by water turbines deliver electricity with constant frequency and voltage. Feedback is commonly used to alleviate disturbances in the process industry, for machine tools and for engine and cruise control in cars. In humans the pupillary reflex is used to make sure that the light intensity of the retina is reasonably constant in spite of large variations in the ambient light. The human body exploits feedback to keep body temperature, blood pressure and other important variables constant. Keeping variables close to a desired, constant reference values in spite of disturbances is called a *regulation problem*.

Disturbance attenuation will be illustrated by control of a process whose dynamics can be approximated by a first order system. A block diagram of the system is shown in Figure 3.2. Since we will focus on the effects of a load disturbance v we will assume that the reference r is zero. The transfer functions G_{yv} and G_{uv} relating the output y and the control u to the load disturbance are given by equation (3.13). For simplicity we will assume that the process is modeled by the first order differential equation

$$\frac{dy}{dt} + ay = bu, \qquad a > 0, \quad b > 0.$$

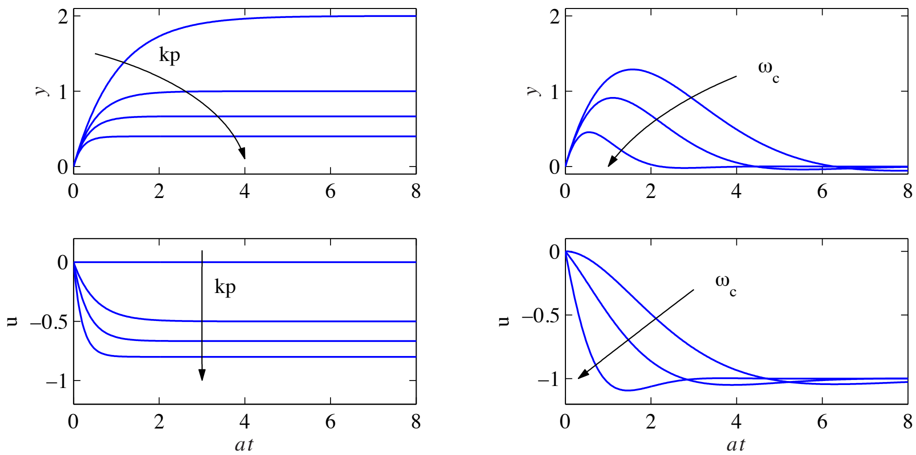
The corresponding transfer function is

$$P(s) = \frac{b}{s+a}. \tag{3.14}$$

A first order system is a reasonable model of a physical system if the storage of mass, momentum or energy can be captured by a single state variable. Typical examples are the velocity of a car on a road, the angular velocity of rotating system and the level of a tank.

Proportional Control

We will first investigate the case of proportional (P) control, when the control signal is proportional to the output error: $u = k_p e$, see Section 1.4. The controller transfer function is then $C(s) = k_p$. The process transfer function is given by (3.14)



(a) Proportional control (P) (b) Proportional-Integral control (PI)

Figure 3.3: Responses of open and closed loop system with proportional control (a) and PI control (b). The process transfer function is $P = 2/(s+1)$. The controller gains for proportional control are $k_p = 0$, 0.5, 1 and 2. The PI controller is designed using equation (3.20) with $\zeta = 0.707$ and $\omega_c = 0.707$, 1 and 2, which gives the controller parameters $k_p = 0$, 0.207, 0.914 and $k_i = 0.25$, 0.50 and 2.

and he effect of the disturbance on the output is then described by the transfer function (3.13)

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{\frac{b}{s+a}}{1+\frac{bk_p}{s+a}} = \frac{b}{s+a+bk_p}.$$

The relation between the disturbance v and the output y is thus given by the differential equation

$$\frac{dy}{dt} + (a+bk_p)y = bv.$$

The closed loop system is stable if $a+bk_p > 0$. A constant disturbance $v = v_0$ then gives an output that approaches the steady state value

$$y_0 = G_{yv}(0) = \frac{b}{a+bk_p} v_0,$$

exponentially with the time constant $T = 1/(a+bk_p)$. Without feedback $k_p = 0$ and a constant disturbance v_0 thus gives the steady state error v_0/a. The steady state error thus decreases when using feedback if $k_p > 0$.

We have thus shown that a constant disturbance gives an error that can be reduced by feedback using a proportional controller. The error decreases with increasing controller gain. Figure 3.3 shows the responses for a few values of controller gain k_p.

Proportional-Integral (PI) Control

The PI controller, introduced in Section 1.4, is described by

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau. \tag{3.15}$$

To determine the transfer function of the controller we differentiate, hence

$$\frac{du}{dt} = k_p \frac{de}{dt} + k_i e$$

and we find by inspection that the transfer function is $C(s) = k_p + k_i/s$. To investigate the effect of the disturbance v on the output we use the block diagram in Figure 3.2 and we find by inspection that the transfer function from v to y is

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{bs}{s^2 + (a+bk_p)s + bk_i}. \tag{3.16}$$

The relation between the load disturbance and the output is thus given by the differential equation

$$\frac{d^2y}{dt^2} + (a+bk_p)\frac{dy}{dt} + bk_i y = b\frac{dv}{dt}. \tag{3.17}$$

Notice that, since the disturbance enters as a derivative in the right hand side, a constant disturbance gives no steady state error. The same conclusion can be drawn from the observation that $G_{yv}(0) = 0$. Compare with the discussion of integral action and steady state error in Section 1.4.

To find suitable values of the controller parameters k_p and k_i we consider the characteristic polynomial of the differential equation (3.17),

$$a_{cl}(s) = s^2 + (a+bk_p)s + bk_i. \tag{3.18}$$

We can assign arbitrary roots to the characteristic polynomial by choosing the controller gains k_p and k_i, and we choose controller parameters that give the characteristic polynomial

$$(s+\sigma+i\omega)(s+\sigma-i\omega) = s^2 + 2\sigma s + \sigma^2 + \omega^2. \tag{3.19}$$

This polynomial has roots at $s = -\sigma \pm i\omega$. The general solution to the homogeneous equation is then a linear combination of the terms

$$e^{-\sigma t}\sin(\omega t), \qquad e^{-\sigma t}\cos(\omega t),$$

which are damped sine and cosine functions, as shown in the lower left plot in Figure 3.1. The coefficient σ determines the decay rate and the parameter ω gives the frequency of the decaying oscillation. Identifying coefficients of equal powers of s in the polynomials (3.18) and (3.19) gives

$$k_p = \frac{2\sigma - a}{b}, \qquad k_i = \frac{\sigma^2+\omega^2}{b}. \tag{3.20}$$

Instead of parameterizing the closed loop system in terms of σ and ω it is common practice to use the *undamped natural frequency* $\omega_c = \sqrt{\sigma^2+\omega^2}$ and the *damping ratio* $\zeta = \sigma/\omega_c$. The closed loop characteristic polynomial is then

$$a_{cl}(s) = s^2 + 2\sigma s + \sigma^2 + \omega^2 = s^2 + 2\zeta\omega_c s + \omega_c^2.$$

This parameterization has the advantage that ζ, which is in the range $[-1, 1]$, determines the shape of the response and that ω_c gives the response speed.

Figure 3.3 shows the output y and the control signal u for $\zeta = 1/\sqrt{2} = 0.707$ and different values of ω_c. Proportional control gives a steady-state error which decreases with increasing controller gain k_p. With PI control the steady-state error is zero. Both the decay rate and the peak error decrease when the design parameter ω_c is increased. Larger controller gain give smaller errors and control signals that react faster to the disturbance.

With the controller parameters (3.20) the transfer function (3.16) from disturbance v to process output becomes

$$G_{yv}(s) = \frac{P(s)}{1 + P(s)C(s)} = \frac{bs}{s^2 + 2\zeta\omega_c s + \omega_c^2}.$$

For efficient attenuation of disturbances is desirable that $|G_{yv}(i\omega)|$ is small for all ω. For small values of ω we have $|G_{yv}(i\omega)| \approx b\omega/\omega_c$, for large ω we have $|G_{yv}(i\omega)| \approx b/\omega$. The largest value of $|G_{yv}(i\omega)|$ is $b/(2\zeta\omega_c$ for $\omega = \omega_c$. It thus follows that a large value of ω_c gives good load disturbance attenuation.

In summary, we find that the analysis gives a simple way to find the parameters of PI controllers for processes whose dynamics can be approximated by a first order system. The technique can be generalized to more complicated systems but the controller will be more complex. To achieve the benefits of large control gains the model must be accurate over wide frequency ranges as will be discussed next.

