Feedback Systems

An Introduction for Scientists and Engineers

SECOND EDITION

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Chapter 6

Linear Systems

Few physical elements display truly linear characteristics. For example the relation between force on a spring and displacement of the spring is always nonlinear to some degree. The relation between current through a resistor and voltage drop across it also deviates from a straight-line relation. However, if in each case the relation is reasonably *linear, then it will be found that the system behavior will be very close to that obtained by assuming an ideal, linear physical element, and the analytical simplification is so enormous that we make linear assumptions wherever we can possibly do so in good conscience.*

Robert H. Cannon, *Dynamics of Physical Systems*, 1967 [Can03].

In Chapters 3–5 we considered the construction and analysis of differential equation models for dynamical systems. In this chapter we specialize our results to the case of linear, time-invariant input/output systems. Two central concepts are the matrix exponential and the convolution equation, through which we can completely characterize the behavior of a linear system. We also describe some properties of the input/output response and show how to approximate a nonlinear system by a linear one.

6.1 Basic Definitions

We have seen several instances of linear differential equations in the examples in the previous chapters, including the spring–mass system (damped oscillator) and the operational amplifier in the presence of small (nonsaturating) input signals. More generally, many dynamical systems can be modeled accurately by linear differential equations. Electrical circuits are one example of a broad class of systems for which linear models can be used effectively. Linear models are also broadly applicable in mechanical engineering, for example, as models of small deviations from equilibrium points in solid and fluid mechanics. Signal-processing systems, including digital filters of the sort used in MP3 players and streaming audio, are another source of good examples, although these are often best modeled in discrete time (as described in more detail in the exercises).

In many cases, we *create* systems with a linear input/output response through the use of feedback. Indeed, it was the desire for linear behavior that led Harold S. Black to the invention of the negative feedback amplifier. Almost all modern signal processing systems, whether analog or digital, use feedback to produce linear or near-linear input/output characteristics. For these systems, it is often useful to represent the input/output characteristics as linear, ignoring the internal details required to get that linear response.

For other systems, nonlinearities cannot be ignored, especially if one cares about the global behavior of the system. The predator–prey problem is one example of this: to capture the oscillatory behavior of the interdependent populations we must include the nonlinear coupling terms. Other examples include switching behavior and generating periodic motion for locomotion. However, if we care about what happens near an equilibrium point, it often suffices to approximate the nonlinear dynamics by their local linearization, as we already explored briefly in Section 5.3. The linearization is essentially an approximation of the nonlinear dynamics around the desired operating point.

Linearity

We now proceed to define linearity of input/output systems more formally. Consider a state space system of the form

$$\frac{dx}{dt} = f(x,u), \qquad y = h(x,u), \tag{6.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$. As in the previous chapters, we will usually restrict ourselves to the single-input, single-output case by taking $p = q = 1$. We also assume that all functions are smooth and that for a reasonable class of inputs (e.g., piecewise continuous functions of time) the solutions of equation (6.1) exist for all time.

It will be convenient to assume that the origin $x = 0$, $u = 0$ is an equilibrium point for this system ($\dot{x} = 0$) and that $h(0,0) = 0$. Indeed, we can do so without loss of generality. To see this, suppose that $(x_\mathrm{e}, u_\mathrm{e}) \neq (0,0)$ is an equilibrium point of the system with output $y_\mathrm{e} = h(x_\mathrm{e}, u_\mathrm{e})$. Then we can define a new set of states, inputs, and outputs,

$$\tilde{x} = x - x_\mathrm{e}, \qquad \tilde{u} = u - u_\mathrm{e}, \qquad \tilde{y} = y - y_\mathrm{e},$$

and rewrite the equations of motion in terms of these variables:

$$\begin{aligned}\frac{d}{dt}\tilde{x} &= f(\tilde{x} + x_\mathrm{e}, \tilde{u} + u_\mathrm{e}) =: \tilde{f}(\tilde{x}, \tilde{u}),\\ \tilde{y} &= h(\tilde{x} + x_\mathrm{e}, \tilde{u} + u_\mathrm{e}) - y_\mathrm{e} =: \tilde{h}(\tilde{x}, \tilde{u}).\end{aligned}$$

In the new set of variables, the origin is an equilibrium point with output 0, and hence we can carry out our analysis in this set of variables. Once we have obtained our answers in this new set of variables, we simply "translate" them back to the original coordinates using $x = \tilde{x} + x_\mathrm{e}$, $u = \tilde{u} + u_\mathrm{e}$, and $y = \tilde{y} + y_\mathrm{e}$.

Returning to the original equations (6.1), now assuming without loss of generality that the origin is the equilibrium point of interest, we write the output $y(t)$

corresponding to the initial condition $x(0) = x_0$ and input $u(t)$ as $y(t; x_0, u)$. Using this notation, a system is said to be a *linear input/output system* if the following conditions are satisfied:

$$\begin{aligned} &\text{(i)} \quad y(t; \alpha x_1 + \beta x_2, 0) = \alpha y(t; x_1, 0) + \beta y(t; x_2, 0),\\ &\text{(ii)} \quad y(t; \alpha x_0, \delta u) = \alpha y(t; x_0, 0) + \delta y(t; 0, u),\\ &\text{(iii)} \quad y(t; 0, \delta u_1 + \gamma u_2) = \delta y(t; 0, u_1) + \gamma y(t; 0, u_2). \end{aligned} \tag{6.2}$$

Thus, we define a system to be linear if the outputs are jointly linear in the initial condition response ($u = 0$) and the forced response ($x(0) = 0$). Property (iii) is a statement of the *principle of superposition*: the response of a linear system to the sum of two inputs u_1 and u_2 is the sum of the outputs y_1 and y_2 corresponding to the individual inputs.

The general form of a linear state space system is

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{6.3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{q\times p}$. In the special case of a single-input, single-output system, B is a column vector, C is a row vector, and D is scalar. Equation (6.3) is a system of linear first-order differential equations with input u, state x, and output y. It is easy to show that given solutions $x_1(t)$ and $x_2(t)$ for this set of equations, the corresponding outputs satisfy the linearity conditions (6.2).

We define $x_\mathrm{h}(t)$ to be the solution with zero input (the general solution to the *homogeneous system*),

$$\frac{dx_\mathrm{h}}{dt} = Ax_\mathrm{h}, \qquad x_\mathrm{h}(0) = x_0,$$

and the solution $x_\mathrm{p}(t)$ to be the input dependent solution with zero initial condition (the *particular solution* or *forced solution*),

$$\frac{dx_\mathrm{p}}{dt} = Ax_\mathrm{p} + Bu, \qquad x_\mathrm{p}(0) = 0.$$

Figure 6.1 illustrates how these two individual solutions can be superimposed to form the complete solution.

It is also possible to show that if a dynamical system with a finite number of states is input/output linear in the sense we have described, it can always be represented by a state space equation of the form (6.3) through an appropriate choice of state variables. In Section 6.2 we will give an explicit solution of equation (6.3), but we illustrate the basic form through a simple example.

Example 6.1 Linearity of solutions for a scalar system
Consider the first-order differential equation

$$\frac{dx}{dt} = ax + u, \qquad y = x,$$

with $x(0) = x_0$. Let $u_1 = A \sin \omega_1 t$ and $u_2 = B \cos \omega_2 t$. The solution to the homogeneous system is $x_\mathrm{h}(t) = e^{at} x_0$, and two particular solutions with $x(0) = 0$

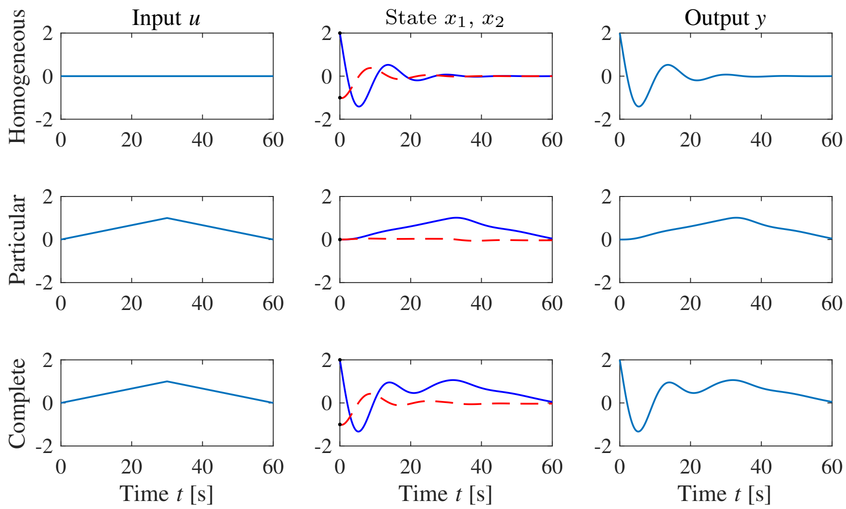


Figure 6.1: Superposition of homogeneous and particular solutions. The first row shows the input, state, and output corresponding to the initial condition response. The second row shows the same variables corresponding to zero initial condition but nonzero input. The third row is the complete solution, which is the sum of the two individual solutions.

are

$$x_{\mathrm{p}1}(t) = -A\frac{-\omega_1 e^{at} + \omega_1 \cos\omega_1 t + a\sin\omega_1 t}{a^2+\omega_1^2},$$
$$x_{\mathrm{p}2}(t) = B\frac{ae^{at} - a\cos\omega_2 t + \omega_2\sin\omega_2 t}{a^2+\omega_2^2}.$$

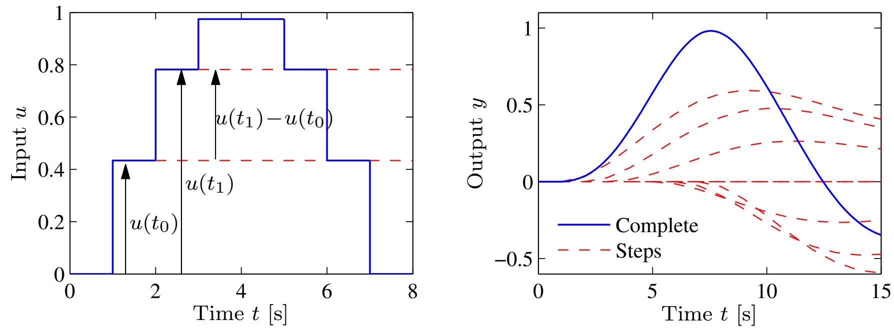
Suppose that we now choose $x(0) = \alpha x_0$ and $u = u_1 + u_2$. Then the resulting solution is the weighted sum of the individual solutions:

$$\begin{aligned} x(t) = e^{at}\left(\alpha x_0 + \frac{A\omega_1}{a^2+\omega_1^2} + \frac{Ba}{a^2+\omega_2^2}\right) \\ - A\frac{\omega_1\cos\omega_1 t + a\sin\omega_1 t}{a^2+\omega_1^2} + B\frac{-a\cos\omega_2 t + \omega_2\sin\omega_2 t}{a^2+\omega_2^2}. \end{aligned} \tag{6.4}$$

To see this, substitute equation (6.4) into the differential equation. Thus, the properties of a linear system are satisfied. ∇

Time Invariance

Time invariance is an important concept that is used to describe a system whose properties do not change with time. More precisely, for a time-invariant system if the input $u(t)$ gives output $y(t)$, then if we shift the time at which the input is applied by a constant amount a, $u(t+a)$ gives the output $y(t+a)$. Systems that are linear and time-invariant, often called *LTI systems*, have the interesting property that their response to an arbitrary input is completely characterized by their response to step inputs or their response to short “impulses.”