Unmodeled Dynamics

The analysis we have made so far indicates that there are no limits to the performance that can be achieved. Figure 3.3 shows that arbitrarily fast response can be obtained simply by making ω_c sufficiently large. In reality there are of course limitations to what can be achieved. One reason is that the controller gains increase with ω_c, the proportional gain is $k_p = (2\zeta\omega_c - a)/b$ and the integral gain is $k_i = \omega_c^2/b$. A large value of ω_c thus gives large controller gains and the control signal may be so large that actuator saturates. Another reason is that the model (3.14) is a simplification, it is only valid in a given frequency range. If the model is instead

$$P(s) = \frac{b}{(s+a)(1+sT)}, \tag{3.21}$$

where the term $1 + sT$ represents dynamics in sensors or actuators or other dynamics that was neglected when deriving (3.14), so-called *unmodeled dynamics*, the closed loop characteristic polynomial for the closed loop system becomes

$$a_{cl} = s(s+a)(1+sT) + k_p s + k_i = s^3 T + s^2(1+aT) + 2\zeta\omega_c s + \omega_c^2.$$

It follows from the Routh-Hurwitz criterion (3.8) that the closed loop system is stable if $\omega_c^2 T < 2\zeta\omega_c(1+aT)$ or if

$$\omega_c < \frac{2\zeta(1+aT)}{T}.$$

The frequency ω_c and the achievable response time are thus limited by the unmodeled dynamics represented by T. When models are developed for control it is therefore important to also consider the unmodeled dynamics.

3.3 Using Feedback to Follow Command Signals

Another major application of feedback is to make a system output follow a command signal. It is called the *servo problem*. Cruise control and steering of a car, tracking a satellite with an antenna or a star with a telescope are some examples. Other examples are high performance audio amplifiers, machine tools and industrial robots.

To illustrate command signal following we will consider the system in Figure 3.2 where the process is a first order system and the controller is a PI controller. The transfer functions of the process and the controller are

$$P(s) = \frac{b}{s+a}, \qquad C(s) = \frac{k_p s + k_i}{s}. \tag{3.22}$$

Since we will focus on following the command signal r we will neglect the load disturbance, $v = 0$. It follows from equation (3.12) that the transfer function from the command signal r to the output y is

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{bk_p s + bk_i}{s^2 + (a+bk_p)s + bk_i}. \tag{3.23}$$

Since $G_{yr}(0) = 1$ it follows that $r = y$ when r and y are constant, independent of the values of the parameters a and b, as long as the closed loop is stable. The steady state output is thus equal to the reference, a useful property of controllers with integral action.

To determine suitable values of the controller parameters k_p and k_i we proceed as in Section 3.2 by choosing controller parameters that makes the closed-loop characteristic polynomial

$$a_{cl}(s) = s^2 + (a+bk_p)s + bk_i \tag{3.24}$$

equal to $s^2 + 2\zeta\omega_c s + \omega_c^2$ with $\zeta > 0$ and $\omega > 0$. Identifying coefficients of equal powers of s in these polynomials give

$$k_p = \frac{2\zeta\omega_c - a}{b}, \qquad k_i = \frac{\omega_c^2}{b}. \tag{3.25}$$

Compare with (3.20). Notice that integral gain increases with the square of ω_c. Figure 3.4 shows the output signal y and the control signal u for different values of the design parameters ζ and ω_c. The response time decreases with increasing

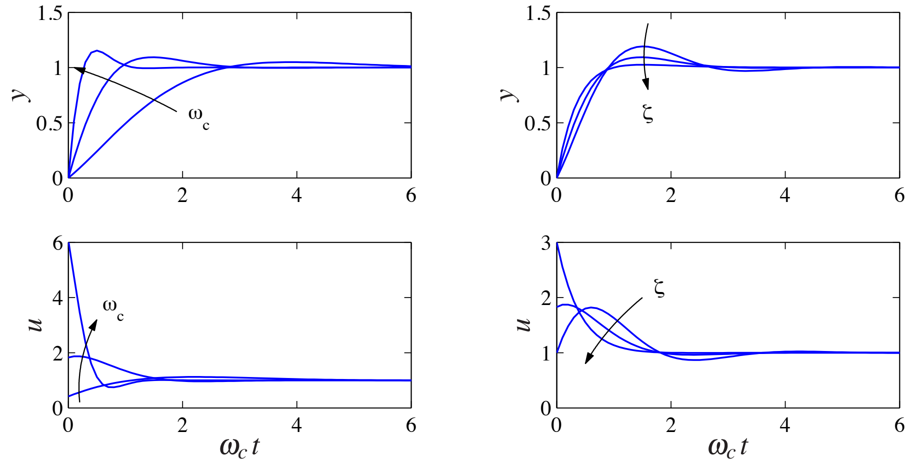


Figure 3.4: Responses to a unit step change in the command signal for different values of the design parameters ω_c and ζ. The left figure shows responses for fixed $\zeta = 0.707$ and $\omega_c = 1$, 2 and 5. The right figure shows responses for $\omega_c = 2$ and $\zeta = 0.5, 0.707$, and 1. The process parameters are $a = b = 1$. The initial value of the control signal is k_p.

ω_c and the initial value of the control signal also increases because it takes more effort to move rapidly. The overshoot decreases with increasing ζ. For $\omega_c = 2$, the design choice $\zeta = 1$ gives a short settling time and a response without overshoot.

It is desirable that the output y will track the reference r for time-varying references. This means that we would like the transfer function $G_{yr}(s)$ to be close to 1 for large frequency ranges. With the controller parameters (3.25) it follows from (3.23) that

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{(2\zeta\omega_c - a)s + \omega_c^2}{s^2 + 2\zeta\omega_c s + \omega_c^2}.$$

It is thus desirable to have a large ω_c to be able to track fast changes in the reference signal. The frequency response of G_{yr} gives a quantitative representation of the tracking abilities.

Controllers with Two Degrees of Freedom

The control law in Figure 3.2 has *error feedback* because the control signal u is generated from the error $e = r - y$. With proportional control, a step in the reference signal r gives an immediate step change in the control signal u. This rapid reaction can be advantageous, but it may give large overshoot, which can be avoided by a replacing the PI controller in equation (3.15) with

$$u(t) = k_p\big(\beta r(t) - y(t)\big) + k_i \int_0^t (r(\tau) - y(\tau))d\tau, \tag{3.26}$$

In this modified PI algorithm, the proportional action only acts on the fraction β of the reference signal. The transfer functions from reference r to u and from output

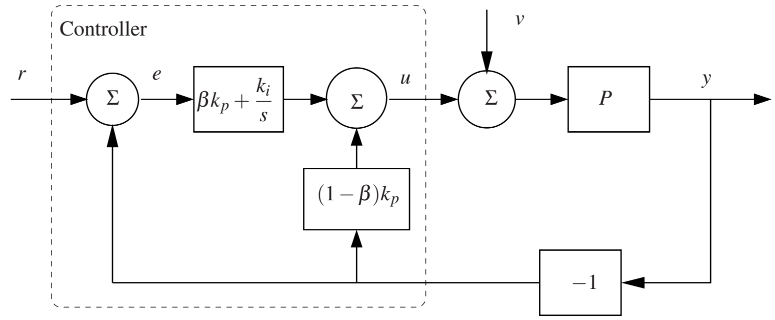


Figure 3.5: Block diagram of a closed-loop system with a PI controller having an architecture with two degrees of freedom.

y to u are

$$C_{ur}(s) = \beta k_p + \frac{k_i}{s}, \qquad C_{uy}(s) = k_p + \frac{k_i}{s} = C(s). \tag{3.27}$$

The controller (3.26) is called a controller with *two degrees of freedom* since the transfer functions $C_{ur}(s)$ and $C_{uy}(s)$ are different.

A block diagram of a closed loop system with a PI controller having two degrees of freedom is shown in Figure 3.5. Let the process transfer function be $P(s) = b/(s+a)$. The transfer functions from reference r and disturbance v to output y are

$$G_{yr}(s) = \frac{b\beta k_p s + bk_i}{s^2 + (a + bk_p)s + bk_i}, \qquad G_{yv}(s) = \frac{s}{s^2 + (a + bk_p)s + bk_i}. \tag{3.28}$$

Comparing with the corresponding transfer function for a controller with error feedback in equations (3.16) and (3.23) we find that the responses to the load disturbances are the same but the response to reference values are different.

A simulation of the closed loop system for $a = 0$ and $b = 1$ is shown in Figure 3.6. The figure shows that the parameter β has a significant effect on the responses. Comparing the system with error feedback ($\beta = 1$) with the system with smaller values of β we find that using a system with two degrees of freedom gives the same settling time with less overshoot and gentler control actions.

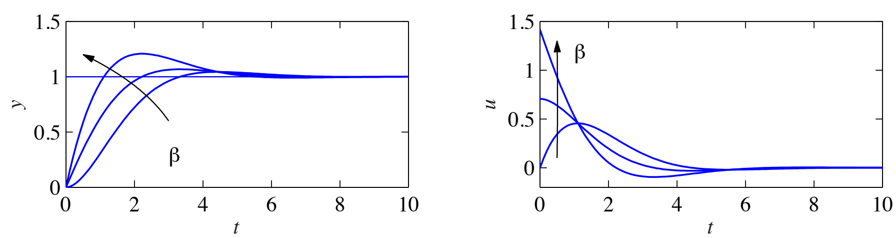


Figure 3.6: Response to a step change in the command signal for a system with a PI controller having two degrees of freedom. The process transfer function is $P(s) = 1/s$ and the controller gains are $k_p = 1.414$, $k_i = 1$ and $\beta = 0$, 0.5 and 1.

The example shows that command signal response can be improved by using a controller architecture having two degrees of freedom. In Section 3.7 we will show that the responses to command signals and disturbances can be completely separated by using a more general system architecture. To use a system with two degrees of freedom both the reference signal r and the output signal y must be measured. There are situations where only the error signal $e = r - y$ can be measured, typical examples are DVD players, optical memories and atomic force microscopes.

3.4 Using Feedback to Provide Robustness

Feedback can be used to make good systems from poor components. The development of the electric feedback amplifier for transmission of telephone signals is an early example [Ben93]. Design of amplifiers with constant linear gain was a major problem. The basic component in the amplifier was the vacuum tube, which was nonlinear and time varying. A major accomplishment was the invention of the feedback amplifier. The idea is to close a feedback loop by arranging a feedback loop around the vacuum tube, which gives a closed loop system with a linear input/output relation with constant gain.

The idea to use feedback to linearize input/output characteristics and to make it insensitive to process variations is common. The recipe is to localize the source of the variations and to close feedback loops around them. This idea is used extensively to obtain linear amplifiers and actuators, and to reduce effects of friction in mechanical systems. We will illustrate with a simple model of an electronic amplifier.

A Nonlinear Amplifier

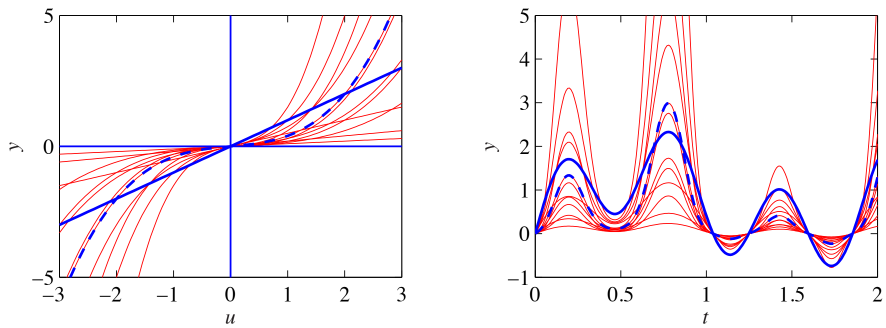
Consider an amplifier with a static, nonlinear input/output relation with considerable variability as illustrated in Figure 3.7a. The nominal input/output characteristics is shown in heavy dashed line and examples of variations in thin lines. The nonlinearity in the figure is actually

$$y = f(u) = \alpha(u + \beta u^3), \qquad -3 \le u \le 3. \tag{3.29}$$

The nominal values corresponding to the dashed line are $\alpha = 0.2$ and $\beta = 1$. The variations of the parameters α and β are in the ranges $0.1 \le \alpha \le 0.5$, $0 \le \beta \le 2$. The responses of the system to the input $u = r$ with

$$r(t) = \sin(t) + \sin(\pi t) + \sin(\pi^2 t). \tag{3.30}$$

are shown in Figure 3.7b. The desired response $y = u$ is shown in heavy full lines and responses for a range of parameters are shown in thin lines. The nominal response of the nonlinear system is shown in heavy dashed lines. It is distorted due to the nonlinearity. Notice in particular the heavy distortion both for small and large signal amplitudes. The behavior of the system is clearly not satisfactory.



2

(a) Input/output relationship (b) Output response

Figure 3.7: Responses of a static nonlinear system. The left figure shows the input/output relation of the open-loop system and the right figure shows responses to the input signal (3.30). The ideal response is shown in full thick lines. The nominal response of the nonlinear system is shown in dashed thick lines and the responses for different parameter values are shown in thin lines.

The behavior of the system can be improved significantly by introducing feedback. A block diagram of a system with a simple integral controller is shown in Figure 3.8. Figure 3.9 shows the behavior of the closed loop system with the same parameter variations as in Figure 3.7. The input/output plot in Figure 3.9a is a scatter plot of the inputs and the outputs of the feedback system. The input/output relation is practically linear and close to the desired response. There is some variability because of the dynamics introduced by the feedback. Figure 3.9b shows the responses to the reference signal, notice the dramatic improvement compared with Figure 3.7b. The tracking error is shown in Figure 3.9c.

Analysis

Analysis of the closed loop system is difficult because it is nonlinear. We can however obtain significant insight by using approximations. We first observe that the system is linear when $\beta = 0$. In other situations we will approximate the nonlinear function by a straight line around an operating point $u = u_0$. The slope of the nonlinear function at $u = u_0$ is $f'(u_0)$ and we will approximate the process with a linear system with the gain $f'(u_0)$. The transfer functions of the process and the

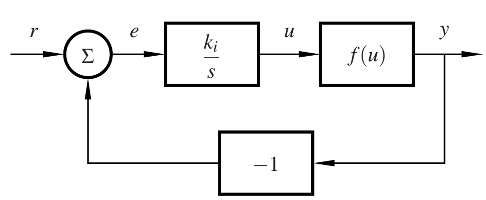
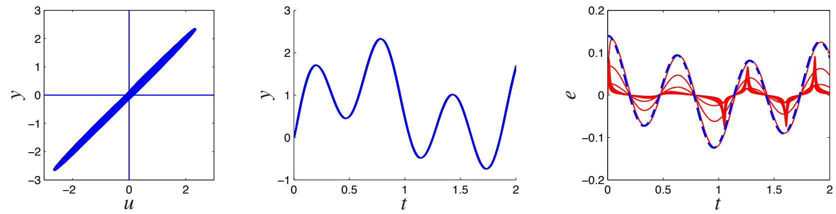


Figure 3.8: Block diagram of a nonlinear system with integral feedback.



(a) (b) (c)

Figure 3.9: Responses of the system with integral feedback ($k_i = 1000$). The left plot, is a scatter plot of inputs and outputs. The center plot shows the response of the closed loop system to the input signal r, and the right plot shows the control error. The parameter variations are the same as in Figure 3.7. Notice the dramatic improvement compared to Figure 3.7b

controller are

$$P(s) = b = f'(u_0) = \alpha(1+3\beta u_0^2), \qquad C(s) = \frac{k_i}{s}, \tag{3.31}$$

where u_0 denotes the operating condition. The process gain $b = \alpha(1+3\beta u_0^2)$ is in the range 0.1–27.5 depending on the values of α, β and u_0. It follows from equation (3.13) that the transfer functions relating the output y and the error e to the reference signal are

$$G_{yr}(s) = \frac{bk_i}{s+bk_i}, \qquad G_{er}(s) = 1 - G_{yr} = \frac{s}{s+bk_i}. \tag{3.32}$$

The closed loop system is a first-order system with the pole $s = -bk_i$ and the time constant $T_{cl} = 1/(bk_i)$. The integral gain is chosen as $k_i = 1000$. The closed loop pole ranges from 100 rad/s to 2.75×10^4 rad/s, which is fast compared to the high frequency component 9.86 rad/s of the input signal.

The error for the approximated system is described by the differential equation

$$\frac{de}{dt} = -bk_i e + \frac{dr}{dt}, \qquad \frac{dr}{dt} = \cos(t) + \pi\cos(\pi t) + \pi^2\cos(\pi^2 t). \tag{3.33}$$

The fast frequency component of the input (3.30) has the frequency $\pi^2 = 9.86$; it is slower than the process dynamics for all parameter variations. Neglecting the term de/dt in (3.33) gives

$$e \approx \frac{1}{bk_i}\frac{dr}{dt}. \tag{3.34}$$

The largest error is obtained when b has ist smallest value 0.1. The error is then approximately $\pi^2/(bk_i)\cos(\pi^2 t) \approx 0.1\cos(\pi^2 t)$ which is shown as the dashed line in Figure 3.9c.

This analysis has shown that it is possible to design an integrating controller for a system whose dynamics can be approximated by a static model. Design is essentially the choice of a single parameter: the integral gain k_i of the controller. The closed loop transfer function from reference to output is given by equation (3.32) where integral gain is $k_i = 1/(bT_{cl})$ where T_{cl} is the desired time constant of the

Figure 3.10: Schematic diagram of xxx which has both positive and negative feedback. An exhibitory connection (positive feedback) is denoted with a line ending with an arrow, an inhibitory interaction (negative feedback) iis denoted with an arrow ending with a circle.

closed loop system. The integral gain is inversely proportional to T_{cl} and the largest integral gain is limited by unmodeled dynamics.