(a) Piecewise constant input

(b) Output response

Figure 6.2: Response to piecewise constant inputs. A piecewise constant signal can be represented as a sum of step signals (a), and the resulting output is the sum of the individual outputs (b).

To explore the consequences of time invariance, we first compute the response to a piecewise constant input. Assume that the system has zero initial condition and consider the piecewise constant input shown in Figure 6.2a. The input has jumps at times t_k, and its values after the jumps are $u(t_k)$. The input can be viewed as a combination of steps: the first step at time t_0 has amplitude $u(t_0)$, the second step at time t_1 has amplitude $u(t_1) - u(t_0)$, etc.

Assuming that the system is initially at an equilibrium point (so that the initial condition response is zero), the response to the input can be obtained by superimposing the responses to a combination of step inputs. Let $H(t)$ be the response to a unit step applied at time 0, and assume that $H(0) = 0$. The response to the first step is then $H(t-t_0)u(t_0)$, the response to the second step is $H(t-t_1)\big(u(t_1)-u(t_0)\big)$, and we find that the complete response is given by

$$\begin{aligned}
y(t) &= H(t-t_0)u(t_0) + H(t-t_1)\big(u(t_1) - u(t_0)\big) + \cdots \\
&= \big(H(t-t_0) - H(t-t_1)\big)u(t_0) + \big(H(t-t_1) - H(t-t_2)\big)u(t_1) + \cdots \\
&= \sum_{k=1}^{n} \big(H(t-t_{k-1}) - H(t-t_k)\big)u(t_{k-1}) + H(t-t_n)u(t_n) \\
&= \sum_{k=1}^{n} \frac{H(t-t_{k-1}) - H(t-t_k)}{t_k - t_{k-1}} u(t_{k-1})\big(t_k - t_{k-1}\big) + H(t-t_n)u(t_n),
\end{aligned}$$

where n is such that $t_n \le t$. An example of this computation is shown in Figure 6.2b.

The response to a continuous input signal is obtained by taking the limit $n \to \infty$ in such a way that $t_k - t_{k-1} \to 0$ and $t_n \to t$, which gives

$$y(t) = \int_0^t H'(t-\tau)u(\tau)d\tau, \tag{6.5}$$

where H' is the derivative of the step response, also called the *impulse response*. The response of a linear time-invariant system to any input can thus be computed from the step response. Notice that the output depends only on the input since we assumed the system was initially at rest, $x(0) = 0$. We will derive equation (6.5) in a slightly different way in Section 6.3.

6.2 The Matrix Exponential

Equation (6.5) shows that the output of a linear system with zero initial state can be written as an integral over the inputs $u(t)$. In this section and the next we derive a more general version of this formula, which includes nonzero initial conditions. We begin by exploring the initial condition response using the matrix exponential.

Initial Condition Response

We will now explicitly show that the output of a linear system depends linearly on the input and the initial conditions. We begin by considering the general solution to the homogeneous system corresponding to the dynamics

$$\frac{dx}{dt} = Ax. \tag{6.6}$$

For the *scalar* differential equation

$$\frac{dx}{dt} = ax, \qquad x \in \mathbb{R},\, a \in \mathbb{R},$$

the solution is given by the exponential

$$x(t) = e^{at}x(0).$$

We wish to generalize this to the vector case, where A becomes a matrix. We define the *matrix exponential* as the infinite series

$$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k, \tag{6.7}$$

where $X \in \mathbb{R}^{n\times n}$ is a square matrix and I is the $n \times n$ identity matrix. We make use of the notation

$$X^0 = I, \qquad X^2 = XX, \qquad X^n = X^{n-1}X,$$

which defines what we mean by the "power" of a matrix. Equation (6.7) is easy to remember since it is just the Taylor series for the scalar exponential, applied to the matrix X. It can be shown that the series in equation (6.7) converges for any matrix $X \in \mathbb{R}^{n\times n}$ in the same way that the normal exponential is defined for any scalar $a \in \mathbb{R}$.

Replacing X in equation (6.7) by At, where $t \in \mathbb{R}$, we find that

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^kt^k,$$

and differentiating this expression with respect to t gives

$$\frac{d}{dt}e^{At} = A + A^2 t + \frac{1}{2}A^3 t^2 + \cdots = A\sum_{k=0}^{\infty}\frac{1}{k!}A^k t^k = Ae^{At}. \tag{6.8}$$

Multiplying by $x(0)$ from the right, we find that $x(t) = e^{At}x(0)$ is the solution to the differential equation (6.6) with initial condition $x(0)$. We summarize this important result as a proposition.

Proposition 6.1. *The solution to the homogeneous system of differential equations* (6.6) *is given by*

$$x(t) = e^{At}x(0).$$

Notice that the form of the solution is exactly the same as for scalar equations, but we must be sure to put the vector $x(0)$ on the right of the matrix e^{At}.

The form of the solution immediately allows us to see that the solution is linear in the initial condition. In particular, if $x_{\mathrm{h1}}(t)$ is the solution to equation (6.6) with initial condition $x(0) = x_{01}$ and $x_{\mathrm{h2}}(t)$ with initial condition $x(0) = x_{02}$, then the solution with initial condition $x(0) = \alpha x_{01} + \beta x_{02}$ is given by

$$x(t) = e^{At}\bigl(\alpha x_{01} + \beta x_{02}\bigr) = \bigl(\alpha e^{At}x_{01} + \beta e^{At}x_{02}\bigr) = \alpha x_{\mathrm{h1}}(t) + \beta x_{\mathrm{h2}}(t).$$

Similarly, we see that the corresponding output is given by

$$y(t) = Cx(t) = \alpha y_{\mathrm{h1}}(t) + \beta y_{\mathrm{h2}}(t),$$

where $y_{\mathrm{h1}}(t)$ and $y_{\mathrm{h2}}(t)$ are the outputs corresponding to $x_{\mathrm{h1}}(t)$ and $x_{\mathrm{h2}}(t)$.

We illustrate computation of the matrix exponential by two examples.

Example 6.2 Double integrator

A very simple linear system that is useful in understanding basic concepts is the second-order system given by

$$\ddot{q} = u, \qquad y = q.$$

This system is called a *double integrator* because the input u is integrated twice to determine the output y.

In state space form, we write $x = (q, \dot{q})$ and

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

The dynamics matrix of a double integrator is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and we find by direct calculation that $A^2 = 0$ and hence

$$e^{At} = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Thus the solution of the homogeneous system ($u = 0$) for the double integrator is given by

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_1(0) + tx_2(0) \\ x_2(0) \end{pmatrix},$$
$$y(t) = x_1(0) + tx_2(0).$$

∇

Example 6.3 Undamped oscillator
A model for an oscillator, such as the spring–mass system with zero damping, is

$$\ddot{q} + \omega_0^2 q = u.$$

Putting the system into state space form using $x_1 = q$, $x_2 = \dot{q}/\omega_0$, the dynamics matrix for this system can be written as

$$A = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \quad \text{and} \quad e^{At} = \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix}.$$

This expression for e^{At} can be verified by differentiation:

$$\begin{aligned} \frac{d}{dt}e^{At} &= \begin{pmatrix} -\omega_0 \sin\omega_0 t & \omega_0 \cos\omega_0 t \\ -\omega_0 \cos\omega_0 t & -\omega_0 \sin\omega_0 t \end{pmatrix} \\ &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix} = Ae^{At}. \end{aligned}$$

The solution to the initial value problem is then given by

$$x(t) = e^{At}x(0) = \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}.$$

The solution is more complicated if the system has damping:

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u.$$

If $\zeta < 1$ we have

$$\exp\begin{pmatrix} -\zeta\omega_0 & \omega_\mathrm{d} \\ -\omega_\mathrm{d} & -\zeta\omega_0 \end{pmatrix} t = e^{-\zeta\omega_0 t} \begin{pmatrix} \cos\omega_\mathrm{d} t & \sin\omega_\mathrm{d} t \\ -\sin\omega_\mathrm{d} t & \cos\omega_\mathrm{d} t \end{pmatrix},$$

where $\omega_\mathrm{d} = \omega_0\sqrt{1-\zeta^2}$. The result can be proven by differentiating the exponential matrix. The corresponding results for $\zeta \geq 1$ are given in Exercise 6.4. ∇

An important class of linear systems are those that can be converted into diagonal form by a linear change of coordinates. Suppose that we are given a system

$$\frac{dx}{dt} = Ax$$

such that all the eigenvalues of A are distinct. It can be shown (Exercise 5.14) that there exists an invertible matrix T such that TAT^{-1} is diagonal. If we choose a set of coordinates $z = Tx$, then in the new coordinates the dynamics become

$$\frac{dz}{dt} = T\frac{dx}{dt} = TAx = TAT^{-1}z.$$

By definition of T, this system will be diagonal.

Now consider a diagonal matrix A and the corresponding kth power of At, which is also diagonal:

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}, \qquad (At)^k = \begin{pmatrix} \lambda_1^k t^k & & & 0 \\ & \lambda_2^k t^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k t^k \end{pmatrix}.$$

It follows from the series expansion that the matrix exponential is given by

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}.$$

A similar expansion can be done in the case where the eigenvalues are complex, using a block diagonal matrix, similar to what was done in Section 5.3.

Given the solution to the dynamics in the z coordinates, the solution in the original x coordinates can be obtained using the expression $x = T^{-1}z$. We can thus obtain an explicit solution for a linear system whose dynamics matrix is diagonalizable.

Jordan Form

Some matrices with repeated eigenvalues cannot be transformed to diagonal form. They can, however, be transformed to a closely related form, called the *Jordan form*, in which the dynamics matrix has the eigenvalues along the diagonal. When there are equal eigenvalues, there may be 1's appearing in the superdiagonal indicating that there is coupling between the states.

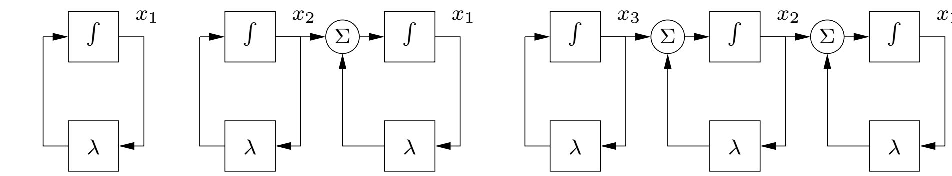
Specifically, we define a matrix to be in Jordan form if it can be written as

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{pmatrix}, \quad \text{where} \quad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}, \tag{6.9}$$

and λ_i is an eigenvalue of J_i. Each matrix J_i is called a *Jordan block*. A first-order Jordan block can be represented as a system consisting of an integrator with feedback λ. A Jordan block of higher order can be represented as series connections of such systems, as illustrated in Figure 6.3.

Theorem 6.2 (Jordan decomposition)**.** *Any matrix $A \in \mathbb{R}^{n\times n}$ can be transformed into Jordan form with the eigenvalues of A determining λ_i in the Jordan form.*

Proof. See any standard text on linear algebra, such as Strang [Str88]. The special case where the eigenvalues are distinct is examined in Exercise 5.14. ☐



(a) 1×1 block (b) 2×2 block (c) 3×3 block

Figure 6.3: Representations of linear systems where the dynamics matrices are Jordan blocks. A 1×1 Jordan block corresponds to an integrator with feedback λ, as shown on the left. 2×2 and 3×3 Jordan blocks correspond to cascade connections of integrators with identical feedback, as shown in the middle and right diagrams.

Converting a matrix into Jordan form can be complicated, although MATLAB can do this conversion for numerical matrices using the `jordan` function. There is no requirement that the individual λ_i's be distinct, and hence for a given eigenvalue we can have one or more Jordan blocks of different sizes.

Once a matrix is in Jordan form, the exponential of the matrix can be computed in terms of the Jordan blocks:

$$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & & 0 \\ & e^{J_2 t} & & \\ & & \ddots & \\ 0 & & & e^{J_k t} \end{pmatrix}. \tag{6.10}$$

This follows from the block diagonal form of J. The exponentials of the Jordan blocks can in turn be written as

$$e^{J_i t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{t} \\ 0 & & & & 1 \end{pmatrix} e^{\lambda_i t}. \tag{6.11}$$

As before, we can express the solution to a linear system that can be converted into this form by making use of the transformations $z = Tx$ and $x = T^{-1}z$.

When there are multiple eigenvalues, the invariant subspaces associated with each eigenvalue correspond to the Jordan blocks of the matrix A. Note that some eigenvalues of A may be complex, in which case the transformation T that converts a matrix into Jordan form will also be complex. When λ has a nonzero imaginary component, the solutions will have oscillatory components since

$$e^{(\sigma + i\omega)t} = e^{\sigma t}(\cos \omega t + i \sin \omega t).$$

We can now use these results to prove Theorem 5.1, which states that the equilibrium point $x_{\mathrm{e}} = 0$ of a linear system is asymptotically stable if and only if $\operatorname{Re} \lambda_i < 0$ for all i.

Proof of Theorem 5.1. Let $T \in \mathbb{C}^{n\times n}$ be an invertible matrix that transforms A into Jordan form, $J = TAT^{-1}$. Using coordinates $z = Tx$, we can write the solution $z(t)$ as

$$z(t) = e^{Jt} z(0),$$

where $z(0) = Tx(0)$, so that $x(t) = T^{-1}e^{Jt}z(0)$.

The solution $z(t)$ can be written in terms of the elements of the matrix exponential. From equation (6.11) these elements all decay to zero for arbitrary $z(0)$ if and only if $\operatorname{Re}\lambda_i < 0$ for all i. Furthermore, if any λ_i has positive real part, then there exists an initial condition $z(0)$ such that the corresponding solution increases without bound. Since we can scale this initial condition to be arbitrarily small, it follows that the equilibrium point is unstable if any eigenvalue has positive real part. ☐

The existence of a canonical form allows us to prove many properties of linear systems by changing to a set of coordinates in which the A matrix is in Jordan form. We illustrate this in the following proposition, which follows along the same lines as the proof of Theorem 5.1.

Proposition 6.3. *Suppose that the system*

$$\frac{dx}{dt} = Ax$$

has no eigenvalues with strictly positive real part and one or more eigenvalues with zero real part. Then the system is stable (in the sense of Lyapunov) if and only if the Jordan blocks corresponding to each eigenvalue with zero real part are scalar (1×1) *blocks.*

Proof. See Exercise 6.6b. ☐

The following example illustrates the use of the Jordan form.

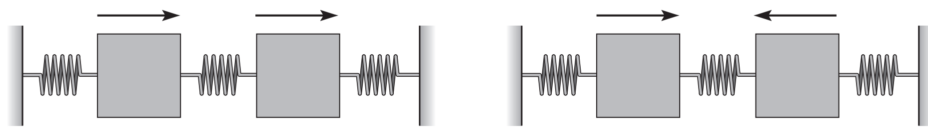
Example 6.4 Linear model of a vectored thrust aircraft
Consider the dynamics of a vectored thrust aircraft such as that described in Example 3.12. Suppose that we choose $u_1 = u_2 = 0$ so that the dynamics of the system become

$$\frac{dz}{dt} = \begin{pmatrix} z_4 \\ z_5 \\ z_6 \\ -g\sin z_3 - \frac{c}{m} z_4 \\ g(\cos z_3 - 1) - \frac{c}{m} z_5 \\ 0 \end{pmatrix}, \tag{6.12}$$

where $z = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$. The equilibrium points for the system are given by setting the velocities $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ to zero and choosing the remaining variables to satisfy

$$\begin{aligned} -g\sin z_{3,e} &= 0 \\ g(\cos z_{3,e} - 1) &= 0 \end{aligned} \qquad \Longrightarrow \qquad z_{3,e} = \theta_{\mathrm{e}} = 0.$$

This corresponds to the upright orientation for the aircraft. Note that x_{e} and y_{e} are not specified. This is because we can translate the system to a new (upright) position and still obtain an equilibrium point.



(a) Mode 1 (b) Mode 2

Figure 6.4: Modes of vibration for a system consisting of two masses connected by springs. In (a) the masses move left and right in synchronization in (b) they move toward or against each other.

To compute the stability of the equilibrium point, we compute the linearization using equation (5.13):

$$A = \left.\frac{\partial F}{\partial z}\right|_{z_e} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & -c/m & 0 & 0 \\ 0 & 0 & 0 & 0 & -c/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of the system can be computed as

$$\lambda(A) = \{0, 0, 0, 0, -c/m, -c/m\}.$$

We see that the linearized system is not asymptotically stable since not all of the eigenvalues have strictly negative real part.

To determine whether the system is stable in the sense of Lyapunov, we must make use of the Jordan form. It can be shown that the Jordan form of A is given by

$$J = \left(\begin{array}{c|ccc|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -c/m & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -c/m \end{array}\right).$$

Since the second Jordan block has eigenvalue 0 and is not a simple eigenvalue, the linearization is unstable (Exercise 6.6). ∇

Eigenvalues and Modes

The eigenvalues and eigenvectors of a system provide a description of the types of behavior the system can exhibit. For oscillatory systems, the term *mode* is often used to describe the vibration patterns that can occur. Figure 6.4 illustrates the modes for a system consisting of two masses connected by springs. One pattern is when both masses oscillate left and right in unison, and another is when the masses move toward and away from each other.

The initial condition response of a linear system can be written in terms of a matrix exponential involving the dynamics matrix A. The properties of the matrix A therefore determine the resulting behavior of the system. Given a matrix $A \in \mathbb{R}^{n\times n}$, recall that v is an eigenvector of A with eigenvalue λ if

$$Av = \lambda v.$$

In general λ and v may be complex-valued, although if A is real-valued, then for any eigenvalue λ its complex conjugate λ^* will also be an eigenvalue (with v^* as the corresponding eigenvector).

Suppose first that λ and v are a real-valued eigenvalue/eigenvector pair for A. If we look at the solution of the differential equation for $x(0) = v$, it follows from the definition of the matrix exponential that

$$e^{At}v = \bigl(I + At + \frac{1}{2}A^2t^2 + \cdots\bigr)v = v + \lambda t v + \frac{\lambda^2 t^2}{2}v + \cdots = e^{\lambda t}v.$$

The solution thus lies in the subspace spanned by the eigenvector. The eigenvalue λ describes how the solution varies in time, and this solution is often called a *mode* of the system. (In the literature, the term "mode" is also often used to refer to the eigenvalue rather than the solution.)

If we look at the individual elements of the vectors x and v, it follows that

$$\frac{x_i(t)}{x_j(t)} = \frac{e^{\lambda t}v_i}{e^{\lambda t}v_j} = \frac{v_i}{v_j},$$

and hence the ratios of the components of the state x are constants for a (real) mode. The eigenvector thus gives the "shape" of the solution and is also called a *mode shape* of the system. Figure 6.5 illustrates the modes for a second-order system consisting of a fast mode and a slow mode. Notice that the state variables have the same sign for the slow mode and different signs for the fast mode.