The example illustrates that feedback can be used to design an amplifier that has practically linear input/output relation even if the basic amplifier is nonlinear with strongly varying characteristics.

3.5 Positive Feedback

Most of this book is focused on negative feedback because of its amazingly good properties, which have been illustrated in the previous sedions. In this section we will discuss positive feedback, which has complementary properites. In spite of this positive feedback has found good use in several contexts.

Systems with negative feedback can be well understood by linear analysis. To understand systems with positive feedback it is necessary to consider nonlinear effects, because without the nonlinearities the instability caused by positive feedback will grow without bounds. The nonliner elements contribute to create interesting and useful effects by limiting the signals.

Positive feedback is common, encouraging a student or a coworker when they have performed well encourages them do to even better. In biologyit is common to distinguish inhibitory connections (negative feedback) from exhibitory feedback (positive feedback) as illustrated in Figure 3.10. Neurons use a combination of positive and negative feedback to generate spikes.

Positive feedback may cause instabilities. Exponential grow is a typical example of positive feedback, where the rate of change of a quantity x is proportional to the x, hence

$$\frac{dx}{dt} = \alpha x,$$

which implies that $x(t) = e^{\alpha t}$ grows exponentially. In nature the exponential growth of a species is limited by the finite amount of food. Another common example is when a microphone is placed close to a speaker in public address systems, resulting in a howling noise. Positve feedback can create stampedes in cattle heard, run on banks, boom-bust behavior. In all these cases there is exponential growth which is finaly limited by by limiting resources.

The notions of positive and negative feedback are highly intuitive but they are also too implistic. Their interpretation is clear when we are not considering dynamics. But if dynamics is considered the sign of the feedback will depend on the frequency, and feedback can change from positive to negative when the frequency changes. A good understanding of the dynamic case can be obtained by tracing sinusoidal signals around the loop as will be done in Section 9.2. We then obtain

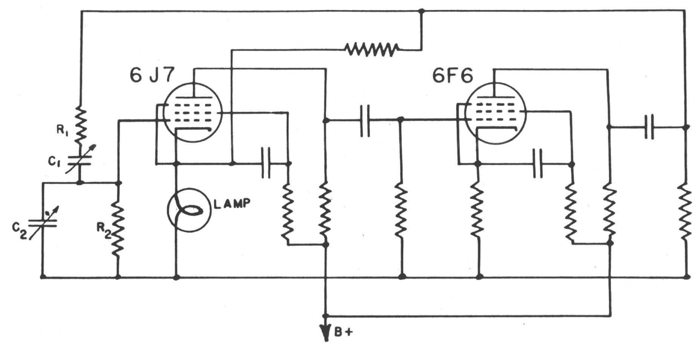


Figure 3.11: Circuit diagram of William Hewletts oscillator that gives a stable oscillation by using positve feedback and a nonlinearity (lamp) that stabilized the amplitude of the oscillation, [].

a more sophisticated notion that a block with feedback is positive if the input and output signals are in phase and feedback is negative if signals are out of phase. With the simple interpretation it appears that a system with negative feedback will always be stable and that a system with positive feedback will always be unstable. None of these statements are true when dynamics is considered.

In spite of the drawbacks mentioned above, positive feedback has several useful applications. A typical example is the generation of periodic signals as illustrated by the following example.

Example 3.1 Hewlett's Oscillator
Since positive feedback tend to generate instability it can be use to construct oscillators. To limit the exponential growth it is necessary to introduce some nonlinearity that limits the amplitude of the oscillation. An example is given in a Figure 3.11 from William Hewlett's 1939 PhD thesis at Stanford University. Hewlett used two vacuum tubes with positive feedback and a nonlinear element in the form of a lamp to maintain constant amplitude of the oscillation. The positive feedback in the basic loop creates an oscillation. The resistance of the lamp decreases as the signal amplitude increases and the amplitude is limited. Hewletts oscillator was the beginning of a very successful company HP, that Hewlett founded with David Packard. ∇

Example 3.2 The Superregenerative Receiver
In the previous sections we have shown that negative feedback has some very useful properties. The negative feedback amplifier was an enabler for long-distance telephony. A key idea was to design an amplifier with a very large open loop gain and to reduce the gain by negative feedback. The result was an amplifier which is robust and linear. Positive feedback has the complementary properties, it is possible to create high gain but the closed loop system is sensitive to parameter variations.

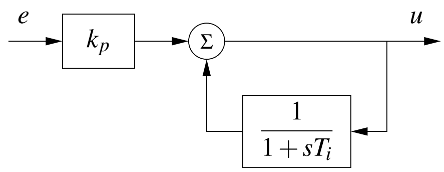


Figure 3.12: Implementation of integral action by *positive feedback* .

To undertstand that positive feedback can generate high gains we consider an amplifer with gain A_{ol}. Neglecting dynamics and closing a feedback loop around the amplifier with positive feedback k gives a closed loop system with the closed loop gain

$$A_{cl} = \frac{A_{ol}}{1 - kA_{ol}}.$$

A very large closed loop gain A_{cl} can be obtained by selecting a feedback gain k which is just below the stability limit $1/A_{ol}$. Choosing the gain so that $kA_{ol} = 0.999$ gives $A_{cl} = 1000A_{ol}$. Using this idea Armstrong constructed a superregenerative radio receiver in 1914 when he was still undergraduate at Columbia University. He built a radio receiver with only one vaccum tube. The drawback by using positive feedback is that the system is highly sensitive and that the gain has to be adjusted carefully to avoid oscillations. It is still used in simple walkie-takies, garage door openers and toys. ∇

Example 3.3 Implementation of Integral Action
Positive feedback was used very in early controllers where integral action was provided by positive feedback around a system with first order dynamics as shown in the block diagram of Figure 3.12. Intuitively the system can be explained as follows. Proportional feedback typically gives a steady state error. The bias is estimated by low pass filtering and it is added to the control signal. The circuit can be understood better by a little analysis. Using block diagram algebra we find that the transfer function of the system is

$$G_{ue} = \frac{k_p}{1 - \dfrac{1}{1+sT}} = k_p + \frac{k_p}{sT},$$

which is a transfer function of a PI controller. This way of implementing integral action is still used in many systems. Notice that in this case the closed loop system is in fact unstable since it has an integrator. Since this is the desired behavior (integral action) it is not necessary to limit the signal. ∇

Positive Feedback combined with Saturation

Positive feedback is often combined with nonlinear elements as in Hewlett's oscillator. To illustrate the combination of positive feedback and nonlinerities we will consider a simple system. Figure 3.13 shows the block diagram of a system with

Figure 3.13: Block diagram of a system with positive feedback.

positive feedback and a nonlinearity in the form of a saturation. The system has a forward path with a gain followed by a block with first order dynamics and a nonlinearity with saturation characteristics. The nonlinearity is introduced to capture the fact that real systems always have some limitation. We assume that

$$y = f(x) = \frac{x}{1-|x|}, \qquad x = f^{-}(y) = \frac{y}{1-|y|} \tag{3.35}$$

The system is described by the equations

$$\frac{dx}{dt} = -x + u = -x + k(r+y) = k(r - G(y)), \qquad G(y) = \frac{y}{k(1-|y|)} - y. \tag{3.36}$$

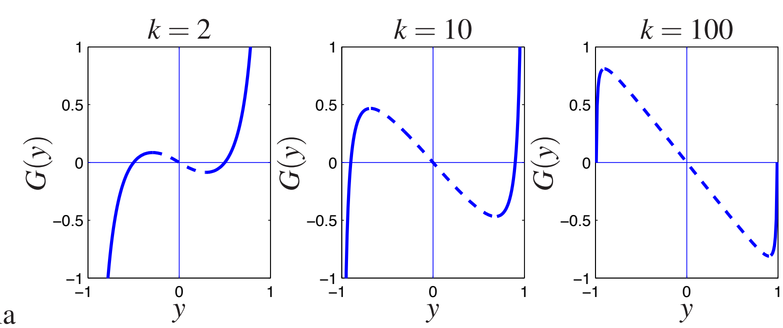
The equilibria for a constant input r are given by

$$r = -y + \frac{1}{k} f^{-}(y) = -y + \frac{y}{k(1-|y|)} = G(y). \tag{3.37}$$

We have

$$G'(y) = -1 + \frac{1}{k(1-|y|)^2} \tag{3.38}$$

Figure 3.14 shows the function G for different values of k. The function G is monotone if $k \leq 1$. If $k > 1$ it has a maximum $r_1 = \frac{2}{\sqrt{k}} - 2$ at $y = -1 + 1/\sqrt{k}$ and a minimum $-r_1$ at $y = -1 + 1/\sqrt{k}$. There is one equilibrium if $k \leq 1$, or if $k > 1$ and $|r| > r_1$, two equilibria if $k > 1$ and $|r| = r_1$ and three equilibria if $k > 1$ and $|r| < r_1$. Since the system is of first order it is easy to understant its behavior between the equilibria. It follows from (3.35) that x is a monotone function of y. Consider the situation when the input r is a constant. It follows from equation (3.36) that dx/dt will be negative to the right of an equilibrium where G' is negative, and positive to the left of the equilibrium. The solution will therefore move towards the equilibria and we find that the equilibria where the slope of G is positive are stable and those where the slope of G are negative are unstable. The unstable equilibria are



hopla

Figure 3.14: Graphs of the function $G(y)$ for different values of the gain k.