The situation is more complicated when the eigenvalues of A are complex. Since A has real elements, the eigenvalues and the eigenvectors are complex conjugates $\lambda = \sigma \pm i\omega$ and $v = u \pm iw$, which implies that

$$u = \frac{v + v^*}{2}, \qquad w = \frac{v - v^*}{2i}.$$

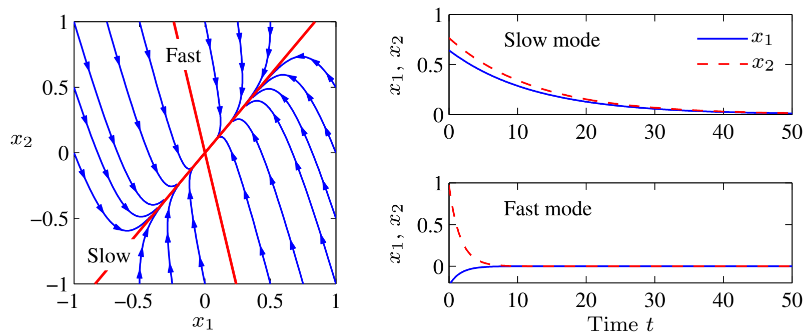
Making use of the matrix exponential, we have

$$e^{At}v = e^{\lambda t}(u + iw) = e^{\sigma t}\bigl((u\cos\omega t - w\sin\omega t) + i(u\sin\omega t + w\cos\omega t)\bigr),$$

from which it follows that

$$\begin{aligned} e^{At}u &= \frac{1}{2}\Bigl(e^{At}v + e^{At}v^*\Bigr) = ue^{\sigma t}\cos\omega t - we^{\sigma t}\sin\omega t, \\ e^{At}w &= \frac{1}{2i}\Bigl(e^{At}v - e^{At}v^*\Bigr) = ue^{\sigma t}\sin\omega t + we^{\sigma t}\cos\omega t. \end{aligned}$$

A solution with initial conditions in the subspace spanned by the real part u and imaginary part w of the eigenvector will thus remain in that subspace. The solution



(a) Phase portrait (b) Simulation

Figure 6.5: The notion of modes for a second-order system with real eigenvalues. The left figure shows the phase portrait and the modes corresponding to solutions that start on the eigenvectors (bold lines). The corresponding time functions are shown on the right.

will be a logarithmic spiral characterized by σ and ω. We again call the solution corresponding to λ a mode of the system and v the mode shape.

If a matrix A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then the initial condition response can be written as a linear combination of the modes. To see this, suppose for simplicity that we have all real eigenvalues with corresponding unit eigenvectors $v_1, \dots, v_n$. From linear algebra, these eigenvectors are linearly independent, and we can write the initial condition $x(0)$ as

$$x(0) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Using linearity, the initial condition response can be written as

$$x(t) = \alpha_1 e^{\lambda_1 t} v_1 + \alpha_2 e^{\lambda_2 t} v_2 + \cdots + \alpha_n e^{\lambda_n t} v_n.$$

Thus, the response is a linear combination of the modes of the system, with the amplitude of the individual modes growing or decaying as $e^{\lambda_i t}$. The case for distinct complex eigenvalues follows similarly (the case for nondistinct eigenvalues is more subtle and requires making use of the Jordan form discussed in the previous section).

Example 6.5 Coupled spring–mass system

Consider the spring–mass system shown in Figure 6.4, but with the addition of dampers on each mass. The equations of motion of the system are

$$m\ddot{q}_1 = -2kq_1 - c\dot{q}_1 + kq_2, \qquad m\ddot{q}_2 = kq_1 - 2kq_2 - c\dot{q}_2.$$

In state space form, we define the state to be $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$, and we can rewrite the equations as

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{2k}{m} & \dfrac{k}{m} & -\dfrac{c}{m} & 0 \\ \dfrac{k}{m} & -\dfrac{2k}{m} & 0 & -\dfrac{c}{m} \end{pmatrix} x.$$

We now define a transformation $z = Tx$ that puts this system into a simpler form. Let $z_1 = \frac{1}{2}(q_1 + q_2)$, $z_2 = \dot{z}_1$, $z_3 = \frac{1}{2}(q_1 - q_2)$ and $z_4 = \dot{z}_3$, so that

$$z = Tx = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} x.$$

In the new coordinates, the dynamics become

$$\frac{dz}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3k}{m} & -\frac{c}{m} \end{pmatrix} z,$$

and we see that the model is now in block diagonal form.

In the z coordinates, the states z_1 and z_2 parameterize one mode with eigenvalues $\lambda \approx -c/(2m) \pm i\sqrt{k/m}$, and the states z_3 and z_4 another mode with $\lambda \approx -c/(2m) \pm i\sqrt{3k/m}$. From the form of the transformation T we see that these modes correspond exactly to the modes in Figure 6.4, in which q_1 and q_2 move either toward or against each other. The real and imaginary parts of the eigenvalues give the decay rates σ and frequencies ω for each mode. ∇

6.3 Input/Output Response

In the previous section we saw how to compute the initial condition response using the matrix exponential. In this section we derive the convolution equation, which includes the inputs and outputs as well.

The Convolution Equation

We return to the general input/output case in equation (6.3), repeated here:

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du. \tag{6.13}$$

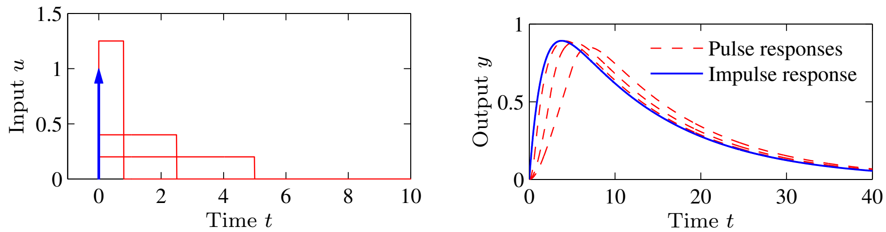
Using the matrix exponential, the solution to equation (6.13) can be written as follows.

Theorem 6.4. *The solution to the linear differential equation* (6.13) *is given by*

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{6.14}$$

Proof. To prove this, we differentiate both sides and use the property (6.8) of the matrix exponential. This gives

$$\frac{dx}{dt} = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t) = Ax + Bu,$$



(a) Pulse and impulse functions

(b) Pulse and impulse responses

Figure 6.6: Pulse response and impulse response. (a) The rectangles show pulses of width 5, 2.5, and 0.8, each with total area equal to 1. The arrow denotes an impulse $\delta(t)$ defined by equation (6.17). The corresponding pulse responses for a linear system with eigenvalues $\lambda = \{-0.08, -0.62\}$ are shown in (b) as dashed lines. The solid line is the true impulse response, which is well approximated by a pulse of duration 0.8.

which proves the result since the initial conditions are also met. Notice that the calculation is essentially the same as for proving the result for a first-order equation. $\square$

It follows from equations (6.13) and (6.14) that the input/output relation for a linear system is given by

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{6.15}$$

It is easy to see from this equation that the output is jointly linear in both the initial conditions and the input, which follows from the linearity of matrix/vector multiplication and integration.

Equation (6.15) is called the *convolution equation*, and it represents the general form of the solution of a system of coupled linear differential equations. We see immediately that the dynamics of the system, as characterized by the matrix A, play a critical role in both the stability and performance of the system. Indeed, the matrix exponential describes *both* what happens when we perturb the initial condition and how the system responds to inputs.

Another interpretation of the convolution equation can be given using the concept of the *impulse response* of a system. Consider the application of an input signal $u(t)$ given by the following equation:

$$u(t) = p_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1/\epsilon & \text{if } 0 \le t < \epsilon, \\ 0 & \text{if } t \ge \epsilon. \end{cases} \tag{6.16}$$

This signal is a *pulse* of duration ϵ and amplitude $1/\epsilon$, as illustrated in Figure 6.6a. We define an *impulse* $\delta(t)$ to be the limit of this signal as $\epsilon \to 0$:

$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t). \tag{6.17}$$

This signal, sometimes called a *delta function,* is not physically achievable but provides a convenient abstraction in understanding the response of a system. Note that the integral of an impulse is 1:

$$\begin{aligned}\int_0^t \delta(\tau)\,d\tau &= \int_0^t \lim_{\epsilon\to 0} p_\epsilon(t)\,d\tau = \lim_{\epsilon\to 0}\int_0^t p_\epsilon(t)\,d\tau\\ &= \lim_{\epsilon\to 0}\int_0^\epsilon 1/\epsilon\,d\tau = 1, \qquad t>0.\end{aligned}$$

In particular, the integral of an impulse over an arbitrarily short period of time that includes the origin is identically 1.

We define the *impulse response* $h(t)$ for a system as the output of the system with zero initial condition and having an impulse as its input:

$$h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)\,d\tau + D\delta(t) = Ce^{At}B + D\delta(t), \tag{6.18}$$

where the second equality follows from the fact that $\delta(t)$ is zero everywhere except the origin and its integral is identically 1. We can now write the convolution equation in terms of the initial condition response and the convolution of the impulse response and the input signal:

$$y(t) = Ce^{At}x(0) + \int_0^t h(t-\tau)u(\tau)\,d\tau. \tag{6.19}$$

One interpretation of this equation, explored in Exercise 6.2, is that the response of the linear system is the superposition of the response to an infinite set of shifted impulses whose magnitudes are given by the input $u(t)$. This is essentially the argument used in analyzing Figure 6.2 and deriving equation (6.5). Note that the second term in equation (6.19) is identical to equation (6.5), and it can be shown that the impulse response is the derivative of the step response.

The use of pulses $p_\epsilon(t)$ as approximations of the impulse function $\delta(t)$ also provides a mechanism for identifying the dynamics of a system from experiments. Figure 6.6b shows the pulse responses of a system for different pulse widths. Notice that the pulse responses approach the impulse response as the pulse width goes to zero. As a general rule, if the fastest eigenvalue of a stable system has real part $-\sigma_{\max}$, then a pulse of length ϵ will provide a good estimate of the impulse response if $\epsilon\sigma_{\max} \ll 1$. Note that for Figure 6.6, a pulse width of $\epsilon = 1$ s gives $\epsilon\sigma_{\max} = 0.62$ and the pulse response is already close to the impulse response.

Coordinate Invariance

The components of the input vector u and the output vector y are determined by the chosen inputs and outputs of a model, but the state variables depend on the coordinate frame chosen to represent the state. This choice of coordinates affects the values of the matrices A, B, and C that are used in the model. (The direct term D is not affected since it maps inputs to outputs.) We now investigate some of the consequences of changing coordinate systems.

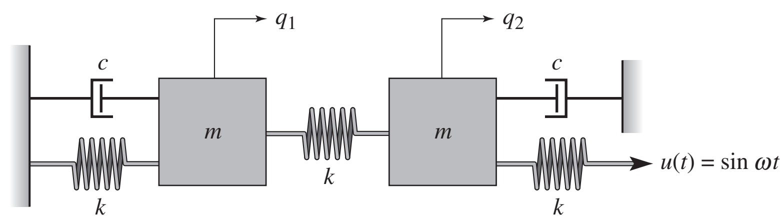


Figure 6.7: Coupled spring mass system. Each mass is connected to two springs with stiffness k and a viscous damper with damping coefficient c. The mass on the right is driven through a spring connected to a sinusoidally varying attachment.

Introduce new coordinates z by the transformation $z = Tx$, where T is an invertible matrix. It follows from equation (6.3) that

$$\frac{dz}{dt} = T(Ax + Bu) = TAT^{-1}z + TBu =: \tilde{A}z + \tilde{B}u,$$
$$y = Cx + Du = CT^{-1}z + Du =: \tilde{C}z + Du.$$

The transformed system has the same form as equation (6.3), but the matrices A, B, and C are different:

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB, \qquad \tilde{C} = CT^{-1}. \tag{6.20}$$

There are often special choices of coordinate systems that allow us to see a particular property of the system, hence coordinate transformations can be used to gain new insight into the dynamics. The eigenvalues of $\tilde{A}$ are the same as those of A, so stability is not affected.

We can also compare the solution of the system in transformed coordinates to that in the original state coordinates. We make use of an important property of the exponential map,

$$e^{TST^{-1}} = Te^{S}T^{-1},$$

which can be verified by substitution in the definition of the matrix exponential. Using this property, it is easy to show that

$$x(t) = T^{-1}z(t) = T^{-1}e^{\tilde{A}t}Tx(0) + T^{-1}\int_0^t e^{\tilde{A}(t-\tau)}\tilde{B}u(\tau)\,d\tau.$$

From this form of the equation, we see that if it is possible to transform A into a form $\tilde{A}$ for which the matrix exponential is easy to compute, we can use that computation to solve the general convolution equation for the untransformed state x by simple matrix multiplications. This technique is illustrated in the following example.

Example 6.6 Coupled spring–mass system
Consider the coupled spring–mass system shown in Figure 6.7. The input to this system is the sinusoidal motion of the position of the rightmost spring, and the output is the position of each mass, q_1 and q_2. The equations of motion are given by

$$m\ddot{q}_1 = -2kq_1 - c\dot{q}_1 + kq_2, \qquad m\ddot{q}_2 = kq_1 - 2kq_2 - c\dot{q}_2 + ku.$$

In state space form, we define the state to be $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$, and we can rewrite the equations as

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{2k}{m} & \frac{k}{m} & -\frac{c}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & 0 & -\frac{c}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{pmatrix} u.$$

This is a coupled set of four differential equations and is quite complicated to solve in analytical form.

The dynamics matrix is the same as in Example 6.5, and we can use the coordinate transformation defined there to put the system in block diagonal form:

$$\frac{dz}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3k}{m} & -\frac{c}{m} \end{pmatrix} z + \begin{pmatrix} 0 \\ \frac{k}{2m} \\ 0 \\ -\frac{k}{2m} \end{pmatrix} u.$$

Note that the resulting matrix equations are decoupled, and we can solve for the solutions by computing the solutions of two sets of second-order systems represented by the states (z_1, z_2) and (z_3, z_4). Indeed, the functional form of each set of equations is identical to that of a single spring–mass system. (The explicit solution is derived in Section 7.3.)

Once we have solved the two sets of independent second-order equations, we can recover the dynamics in the original coordinates by inverting the state transformation and writing $x = T^{-1}z$. We can also determine the stability of the system by looking at the stability of the independent second-order systems. ∇

Steady-State Response

A common practice in evaluating the response of a linear system is to separate out the short-term response from the long-term response. Given a linear input/output system

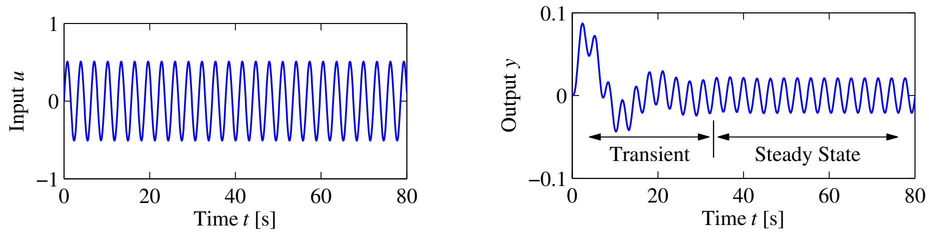
$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du, \tag{6.21}$$

the general form of the solution to equation (6.21) is given by the convolution equation:

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

We see from the form of this equation that the solution consists of an initial condition response and an input response.

The input response, corresponding to the last two terms in the equation above, itself consists of two components—the *transient response* and the *steady-state response*. The transient response occurs in the first period of time after the input



(a) Input (b) Output

Figure 6.8: Transient versus steady-state response. The input to a linear system is shown in (a), and the corresponding output with $x(0) = 0$ is shown in (b). The output signal initially undergoes a transient before settling into its steady-state behavior.

is applied and reflects the mismatch between the initial condition and the steady-state solution. The steady-state response is the portion of the output response that reflects the long-term behavior of the system under the given inputs. For inputs that are periodic the steady-state response will often be periodic, and for constant inputs the response will often be constant. An example of the transient and the steady-state response for a periodic input is shown in Figure 6.8.

A particularly common form of input is a *step input*, which represents an abrupt change in input from one value to another. A *unit step* (sometimes called the Heaviside step function) is defined as

$$u(t) = S(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$$

The *step response* of the system (6.21) is defined as the output $y(t)$ starting from zero initial condition (or the appropriate equilibrium point) and given a step input. We note that the step input is discontinuous and hence is not practically implementable. However, it is a convenient abstraction that is widely used in studying input/output systems.

We can compute the step response to a linear system using the convolution equation. Setting $x(0) = 0$ and using the definition of the step input above, we have

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) = C\int_0^t e^{A(t-\tau)}Bd\tau + D \\ &= C\int_0^t e^{A\sigma}Bd\sigma + D = C\left(A^{-1}e^{A\sigma}B\right)\Big|_{\sigma=0}^{\sigma=t} + D \\ &= CA^{-1}e^{At}B - CA^{-1}B + D. \end{aligned}$$

We can rewrite the solution as

$$y(t) = \underbrace{CA^{-1}e^{At}B}_{\text{transient}} + \underbrace{D - CA^{-1}B}_{\text{steady-state}}, \qquad t > 0. \tag{6.22}$$

The first term is the transient response and it decays to zero as $t \to \infty$ if all eigenvalues of A have negative real parts (implying that the origin is a stable equilibrium

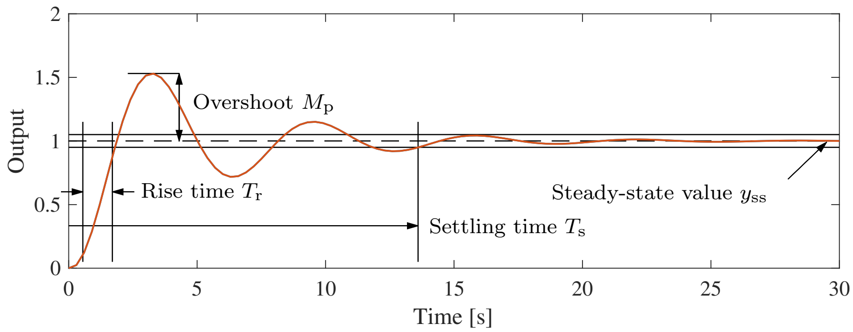


Figure 6.9: Sample step response. The rise time, overshoot, settling time, and steady-state value give the key performance properties of the signal.

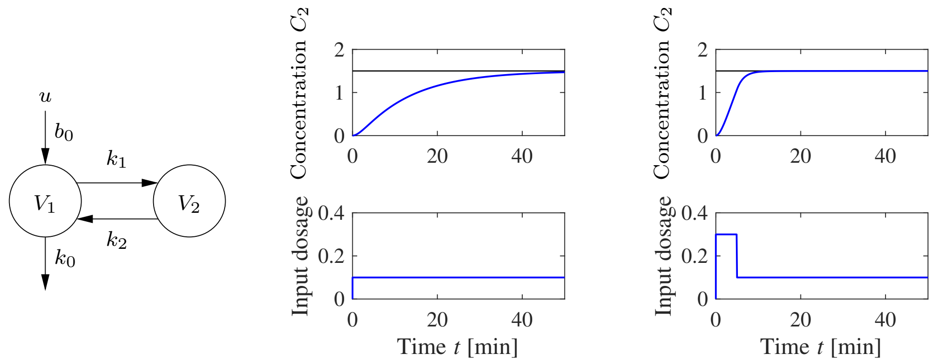
point in the absence of any input). The second term, computed under the assumption that the matrix A is invertible, is the steady-state step response and represents the value of the output for large time.

A sample step response is shown in Figure 6.9. Several key properties are used when describing a step response. The *steady-state value* y_{ss} of a step response is the final level of the output, assuming it converges. The *rise time* T_r is the amount of time required for the signal to first go from 10% of its final value to 90% of its final value. (It is possible to define other limits as well, but in this book we shall use these percentages unless otherwise indicated.) The *overshoot* M_p is the percentage of the final value by which the signal initially rises above the final value. This usually assumes that future values of the signal do not overshoot the final value by more than this initial transient, otherwise the term can be ambiguous. Finally, the *settling time* T_s is the amount of time required for the signal to stay within 2% of its final value for all future times. The settling time is also sometimes defined as reaching 1% or 5% of the final value (see Exercise 6.7). In general these performance measures can depend on the amplitude of the input step, but for linear systems the last three quantities defined above are independent of the size of the step.

Example 6.7 Compartment model
Consider the compartment model illustrated in Figure 6.10 and described in more detail in Section 4.6. Assume that a drug is administered by constant infusion in compartment V_1 and that the drug has its effect in compartment V_2. To assess how quickly the concentration in the compartment reaches steady state we compute the step response, which is shown in Figure 6.10b. The step response is quite slow, with a settling time of 39 min. It is possible to obtain the steady-state concentration much faster by having a faster injection rate initially, as shown in Figure 6.10c. The response of the system in this case can be computed by combining two step responses (Exercise 6.3). ∇

Frequency Response

Another common input signal to a linear system is a sinusoid (or a combination of sinusoids). The *frequency response* of an input/output system measures the way



(a) Schematic (b) Step input (c) Pulse input

Figure 6.10: Response of a compartment model to a constant drug infusion. A simple diagram of the system is shown in (a). The step response (b) shows the rate of concentration buildup in compartment 2. In (c) a pulse of initial concentration is used to speed up the response.

in which the system responds to a sinusoidal excitation on one of its inputs. As we have already seen for scalar systems, the particular solution associated with a sinusoidal excitation is itself a sinusoid at the same frequency. Hence we can compare the magnitude and phase of the output sinusoid to the input.