Figure 3.15: The left figure shows the static input-output map of the system and the righr figure shows time response of the system to a timevarying input.

indicated in dashed lines in Figure 3.14.

We now have a good insight into the behavior of the system. Consider the case $k > 1$, say $k = 100$ which corresponds the rightmost plot in Figure 3.14. It follows from Figure 3.14 that the output y is close to -1 for large negagive values of r. The output increases as the input increases and for and the output flips to a value close to $y = 1$ when the input reaches the value r=1. The output y then continues to increase with increasing r. For large values of k the system thus acts like a saturation with hysteresis. The output switches between the values +1 and -1 and the hysteresis width is 2. The input output behavior of the system is shown in Figure 3.15.

The circuit shown in the figure is commonly used as a trigger to detect changes in a signal (a Smith trigger). It is also used as a memory element in solid state memories.

3.6 Using Feedback to Shape Behavior

The regulation and servo problems discussed in Sections 3.2 and 3.3 are classical applications of feedback. In Section 3.4 it was shown that feedback can be used to obtain essentially linear input/output behavior for a nonlinear system with strong variability. In this section we will show how feedback can be used to shape the dynamic behavior of a system.

Collision avoidance is a useful behavior of moving robots. Feedback is used in automobiles to create behaviors that avoid locking brakes, skids and collision with pedestrians. Feedback is used to make the dynamic behavior of airplanes invariant to operating conditions. Feedback is also an essential element of human balancing and locomotion.

Bacteria use simple feedback mechanisms to search for areas where there is high concentration of food or light. The principle is to sense a variable and to make exploratory moves to see if the concentration increases. A similar mechanism can be used to avoid harmful substances. Optimization is also used in computer systems to maintain high throughput of servers.

Stabilization

Stabilizing an unstable system is a typical example of how feedback can be used to change behavior. Many systems are naturally unstable. The ability to stand up-

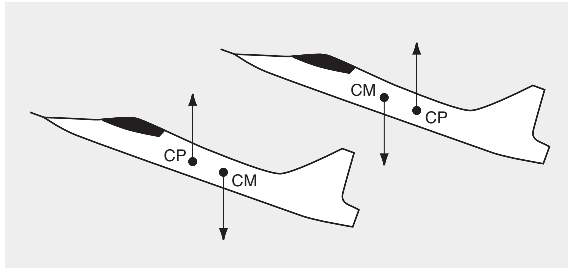


Figure 3.16: Schematic digram of two aircrafts. The aircraft at the top is stable because it has the center of pressure CP behind the center of mass CM. The aircraft at the bottom is unstable because the positions of center of mass and center of pressure are reversed.

right, walk and run has given humans many advantages but it requires stabilization. Stability and maneuverability are conflicting goals in vehicle design. The ship designer Minorsky realized that there was a trade-off between maneuverable and stability and he emphasized that a stable ship is difficult to steer. The Wright Flyer, which was maneuverable but unstable, inspired Sperry to design an autopilot. Feedback has been used extensively in aircraft, from simple systems for stability augmentation to systems that provide full autonomy.

Military airplanes gain significant competitive advantage by being made unstable. Schematic pictures of two airplanes are shown in Figure 3.16. The positions of the center of mass CM and the center of pressure CP are key elements. To be stable the center of pressure must be behind of the center of mass. The center of pressure of an aircraft shifts backwards when a plane goes supersonic. If the plane is stable at subsonic speeds it becomes even more stable at supersonic speeds because of the long distance between CM and CP. Large forces and large control surfaces are then required to maneuver the airplane and the plane will be more sluggish. A more balanced design is obtained by placing the center of pressure in front of the center of mass at subsonic speeds. Such an airplane will have superior performance, but it is unstable at subsonic speeds, typically at takeoff and landing. The control system that stabilizes the aircraft in these operating conditions is mission critical with strong requirements on robustness and reliability.

The evolutionary biologist John Maynard Smith [**?**] has claimed that while early flying animals were inherently stable they later developed unstable configurations when their sensory and nervous systems became more sophisticated and able to stabilize. The unstable configuration had significant advantages in manoevrability both for predator and prey.

Keeping an inverted pendulum in the upright position is a prototype example of stabilization. Consider the cart–pendulum discussed in Examples 2.1 and 2.2. Neglecting damping, assuming that the cart is much heavier than the pendulum and assuming that the tilt angle θ is small, equation (2.10) can be approximated by the differential equation

$$J_t\ddot{\theta} - mgl\theta = u. \tag{3.39}$$

The transfer function of the open loop system is

$$G_{\theta u} = \frac{1}{J_t s^2 - mgl}, \qquad a_{cl}(s) = J_t s^2 - mgl.$$

The system is unstable because it has a pole $s = \sqrt{mgl/J_t} = \omega_0$ in the right half plane. It can be stabilized with a *proportional-derivative (PD) controller* that has the transfer function

$$C(s) = k_d s + k_p. \tag{3.40}$$

The closed-loop characteristic polynomial is

$$a_{cl}(s) = J_t s^2 + k_d s + (k_p - mgl),$$

and all of its roots are in the left half plane if $k_p > mgl$.

One way to find controller parameters is to choose the controller gains so that the characteristic polynomial has natural frequency ω_c and damping ratio ζ, hence

$$k_d = 2\zeta\omega_c J_t, \qquad k_p = J_t\omega_c^2 + mgl.$$

Choosing $\omega_c = \omega_0$ moves the poles from $\pm\omega_0$ in open loop to $-\zeta\omega_0 \pm i\sqrt{1-\zeta^2}\omega_0$ in closed loop. The controller gains are then $k_p = 2mgl$ and $k_d = 2\zeta\sqrt{mglJ_t}$. The control law (3.40) stabilizes the pendulum but is does not stabilize the motion of the cart. To do this it is necessary to introduce feedback from cart position and cart velocity.

The Segway

The Segway discussed in Example 2.1 is essentially a pendulum on a cart and can be modeled by equation (2.9) with an added torque τ on the pendulum that is exerted by the person leaning on the platform. Neglecting the damping terms $c\dot{p}$, $\gamma\dot{\theta}$ and linearizing (2.9) gives

$$M_t\ddot{p} - ml\ddot{\theta} = u, \qquad -ml\ddot{p} + J_t\ddot{\theta} - mgl\theta = -\tau,$$

where u is the horizontal force generated by the motor and τ is the torque generated by the lean of the rider. Since the Segway is similar to the inverted pendulum on a cart, we will explore if the feedback (3.40) can be used to stabilize the system. The closed loop system is then described by

$$M_t\ddot{p} - ml\ddot{\theta} = -k_d\dot{\theta} - k_p\theta, \qquad -ml\ddot{p} + J_t\ddot{\theta} - mgl\theta = -\tau. \tag{3.41}$$

Multiply the first equation by ml, the second by M_t and add the equations, hence

$$(M_tJ_t - m^2l^2)\ddot{\theta} + mlk_d\dot{\theta} + ml(k_p - M_tg)\theta = -M_t\tau.$$

Since $M_tJ_t - m^2l^2 = M_tJ + nMl^2 > 0$, this differential equation is stable if $k_p > M_tg$. Solving for $\ddot{p}$ in (3.41) gives the following transfer functions from torque τ

Figure 3.17: Haptic devices, the left figure shows the PHANTOMTM and the right a system is developed by Quanser.

to tilt angle θ and horizontal acceleration $\ddot{p}$.

$$G_{\ddot{p}\tau}(s) = \frac{m^2l^2s^2 - mlk_ds - mlk_p}{(M_tJ_t - m^2l^2)s^2 + mlk_ds + ml(k_p - M_tg)},$$
$$G_{\theta\tau}(s) = \frac{M_t}{(M_tJ_t - m^2l^2)s^2 + mlk_ds + ml(k_p - M_tg)}.$$

The feedback (3.40), which stabilizes the Segway, thus creates a behavior where the horizontal acceleration $\ddot{p}$ of the Segway can be controlled by the torque τ, which can be generated by leaning on the Segway, see Figure 2.6c.