To see this in more detail, we must evaluate the convolution equation (6.15) for $u = \cos \omega t$. This turns out to be a very messy calculation, but we can make use of the fact that the system is linear to simplify the derivation. It follows from Euler's formula that

$$\cos \omega t = \frac{1}{2}\Big(e^{i\omega t} + e^{-i\omega t}\Big).$$

Since the system is linear, it suffices to compute the response of the system to the complex input $u(t) = e^{st}$ and we can then reconstruct the input to a sinusoid by averaging the responses corresponding to $s = i\omega$ and $s = -i\omega$.

Applying the convolution equation to the input $u = e^{st}$ we have

$$\begin{aligned} y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Be^{s\tau}d\tau + De^{st} \\ &= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}Bd\tau + De^{st}. \end{aligned}$$

If we assume that none of the eigenvalues of A are equal to $\pm i\omega$, then the matrix $sI - A$ is invertible, and we can write

$$\begin{aligned} y(t) &= Ce^{At}x(0) + Ce^{At}\,\Big((sI-A)^{-1}e^{(sI-A)\tau}B\Big)\Big|_0^t + De^{st} \\ &= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\Big(e^{(sI-A)t} - I\Big)B + De^{st} \\ &= Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st}, \end{aligned}$$

and we obtain

$$y(t) = \underbrace{Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big)}_{\text{transient}} + \underbrace{\Big(C(sI - A)^{-1}B + D\Big)e^{st}}_{\text{steady-state}}. \tag{6.23}$$

Notice that once again the solution consists of both a transient component and a steady-state component. The transient component decays to zero if the system is asymptotically stable and the steady-state component is proportional to the (complex) input $u = e^{st}$.

We can simplify the form of the solution slightly further by rewriting the steady-state response as

$$y_{\text{ss}}(t) = Me^{i\theta}e^{st} = Me^{(st+i\theta)},$$

where

$$Me^{i\theta} = G(s) = C(sI - A)^{-1}B + D, \tag{6.24}$$

and M and θ represent the magnitude and phase of the complex number $G(s)$. When $s = i\omega$, we say that $M = |G(i\omega)|$ is the *gain* and $\theta = \arg G(i\omega)$ is the *phase* of the system at a given forcing frequency ω. Using linearity and combining the solutions for $s = +i\omega$ and $s = -i\omega$, we can show that if we have an input $u = A_u \sin(\omega t + \psi)$ and an output $y = A_y \sin(\omega t + \varphi)$, then

$$\text{gain}(\omega) = \frac{A_y}{A_u} = M, \qquad \text{phase}(\omega) = \varphi - \psi = \theta.$$

The steady-state solution for a sinusoid $u = \cos\omega t = \sin(\omega t + \pi/2)$ is now given by

$$y_{\text{ss}}(t) = \text{Re}\big(G(i\omega)e^{i\omega t}\big) = M\cos(\omega t + \theta). \tag{6.25}$$

If the phase θ is positive, we say that the output *leads* the input, otherwise we say it *lags* the input.

A sample steady-state sinusoidal response is illustrated in Figure 6.11a. The dashed line shows the input sinusoid, which has amplitude 1. The output sinusoid is shown as a solid line and has a different amplitude plus a shifted phase. The gain is the ratio of the amplitudes of the sinusoids, which can be determined by measuring the height of the peaks. The phase is determined by comparing the ratio of the time between zero crossings of the input and output to the overall period of the sinusoid:

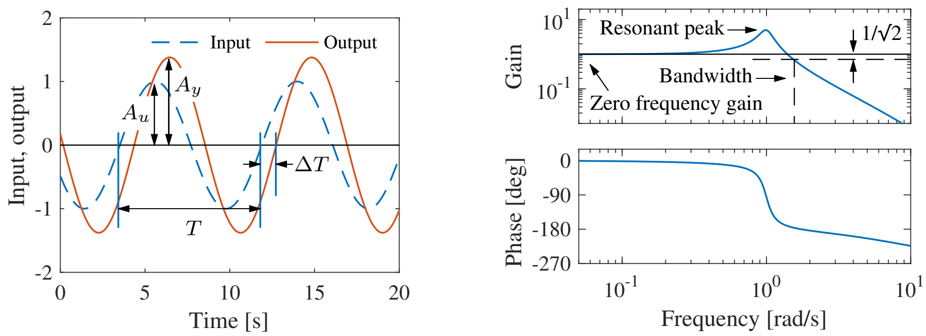
$$\theta = -2\pi \cdot \frac{\Delta T}{T}.$$

A convenient way to view the frequency response is to plot how the gain and phase in equation (6.24) depend on ω (through $s = i\omega$). Figure 6.11b shows an example of this type of representation (called a Bode plot and discussed in more detail in Section 9.6).

Example 6.8 Active band-pass filter

Consider the op amp circuit shown in Figure 6.12a. We can derive the dynamics of the system by writing the *nodal equations*, which state that the sum of the currents at any node must be zero. Assuming that $v_- = v_+ = 0$, as we did in Section 4.3, we have

$$0 = \frac{v_1 - v_2}{R_1} - C_1\frac{dv_2}{dt}, \qquad 0 = C_1\frac{dv_2}{dt} + \frac{v_3}{R_2} + C_2\frac{dv_3}{dt}.$$



(a) Input/output response (b) Frequency response

Figure 6.11: Steady-state response of an asymptotically stable linear system to a sinusoid. (a) A sinusoidal input of magnitude A_u (dashed) gives a sinusoidal output of magnitude A_y (solid), delayed by ΔT seconds. (b) Frequency response, showing gain and phase. The gain is given by the ratio of the output amplitude to the input amplitude, $M = A_y/A_u$. The phase lag is given by $\theta = -2\pi\Delta T/T$; it is negative for the case shown because the output lags the input.

Choosing v_2 and v_3 as our states and using these equations, we obtain

$$\frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1C_1}, \qquad \frac{dv_3}{dt} = \frac{-v_3}{R_2C_2} - \frac{v_1 - v_2}{R_1C_2}.$$

Rewriting these in linear state space form, we obtain

$$\frac{dx}{dt} = \begin{pmatrix} -\dfrac{1}{R_1C_1} & 0 \\ \dfrac{1}{R_1C_2} & -\dfrac{1}{R_2C_2} \end{pmatrix} x + \begin{pmatrix} \dfrac{1}{R_1C_1} \\ \dfrac{-1}{R_1C_2} \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} x, \tag{6.26}$$

where $x = (v_2, v_3)$, $u = v_1$, and $y = v_3$.

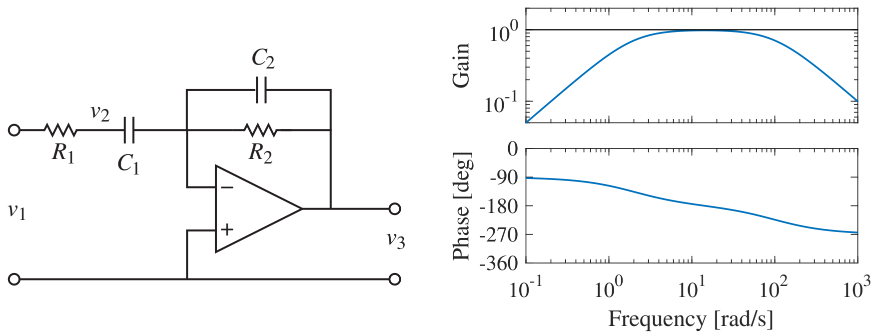
The frequency response for the system can be computed using equation (6.24):

$$Me^{i\theta} = C(sI - A)^{-1}B + D = -\frac{R_2}{R_1}\frac{R_1C_1s}{(1 + R_1C_1s)(1 + R_2C_2s)}, \qquad s = i\omega.$$

The magnitude and phase are plotted in Figure 6.12b for $R_1 = 100\ \Omega$, $R_2 = 5\ \mathrm{k}\Omega$, and $C_1 = C_2 = 100\ \mu\mathrm{F}$. We see that signals with frequencies around 15 rad/s pass through the circuit with small attenuation but that signals below 2 rad/s or above 100 rad/s are attenuated. At 0.1 rad/s the input signal is attenuated by a factor of 20. This type of circuit is called a *band-pass filter* since it passes through signals in the band of frequencies between 5 and 50 rad/s (approximately). ∇

As in the case of the step response, a number of standard properties are defined for frequency responses. The gain of a system at $\omega = 0$ is called the *zero frequency gain* and corresponds to the ratio between a constant input and the steady output:

$$M_0 = G(0) = -CA^{-1}B + D$$



(a) Circuit diagram (b) Frequency response

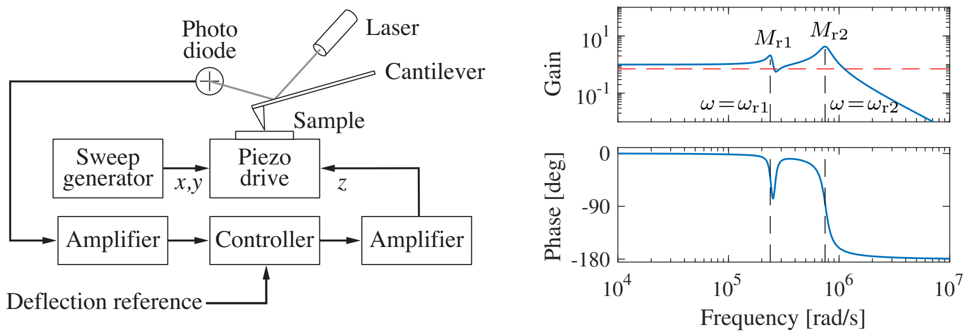
Figure 6.12: Active band-pass filter. The circuit diagram (a) shows an op amp with two *RC* filters arranged to provide a band-pass filter. The plot in (b) shows the gain and phase of the filter as a function of frequency. Note that the phase starts at -90° due to the negative gain of the operational amplifier.

(compare to equation (6.24)). The zero frequency gain is well defined only if A is invertible (i.e., if it does not have eigenvalues at 0). It is also important to note that the zero frequency gain is a relevant quantity only when a system is stable about the corresponding equilibrium point. So, if we apply a constant input $u = r$, then the corresponding equilibrium point $x_\mathrm{e} = -A^{-1}Br$ must be stable in order to talk about the zero frequency gain. (In electrical engineering, the zero frequency gain is often called the *DC gain*. DC stands for direct current and reflects the common separation of signals in electrical engineering into a direct current [zero frequency] term and an alternating current [AC] term.)

The *bandwidth* ω_b of a system is the frequency range over which the gain has decreased by no more than a factor of $1/\sqrt{2}$ from its reference value. For systems with nonzero, finite zero frequency gain, the reference value is taken as the zero frequency gain. For systems that attenuate low frequencies but pass through high frequencies, the reference gain is taken as the high-frequency gain. For a system such as the band-pass filter in Example 6.8, bandwidth is defined as the range of frequencies where the gain is larger than $1/\sqrt{2}$ of the gain at the center of the band. (For Example 6.8 this would give a bandwidth of approximately 2 to 100 rad/s.)

Other important properties of the frequency response are the *resonant peak* M_r, the largest value of the frequency response, and the *peak frequency* ω_mr, the frequency where the maximum occurs. These two properties describe the frequency of the sinusoidal input that produce the largest possible output and the gain at the frequency.

Example 6.9 Atomic force microscope in contact mode
Consider the model for the vertical dynamics of the atomic force microscope in contact mode, discussed in Section 4.5. The basic dynamics are given by equation (4.24). The piezo stack can be modeled by a second-order system with undamped natural frequency ω_3 and damping ratio ζ_3. The dynamics are then de-



(a) AFM block diagram (b) Frequency response

Figure 6.13: AFM frequency response. (a) A block diagram for the vertical dynamics of an atomic force microscope in contact mode. The plot in (b) shows the gain and phase for the piezo stack. The response contains two frequency peaks at resonances of the system, along with an antiresonance at $\omega = 268$ krad/s. The combination of a resonant peak followed by an antiresonance is common for systems with multiple lightly damped modes. The dashed horizontal line represents the gain equal to the zero frequency gain divided by $\sqrt{2}$.

scribed by the linear system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_2/(m_1+m_2) & -c_2/(m_1+m_2) & 1/m_2 & 0 \\ 0 & 0 & 0 & \omega_3 \\ 0 & 0 & -\omega_3 & -2\zeta_3\omega_3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_3 \end{pmatrix} u,$$

$$y = \frac{m_2}{m_1+m_2} \begin{pmatrix} \dfrac{m_1 k_2}{m_1+m_2} & \dfrac{m_1 c_2}{m_1+m_2} & 1 & 0 \end{pmatrix} x,$$

where the input is the drive signal to the amplifier and the output is the elongation of the piezo. The frequency response of the system is shown in Figure 6.13b. The zero frequency gain of the system is $M_0 = 1$. There are two resonant poles with peaks $M_{r1} = 2.12$ at $\omega_{mr1} = 238$ krad/s and $M_{r2} = 4.29$ at $\omega_{mr2} = 746$ krad/s. There is also a dip in the gain $M_d = 0.556$ for $\omega_{md} = 268$ krad/s. This dip, called an *antiresonance*, is associated with a dip in the phase and limits the performance when the system is controlled by simple controllers, as we will see in Chapter 11. The bandwidth is the frequency range over which the gain has decreased by no more than a factor of $1/\sqrt{2}$ from its reference value, which in this case is the zero frequency gain. Neglecting the slight dip at the antiresonance, the bandwidth becomes $\omega_b = 1.12$ Mrad/s.

∇

So far we have used the frequency response to compute the output for a single sinusoid. The transfer function can also be used to compute the output for any periodic signal. Consider a system with the frequency response $G(i\omega)$. Let the input signal $u(t)$ be periodic and decompose it into a sum of a set of sines and cosines,

$$u(t) = \sum_{k=0}^{\infty} a_k \sin(k\omega_{\mathrm{f}}\, t) + b_k \cos(k\omega_{\mathrm{f}}\, t),$$

where ω_{f} is the fundamental frequency of the periodic input. Using equation (6.25) and superposition, we find that the input $u(t)$ generates the steady-state output

$$y(t) = \sum_{k=0}^{\infty} |G(ik\omega_{\mathrm{f}})| \Big(a_k \sin\big(k\omega_{\mathrm{f}}\, t + \arg G(ik\omega_{\mathrm{f}})\big) + b_k \cos\big(k\omega_{\mathrm{f}}\, t + \arg G(ik\omega_{\mathrm{f}})\big)\Big).$$

The gain and phase at each frequency are determined by the frequency response $G(i\omega)$, as given in equation (6.24). If we know the steady-state frequency response $G(i\omega)$, we can thus compute the response to any (periodic) signal using superposition.

We can go even further to approximate the response to a transient signal. Consider a system with the transfer function $G(s)$ and the input u. Approximate the initial part of the function $u(t)$ by the periodic signal

$$u_{\mathrm{p}}(t) = \begin{cases} u(t) & \text{if } 0 \le t < T/2, \\ 0 & \text{if } T/2 \le t < T, \end{cases}$$

with period T. Since u_{p} is periodic it has a Fourier transform $u_{\mathrm{F}}(i\omega)$, and it follows from equation (6.25) that the Fourier transform of y_{p} is $y_{\mathrm{F}}(i\omega) = G(i\omega)u_{\mathrm{F}}(i\omega)$, where u_{F} and y_{F} represent the Fourier transforms of u_{p} and y_{p}, respectively. Taking the inverse Fourier transform then gives the time response $y_{\mathrm{p}}(t)$. Efficient algorithms can be obtained using fast Fourier transforms (Exercise 6.12).

Sampling

It is often convenient to use both differential and difference equations in modeling and control. For linear systems it is straightforward to transform from one to the other. Consider the general linear system described by equation (6.13) and assume that the control signal is constant over a sampling interval of constant length h. It follows from equation (6.14) of Theorem 6.4 that

$$x(t+h) = e^{Ah}x(t) + \int_t^{t+h} e^{A(t+h-\tau)}Bu(\tau)\, d\tau =: \Phi x(t) + \Gamma u(t), \tag{6.27}$$

where we have assumed that the discontinuous control signal is continuous from the right. The behavior of the system at the sampling times $t = kh$ is described by the difference equation

$$x[k+1] = \Phi x[k] + \Gamma u[k], \qquad y[k] = Cx[k] + Du[k], \tag{6.28}$$

where

$$\Phi = e^{Ah}, \qquad \Gamma = \Big(\int_0^h e^{As}\, ds\Big)B.$$

Notice that the difference equation (6.28) is an exact representation of the behavior of the system at the sampling instants. Similar expressions can also be obtained if the control signal is linear over the sampling interval.

The transformation from equation (6.27) to equation (6.28) is called *sampling.* The relations between the system matrices in the continuous and sampled representations are as follows:

$$A = \frac{1}{h}\log\Phi, \qquad B = \Bigl(\int_0^h e^{As}\,ds\Bigr)^{-1}\Gamma. \tag{6.29}$$

Notice that if A is invertible, we have

$$\Gamma = A^{-1}\bigl(e^{Ah} - I\bigr)B.$$

All continuous-time systems can be sampled to obtain a discrete-time version, but there are discrete-time systems that do not have a continuous-time equivalent. The issue is related to logarithms of matrices and there are several subtleties; for example, there may be many solutions. A necessary but not sufficient condition is that the matrix Φ is nonsingular [Gan60]. A key result is that a real matrix has a real logarithm if and only if it is invertible and if each Jordan block associated with a negative eigenvalue occurs an even number of times [Cul66]. This implies that the matrix Φ cannot have isolated eigenvalues on the negative real axis. A detailed discussion of sampling is given in [SÅH84].

Example 6.10 IBM Lotus server
In Example 3.5 we described how the dynamics of an IBM Lotus server were obtained as the discrete-time system

$$x[k+1] = ax[k] + bu[k],$$

where $a = 0.43$, $b = 0.47$, the sampling period is $h = 60$ s, and x denotes the total requests being served. A differential equation model is needed if we would like to design control systems based on continuous-time theory. Such a model is obtained by applying equation (6.29); hence

$$A = \frac{\log a}{h} = -0.0141, \qquad B = \Bigl(\int_0^h e^{At}\,dt\Bigr)^{-1} b = 0.0116,$$

and we find that the difference equation can be interpreted as a sampled version of the ordinary differential equation

$$\frac{dx}{dt} = -0.0141x + 0.0116u.$$

∇

6.4 Linearization

As described at the beginning of the chapter, a common source of linear system models is through the approximation of a nonlinear system by a linear one. It is common practice in control engineering to design controllers based on an approximate linear model and to verify the results by simulating the closed loop system using a nonlinear model. In this section we describe how to locally approximate a nonlinear system by a linear one, and discuss what can be inferred about the stability of the original system. We begin with an illustration that controllers can successfully be designed from approximate linear models using the cruise control example, which is described in more detail in Chapter 4.

Example 6.11 Cruise control
The dynamics for the cruise control system are derived in Section 4.1 and have the form

$$m\frac{dv}{dt} = \alpha_n u T(\alpha_n v) - mgC_\mathrm{r}\,\mathrm{sgn}(v) - \frac{1}{2}\rho C_\mathrm{d} A v|v| - mg\sin\theta, \tag{6.30}$$

where the first term on the right-hand side of the equation is the force generated by the engine and the remaining three terms are the rolling friction, aerodynamic drag, and gravitational disturbance force. There is an equilibrium point $(v_\mathrm{e}, u_\mathrm{e})$ when the force applied by the engine balances the disturbance forces.