Impedance Control and Haptics

Changing behavior of a mechanical system is common in robotics and haptics. Position control is not sufficient when industrial robots are used for grinding, polishing and assembly. The robot can be brought into proximity with the workspace by position control but to carry out the operations it is desirable to shape how the force depends on the distance between the tool and the workspace. A spring-like behavior is an example. The general problem is to create a behavior specified by a given differential equation between force and motion, a procedure called *impedance control*. Similar situations occur in teleoperation in hazardous environment or in telesurgery. In this situation the workpiece is operated remotely using a joystick. It is useful for the operator to have some indication of the forces between the tool and the workpiece. This can be accomplished by generating a force on the operators joystick that mimics the force on the workpiece.

Figure 3.17 shows two haptic input devices. The systems are pen-like with levers or gimbals containing angle sensors and force actuation. By sensing position and orientation, and generating a force depending on position and velocity, it is possible to create a behavior that simulates touching real or virtual objects. Forces that simulate friction and surface structure can also be generated.

We illustrate the principle with a joystick having a low friction joint. Let J be the moment of inertia, and let the actuation torque and the external torque from the

Table 3.1: Properties of feedback and feedforward

Feedback	Feedforward
Closed loop	Open loop
Acts on deviations	Acts on plans
Robust to model uncertainty	Sensitive to model uncertainty
Risk for instability	No risk for instability
Sensitive to measurement noise	Insensitive to measurement noise

operator be T_a and T, respectively. The equation of motion is

$$J\frac{d^2\theta}{dt^2} = T + T_a.$$

By measuring the angle θ and its first two derivatives we can create the feedback

$$T_a = k_p(\theta_r - \theta) - k_d\frac{d\theta}{dt} - k_a\frac{d^2\theta}{dt^2}.$$

The closed loop system is then

$$(J + k_a)\frac{d^2\theta}{dt^2} + k_d\frac{d\theta}{dt} + k_p(\theta - \theta_r) = T.$$

The feedback has thus provided virtual inertia k_a, virtual damping k_d and virtual spring action k_p. If no torque T is applied the joystick will assume the orientation given by the reference signal θ_r. If a the user applies a torque the joystick will behave like a damped spring-mass system.

3.7 Feedback and Feedforward

Feedback and feedforward have complementary properties. Feedback only acts when there are deviations between the actual and the desired behavior, feedforward acts on planned behavior. Some of the properties are summarized in Table 3.1. In economics feedback represents a market economy and feedforward a plan economy. Feedback and feedforward can be combined to improve response to command signals and to reduce the effect of disturbances that can be measured.

Feedforward and System Inversion

To explore feedforward control we will first investigate command signal following. Consider the system modeled by the differential equation (3.1):

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_1\frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_n u.$$

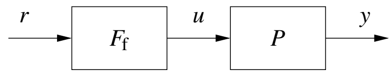


Figure 3.18: Block diagram of a process with pure feedforward compensation. The process transfer function is $P(s)$, the feedforward compensator has the transfer function $F_f(s) = P^{-1}(s)F_m(s)$ and the combined system has transfer function $F_m(s)$.

Assume that we want to find a control signal u that gives the response y_r. It follows from (3.1) that the desired control signal satisfies

$$b_1 \frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_n u = \frac{d^n y_r}{dt^n} + a_1 \frac{d^{n-1}y_r}{dt^{n-1}} + \ldots + a_n y_r, \tag{3.42}$$

This equation is called the *inverse* of equation (3.1) because it is obtained by exchanging inputs and outputs. If the transfer function of the original system is $P(s)$, the transfer function of the inverse system is simply $P^{-1}(s)$.

Consider a system with the transfer function $P(s)$, and assume that we want to find a feedforward controller so that the response to command signals is given by the transfer function $F_m(s)$, as shown in Figure 3.18. The feedforward compensator is then

$$F_f(s) = P^{-1}(s)F_m(s) \tag{3.43}$$

because $P(s)F_f(s) = F_m(s)$. Design of a feedforward compensator is thus closely related to system inversion.

There are problems with system inversion since the inverse may require differentiations and it may be unstable. If $b_1 \neq 0$ we have $P^{-1}(s) \approx s/b_1$ for large s, which implies that to obtain a bounded control signal we must require that the reference signal has a smooth first derivative. If $b_1 = 0$ we must similarly require that the reference signal has a smooth second derivative.

Difficulties with Feedforward Compensation

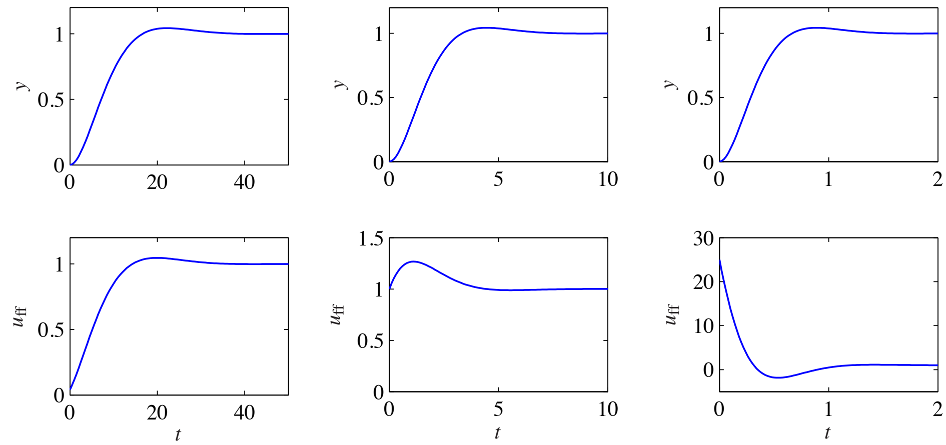
Let the process and the desired response have the transfer functions

$$P(s) = \frac{1}{(s+1)^2}, \qquad F_m(s) = \frac{\omega_c^2}{s^2 + 2\zeta\omega_c s + \omega_c^2}.$$

The feedforward transfer function is then given by equation (3.43), hence

$$F_f(s) = P^{-1}(s)F_m(s) = \frac{\omega_c^2 (s+1)^2}{s^2 + 2\zeta\omega_c s + \omega_c^2}. \tag{3.44}$$

Notice that $F_f(o) = 1$ and $F_f(\infty) = \omega_c^2$, the initial value of the control signal for a unit step command is thus ω_c^2 and the final value is 1. Figure 3.19 shows the outputs y and the feedforward signals u_{ff} for a unit step reference signal r and different values of ω_c. The parameter ω_c determines the response speed, since $F_f(\infty) = \omega_c^2$ very large control signals are required fast responses, Achievable performance is thus limited by the size of admissible control signals.



(a) $\omega_c = 0.2$ (b) $\omega_c = 1$ (c) $\omega_c = 5$

Figure 3.19: Outputs y (top plots) and feedforward signals u_{ff} (lower plots) for a unit step command signal. The values of the design parameter is $\omega_c = 0.2$ (left) 1 (center) and 5 right. The outputs are identical apart from the time scale, but the control signals required to generate the output differs significantly. The largest value of the feedforward signal u_{ff} is ω_c^2, and it increases significantly with increasing ω_c.

Another difficulty with feedforward is that the inverse process dynamics may be unstable. To have a bounded feedforward signal it follows from equation (3.43) that the desired transfer function F_{m} must have the same right half-plane zero as the process transfer function P. Right-half plane process zeros thus limit what can be achieved with feedforward.

Let the process and the desired response be characterized by the transfer functions

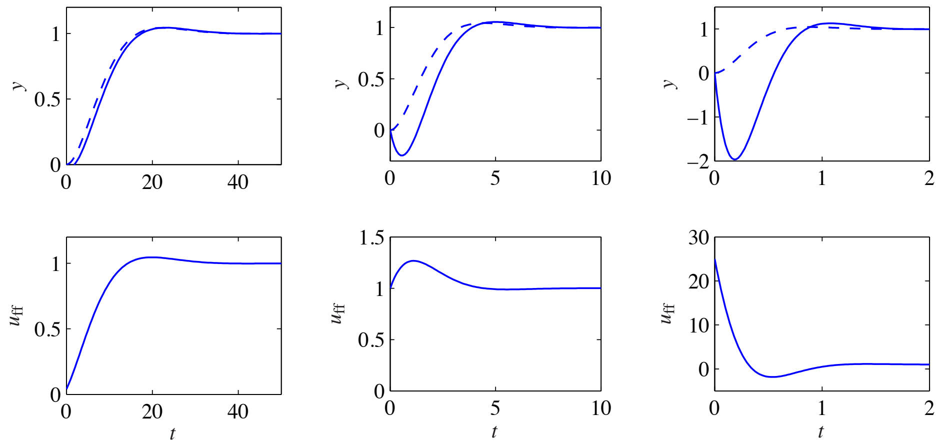
$$P(s) = \frac{1-s}{(s+1)^2}, \qquad F_{\mathrm{m}}(s) = \frac{\omega_c^2(1-s)}{s^2+2\zeta\omega_c s+\omega_c^2}.$$

Since the process has a right half plane zero at $s = 1$ the inverse model is unstable and it follows from equation (3.43) that we must require that the transfer function of the desired response has the same zero. Equation (3.43) gives the feedforward transfer function

$$F_{\mathrm{f}}(s) = \frac{\omega_c^2(s+1)^2}{s^2+2\zeta\omega_c s+\omega_c^2}, \tag{3.45}$$

compare with (3.44). Figure 3.20 shows the outputs y and the feedforward signals u_{ff} for different values of ω_c. The response to the command signal goes in the wrong direction initially because of the right half plane zero at $s = 1$. This effect, called *inverse response*, is barely noticeable if the response is slow ($\omega_c = 1$) but increases with increasing response speed. For $\omega_c = 5$ the undershoot is more than 200%. The right half plane zero thus severely limits the response time.

The behavior of the control signal changes qualitatively with ω_c. To understand what happens we note that the zero frequency gain of the feedforward transfer



(a) $\omega_c = 0.2$ (b) $\omega_c = 1$ (c) $\omega_c = 5$

Figure 3.20: Outputs y (top plots) and feedforward signals $u_{\rm ff}$ (lower plots) for a unit step command signal. The design parameter has the values $\omega_c = 0.2$ (left) 1 (center) and 0.5 (right) for a unit step command in the reference signal. The dashed curve shows the response that could be achieved if the process did not have the right half plane zero.

function (3.45) is $F_{\rm f}(0) = 1$ and that its high frequency gain is $F_{\rm f}(\infty) = \omega_c^2$. For a unit step reference signal $r = 1$, the initial value of the control signal is $u_{\rm ff}(0) = \omega_c^2$ and the final value is $u_{\rm ff}(\infty) = 1$. For $\omega_c = 0.2$ the control signal grows from 0.04 to the final value 1 with a small overshoot. For $\omega_c = 1$ the control signal starts from 1 has an overshoot and settles on the final value 1. For $\omega_c = 5$ the control signals starts at 25 and decays towards the final value 1 with an undershoot.

Sensitivity to Process Variations

The transfer function from reference r to output y of a system with pure feedforward control is

$$F_{\rm m}(s) = P(s)F_{\rm f}(s). \tag{3.46}$$

To find the sensitivity of $F_{\rm m}$ to variations in the process transfer function $P(s)$ we take logarithm of equation (3.46) and differentiate to obtain

$$\frac{dF_{\rm m}(s)}{F_{\rm m}(s)} = \frac{dP(s)}{P(s)}. \tag{3.47}$$

The relative variations in the system with feedforward are the same as those in the process and is thus sensitive to process variations.

Combining Feedforward with Feedback

Since feedback can give systems that are robust to model uncertainties it seems natural to combine feedforward with feedback. The architecture of such a controller is shown in the block diagram of Figure 3.21. The controller has three blocks repre-

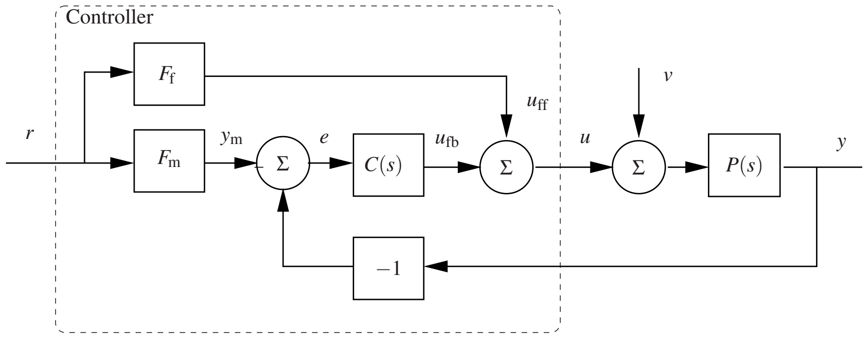


Figure 3.21: Block diagram of a closed loop system where the controller has an architecture with two degrees of freedom. The signals y_m and $u_{\rm ff}$ are generated by feedforward from the reference r. The feedback controller $C(s)$ acts on the control error $e = y_{\rm m} - y$ and generates the feedback control signal $u_{\rm fb}$.

senting the feedback transfer function $C(s)$ and the feedforward transfer functions $F_{\rm m}$ and $F_{\rm f}$.

The desired response is given by the transfer function $F_{\rm m}$. The controller architecture in Figure 3.21 is highly intuitive. The feedforward signal $u_{\rm ff}$ generates the ideal output $y = y_{\rm m}$. If there are no disturbances and no modeling errors the error is then zero and the feedback signal $u_{\rm fb}$ is also zero. All control is thus handled by the feedforward action. If there are disturbances and modeling errors, the error e will not be zero and the feedback controller $C(s)$ will make appropriate corrections.

The controller architecture in Figure 3.21 is a generalization of the controller with two degrees of freedom introduced in Section 3.3 (see Figure 3.5). A nice property is that it gives a separation of command signal following, robustness and disturbance attenuation. Command signal following is dealt with by design of the feedforward transfer functions $F_{\rm m}$ and $F_{\rm f}$. Robustness and disturbance attenuation is dealt with by design of the feedback transfer function $C(s)$.

The transfer function from r to y for the system in Figure 3.21 is

$$G_{yr} = \frac{P(F_{\rm f} + CF_{\rm m})}{1+PC} = F_{\rm m} + \frac{PF_{\rm f} - F_{\rm m}}{1+PC}. \tag{3.48}$$

The transfer function G_{yr} is equal to $F_{\rm m}$ if the feedforward transfer function $F_{\rm f}$ is chosen so that

$$F_{\rm f}(s) = P(s)^{-1} F_{\rm m}(s). \tag{3.49}$$

This condition is the same as the condition (3.43) for pure feedforward.

The transfer functions relating the output y and the feedback signal $u_{\rm fb}$ to the disturbances v are

$$G_{yv}(s) = \frac{P(s)}{1+P(s)C(s)}, \qquad G_{uv}(s) = -\frac{C(s)}{1+P(s)C(s)}. \tag{3.50}$$

These transfer functions do not depend on the feedforward transfer functions.

The controller architecture in Figure 3.21 admits a decoupling of the response

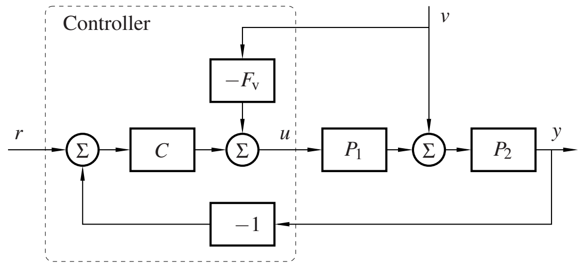


Figure 3.22: Block diagram of a system with a control architecture that combines feedback with feedforward from a disturbance v that can be measured.

to command signals and the response to disturbances. The feedback controller $C(s)$ is designed to give robustness to process variations and attenuation of load disturbances. The desired response to command signals is obtained by design of the feedforward transfer functions F_f and F_m..

To investigate the effect of process uncertainty on the response to reference signals we consider the case of small variations. Taking the logarithm of G_{yr} in equation (3.48) gives

$$\log G_{yr} = \log P + \log\left(F_\mathrm{f} + CF_\mathrm{m}\right) - \log\left(1 + PC\right).$$

Differentiating with respect to P gives the following expression for the sensitivity

$$\frac{dG_{yr}}{G_{yr}} = \frac{dP}{P} - \frac{CdP}{1+PC} = \frac{1}{1+PC}\frac{dP}{P} = S\frac{dP}{P}. \tag{3.51}$$

The relative error in the closed loop transfer function G_{yr} can thus be smaller than the relative error in the process transfer function P for frequencies where the sensitivity function S is small. Compare with the corresponding expression (3.47) for pure feedforward. It is thus useful to combine feedback and feedforward.

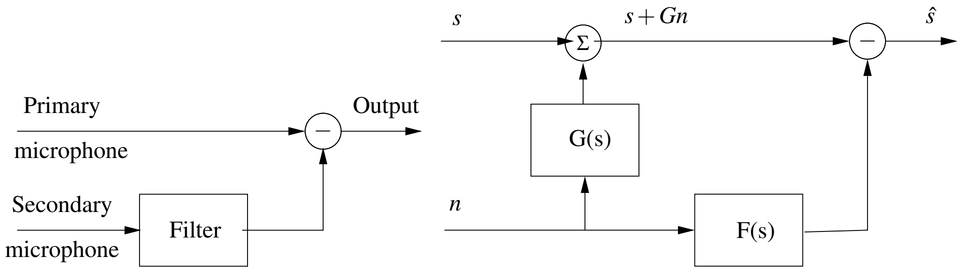
Using Feedforward to Attenuate Measured Disturbances

Feedforward can also be used to mitigate the effect of disturbances that can be measured. Such a scheme is shown in Figure 3.22. The process transfer function P is composed of two factors, $P = P_1P_2$. A measured disturbance v enters at the input of process section P_2. The measured disturbance is fed to the process input via the feedforward transfer function F_v.

The transfer function from the disturbance v to process output y is

$$G_{\mathrm{yv}}(s) = \frac{P_2(1 - P_1F_\mathrm{v})}{1+PC} = SP_2(1 - P_1F_\mathrm{v}). \tag{3.52}$$

This equation shows that there are two ways of reducing the disturbance. The transfer function $1 - P_1F_\mathrm{v}$ can be made small by a proper choice of the feedforward transfer function F_uv. In feedback compensation the effect of the disturbance is instead reduced by making the sensitivity function S small. Feedforward makes



(a) Schematic diagram (b) Block diagram

Figure 3.23: Schematic and block diagrams for noise cancellation.

the error small by subtraction. Feedback instead makes the error small by multiplication with S. An immediate consequence is that feedforward is more sensitive than feedback since we are trying to match two terms. Feedback gives better robustness but there is a risk of instability. Feedback and feedforward are therefore complementary, and we again can see that it is useful to combine them.

Feedforward is most effective when the disturbance v enters early in the process. This occurs when most of the dynamics are in process section P_2. When $P_2 = P$, and therefore $P_2 = 1$, the feedforward compensator is simply a proportional controller.

Noise cancellation is a common example of use of feedforward to cancel effects of disturbances. Consider, for example, a pilot that has to communicate in a noisy cabin. The environmental noise will seriously deteriorate the communication because the pilots microphone will pick up ambient noise. The noise can be reduced significantly by using two microphones as illustrated in Figure 3.23. The primary microphone is directed towards the pilot. It picks up the pilots voice and ambient noise. The second microphone is directed away from the pilot and it picks up the ambient noise. The effect of the noise can be reduced by filtering the signal from the secondary microphone and subtracting it from the signal from the primary microphone. A block diagram of the system is shown in Figure 3.23b. The transfer function $G(s)$ represents the dynamics of the acoustic transmission from the secondary microphone to the first microphone. The transfer function $F(s)$ is the transfer function of the filter. To cancel the effect of the noise the transfer function $F(s)$ should be close to $G(s)$. Since the noise transmission depends on the situation, for example how the pilot turns his head, it is common to let the filter be adaptive so that it can adjust, as described later in Example 4.16. Noise cancellation has many applications, in headphones, to create noise-free spaces by active noise control, or to measure electrocardiogram and heartbeat of mother and fetus.

3.8 Further Reading

The books by Bennett [Ben79, Ben93] and Mindel [Min02, Min08] give interesting perspective on the development of control. Much of the material touched upon in this chapter is classical control see [JNP47], [CM51] and [Tru55].The notion of controllers with two degrees of freedom was introduced by Horowitz [Hor63]. The analysis will be elaborated in the rest of the book. Transfer functions and other descriptions of dynamics are discussed in Chapters 5 and 8, methods to investigate stability in Chapter 9. The simple method to find parameters of controllers based on matching of coefficients of the closed loop characteristic polynomial is developed further in Chapters 6, 7 and 12. Feedforward control is discussed in Section 7.5.

Exercises

3.1 Let $y \in \mathbb{R}$ and $u \in \mathbb{R}$. Solve the differential equations

$$\frac{dy}{dt} + ay = bu, \qquad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\frac{du}{dt} + u.$$

Determine the responses to a unit step $u(t) = 1$ and the exponential signal $u(t) = e^{st}$ when the initial condition is zero. Derive the transfer functions of the systems.

3.2 Let $y_0(t)$ be the response of a system with the transfer function $G_0(s)$ to a given input. The transfer function $G(s) = (1+sT)G_0(s)$ has the same zero frequency gain but it has an additional zero at $z = -1/T$. Let $y(t)$ be the response of the system with the transfer function $G(s)$, show that

$$y(t) = y_0(t) + T\frac{dy_0}{dt}, \tag{3.53}$$

Then consider the system with the transfer function

$$G(s) = \frac{s+a}{a(s^2+2s+1)},$$

which has unit zero-frequency-gain ($G(0) = 1$). Use the result in equation (3.53) to explore the effect of the zero $s = -1/T$ on the step response of a system

3.3 Consider a closed loop system with process and a PI controller modeled by

$$\frac{dy}{dt} + ay = bu, \qquad u = k_p(r-y) + k_i \int_0^t (r(\tau) - y(\tau))d\tau,$$

where r is the reference, u the control variable and y the process output. Derive a differential equation relating the output y to the difference by direct manipulation of the equations. Draw a block diagram of the system. Derive the transfer functions of the process and the controller. Compute the transfer function from reference r to output y of the closed loop system. Make the derivations both by direct manipulation of the system equations and by polynomial algebra. Compare the results with a direct determination of the transfer functions by inspection of the block diagram.

3.4 The dynamics of the pupillary reflex is approximated by a linear system with the transfer function

$$P(s) = \frac{0.2(1-0.1s)}{(1+0.1s)^3}.$$

Assume that the nerve system that controls the pupil opening is modeled as a proportional controller with the gain k. Use Routh-Hurwitz theorem to determine the largest gain that gives a stable closed loop system.

3.5 A simple model for the relation between speed v and throttle u for a car is given by the transfer function

$$G_{vu} = \frac{b}{s+a}$$

where $b = 1\ \mathrm{m/s^2}$ and $a = 0.025\ \mathrm{rad/s}$, see Appendix A.3. The control signal is normalized to the range $0 \le u \le 1$. Design a PI controller for the system that gives a closed loop system with the characteristic polynomial

$$a_{cl}(s) = s^2 + 2\zeta\omega_c s + \omega_c^2.$$

What are the consequences of choosing different values of the design parameters ζ and ω_c. Use you judgment to find suitable values. Hint: Investigate maximum acceleration and maximum velocity for step changes in the velocity reference.

3.6 Consider the feedback system in Figure 3.2. Let the disturbance $v = 0$, $P(s) = 1$ and $C(s) = k_i/s$. Determine the transfer function G_{yr} from reference r to output y. Also determine how much G_{yr} is changed when the process gain changes by 10%.

3.7 The calculations in Section 3.2 can be interpreted as a design method for a PI controller for a first order system. A similar calculation can be made for PID control of the second order system. Let the transfer functions of the process and the controller be

$$P(s) = \frac{b}{s^2 + a_1 s + a_2}, \qquad C(s) = k_p + \frac{k_i}{s} + k_d s.$$

Show that the controller parameters

$$k_p = \frac{(1+2\alpha\zeta)\omega_c^2 - a_2}{b}, \qquad k_i = \frac{\alpha\omega_c^3}{b}, \qquad k_d = \frac{(\alpha+2\zeta)\omega_c - a_1}{b}.$$

give a closed loop system with the characteristic polynomial

$$(s^2 + 2\zeta\omega_c s + \omega_c^2)(s + \alpha\omega_c).$$

3.8 Consider an open loop system with the nonlinear input-output relation $y = f(u)$. Assume that the system is closed with the proportional controller $u = k(r - y)$. Show that the input-output relation of the closed loop system is

$$y + \frac{1}{k} f^{-1}(y) = r.$$

Estimate the largest deviation from ideal linear response $y = r$. Illustrate by plotting the input output responses for a) $f(u) = \sqrt{u}$ and b) $f(u) = u^2$ with $0 \le u \le 1$ and $k = 5, 10$ and 100.

3.9 Consider the system in Section 3.2 where the controller was designed to give a closed loop system characterized by $\omega_c = 1$ and $\zeta = 0.707$. The transfer functions of the process and the controller are

$$P(s) = \frac{2}{s+1}, \qquad C(s) = \frac{0.207s+0.5}{s}.$$

The response of the closed loop system to step command signals has a settling time (time required to stay within 2% of the final value, see Figure 5.9) of $4/\zeta\omega_c \approx 5.66$ Assume that the attenuation of the load disturbances is satisfactory but that we want a closed loop system system that responds five times faster to command signals without overshoot. Determine the transfer functions of a controller with the architecture in Figure 3.21 that gives a response to command signals with a first order dynamics. Simulate the system in the nominal case of a perfect model and explore the effects of modeling errors by simulation.

3.10 Consider a queuing system modeled by

$$\frac{dx}{dt} = \lambda - \mu_{\max}\frac{x}{x+1}.$$

The model is nonlinear and the dynamics of the system changes significantly with the queuing length, see Example 2.12. Investigate the situation when a PI controller is used for admission control. The arrival intensity λ is then given by

$$\lambda = k_p(r-x) + k_i \int^t (r(t) - x(t))dt.$$

The controller parameters are determined from the approximate model

$$\frac{dx}{dt} = \lambda.$$

Find controller parameters that give the closed loop characteristic polynomial $s^2 + 2s + 1$ for the approximate model. Investigate the behavior of the control strategy for the nonlinear model by simulation for the input $r = 5 + 4\sin(0.1t)$.