To explore the behavior of the system near the equilibrium point we will linearize the system. A Taylor series expansion of equation (6.30) around the equilibrium point gives

$$\frac{d(v-v_\mathrm{e})}{dt} = -a(v-v_\mathrm{e}) - b_\mathrm{g}(\theta-\theta_\mathrm{e}) + b(u-u_\mathrm{e}) + \text{higher-order terms}, \tag{6.31}$$

where

$$a = \frac{\rho C_\mathrm{d} A|v_\mathrm{e}| - u_\mathrm{e}\alpha_n^2 T'(\alpha_n v_\mathrm{e})}{m}, \qquad b_\mathrm{g} = g\cos\theta_\mathrm{e}, \qquad b = \frac{\alpha_n T(\alpha_n v_\mathrm{e})}{m}. \tag{6.32}$$

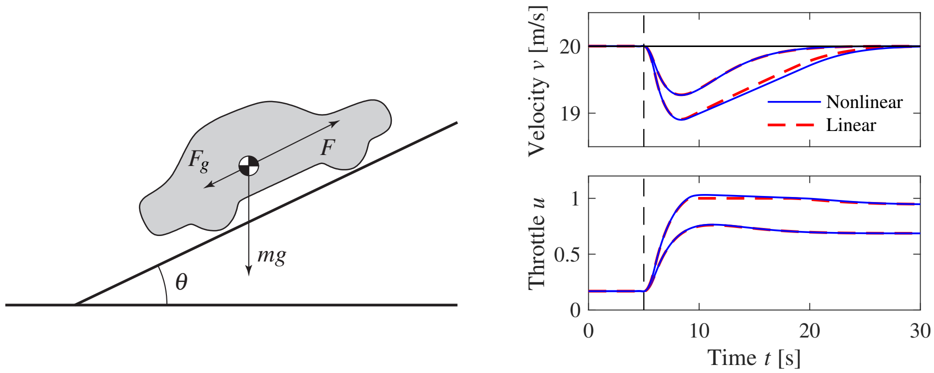
Notice that the term corresponding to rolling friction disappears if $v > 0$. For a car in fourth gear with $v_\mathrm{e} = 20$ m/s, $\theta_\mathrm{e} = 0$, and the numerical values for the car from Section 4.1, the equilibrium value for the throttle is $u_\mathrm{e} = 0.1687$ and the parameters are $a = 0.01$, $b = 1.32$, and $b_\mathrm{g} = 9.8$. This linear model describes how small perturbations in the velocity about the nominal speed evolve in time.

We will later describe how to design a proportional-integral (PI) controller for the system. Here we will simply assume that we have obtained a good controller and we will compare the behaviors when the closed loop system is simulated using the nonlinear model and the linear approximation. The simulation scenario is that the car is running with constant speed on a horizontal road and the system has stabilized so that the vehicle speed and the controller output are constant. Figure 6.14 shows what happens when the car encounters a hill with a slope of 4° and a hill with a slope of 6° at time t =5 s. The results for the nonlinear model are solid curves and those for the linear model are dashed curves. The differences between the curves are very small (especially for $\theta = 4°$), and control design based on the linearized model is thus validated. ∇

Jacobian Linearization Around an Equilibrium Point

To proceed more formally, consider a single-input, single-output nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= f(x,u), && x\in\mathbb{R}^n, u\in\mathbb{R}, \\ y &= h(x,u), && y\in\mathbb{R}, \end{aligned} \tag{6.33}$$



(a) Vehicle diagram (b) Closed loop response

Figure 6.14: Simulated response of a vehicle with PI cruise control as it climbs a hill with a slope of 4° (smaller velocity deviation/throttle) and a slope of 6° (larger velocity deviation/throttle). The solid line is the simulation based on a nonlinear model, and the dashed line shows the corresponding simulation using a linear model. The controller gains are $k_\mathrm{p} = 0.5$ and $k_\mathrm{i} = 0.1$ and include anti-windup compensation (described in more detail in Example 11.6).

with an equilibrium point at $x = x_\mathrm{e}$, $u = u_\mathrm{e}$. Without loss of generality we can assume that $x_\mathrm{e} = 0$ and $u_\mathrm{e} = 0$, although initially we will consider the general case to make the shift of coordinates explicit.

To study the *local* behavior of the system around the equilibrium point $(x_\mathrm{e}, u_\mathrm{e})$, we suppose that $x - x_\mathrm{e}$ and $u - u_\mathrm{e}$ are both small, so that nonlinear perturbations around this equilibrium point can be ignored compared with the (lower-order) linear terms. This is roughly the same type of argument that is used when we do small-angle approximations, replacing $\sin\theta$ with θ and $\cos\theta$ with 1 for θ near zero.

We define a new set of state variables z, as well as inputs v and outputs w:

$$z = x - x_\mathrm{e}, \qquad v = u - u_\mathrm{e}, \qquad w = y - h(x_\mathrm{e}, u_\mathrm{e}).$$

These variables are all close to zero when we are near the equilibrium point, and so in these variables the nonlinear terms can be thought of as the higher-order terms in a Taylor series expansion of the relevant vector fields (assuming for now that these exist).

Formally, the *Jacobian linearization* of the nonlinear system (6.33) is

$$\frac{dz}{dt} = Az + Bv, \qquad w = Cz + Dv, \tag{6.34}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_\mathrm{e},u_\mathrm{e})}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_\mathrm{e},u_\mathrm{e})}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{(x_\mathrm{e},u_\mathrm{e})}, \quad D = \left.\frac{\partial h}{\partial u}\right|_{(x_\mathrm{e},u_\mathrm{e})}. \tag{6.35}$$

The system (6.34) approximates the original system (6.33) when we are near the equilibrium point about which the system was linearized. It follows from Theorem 5.3 that if the linearization is asymptotically stable, then the equilibrium point x_e is locally asymptotically stable for the full nonlinear system.

Example 6.12 Cruise control using Jacobian linearization
Consider again the cruise control system from Example 6.11 with θ taken as a constant θ_e. We can write the dynamics as a first-order, nonlinear differential equation:

$$\frac{dx}{dt} = f(x,u) = \frac{\alpha_n}{m} u T(\alpha_n x) - g C_r \operatorname{sgn}(x) - \frac{1}{2}\frac{\rho C_d A}{m} x|x| - g \sin\theta_e,$$
$$y = h(x,u) = x,$$

where $x = v$ is the velocity of the vehicle and u is the throttle. We use the velocity as the output of the system (since this is what we are trying to control).

If we linearize the dynamics of the system about an equilibrium point $x = v_e > 0$, $u = u_e$, using equation (6.35) and the previous formula we obtain

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} = \frac{u_e \alpha_n^2 T'(\alpha_n x_e) - \rho C_d A |x_e|}{m}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)} = \frac{\alpha_n T(\alpha_n x_e)}{m},$$
$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} = 1 \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)} = 0,$$

where we have used the fact that $\operatorname{sgn}(x) = 1$ for $x > 0$. This matches the results in Example 6.11, remembering that we have used x as the system state (vehicle velocity). ∇

It is important to note that we can define the linearization of a system only near a solution of the differential equations for the system, of which an equilibrium point is a particularly common case. To see this, consider a polynomial system

$$\frac{dx}{dt} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + u,$$

where $a_0 \neq 0$. A set of equilibrium points for this system is given by $(x_e, u_e) = (x_e, -a_0 - a_1 x_e - a_2 x_e^2 - a_3 x_e^3)$, and we can linearize around any of them. Suppose that we try to linearize around the origin of the system $x = 0$, $u = 0$ (which does not correspond to a solution of the differential equation if $a_0 \neq 0$). If we drop the higher-order terms in x, then we get

$$\frac{dx}{dt} = a_0 + a_1 x + u,$$

which is *not* the Jacobian linearization if $a_0 \neq 0$. The constant term must be kept, and it is not present in equation (6.34). Furthermore, even if we kept the constant term in the approximate model, the system would quickly move away from this point (since it is "driven" by the constant term a_0), and hence the approximation could soon fail to hold.

Software for modeling and simulation frequently has facilities for performing linearization symbolically or numerically. The MATLAB command `trim` finds the equilibrium point, and `linmod` extracts linear state space models from a SIMULINK system around an equilibrium point. The more general case of linearizing around a trajectory leads to a time-varying linear system.

Example 6.13 Vehicle steering

Consider the vehicle steering system introduced in Example 3.11. The nonlinear equations of motion for the system are given by equations (3.25)–(3.27) and can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} v\,\cos\left(\alpha(\delta)+\theta\right) \\ v\,\sin\left(\alpha(\delta)+\theta\right) \\ \dfrac{v\,\sin\alpha(\delta)}{a} \end{pmatrix}, \qquad \alpha(\delta) = \arctan\Big(\frac{a\tan\delta}{b}\Big).$$

The state of the system is the position x, y of the center of mass and the orientation θ of the vehicle. The control variable is the steering angle δ. Furthermore b is the wheelbase and a is the distance between the center of mass and the rear wheel.

We are interested in the motion of the vehicle about a straight-line path ($\theta = \theta_0$) with constant velocity $v_0 \neq 0$. To find the relevant equilibrium point, we first set $\dot\theta = 0$ and we see that we must have $\delta = 0$, corresponding to the steering wheel being straight. This also yields $\alpha = 0$. Looking at the first two equations in the dynamics, we see that the motion in the xy plane is by definition *not* at equilibrium since $\dot x^2 + \dot y^2 = v^2 \neq 0$. Therefore we cannot formally linearize the full model.

Suppose instead that we are concerned with the lateral deviation of the vehicle from a straight line. For simplicity, we let $\theta_e = 0$, which corresponds to driving along the x axis. We can then focus on the equations of motion in the y and θ directions. With some abuse of notation we introduce the state $x = (y, \theta)$ and $u = \delta$. The system is then in standard form with

$$f(x,u) = \begin{pmatrix} v_0 \sin\left(\alpha(u) + x_2\right) \\ \dfrac{v_0 \sin\alpha(u)}{a} \end{pmatrix}, \qquad \alpha(u) = \arctan\Big(\frac{a\tan u}{b}\Big), \qquad h(x,u) = x_1.$$

The equilibrium point of interest is given by $x = (0,0)$ and $u = 0$. To compute the linearized model around this equilibrium point, we make use of the formulas (6.35). A straightforward calculation yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=0\\u=0}} = \begin{pmatrix} 0 & v_0 \\ 0 & 0 \end{pmatrix}, \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=0\\u=0}} = \begin{pmatrix} av_0/b \\ v_0/b \end{pmatrix},$$
$$C = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=0\\u=0}} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = \left.\frac{\partial h}{\partial u}\right|_{\substack{x=0\\u=0}} = 0,$$

and the linearized system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du \tag{6.36}$$

thus provides an approximation to the original nonlinear dynamics.

The linearized model can be simplified further by introducing normalized variables, as discussed in Section 3.3. For this system, we choose the wheelbase b as the length unit and the time unit as the time required to travel a wheelbase. The normalized state is thus $z = (x_1/b, x_2)$, and the new time variable is $\tau = v_0 t/b$. The model (6.36) then becomes

$$\frac{dz}{d\tau} = \begin{pmatrix} z_2 + \gamma u \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} \gamma \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} z, \tag{6.37}$$

where $\gamma = a/b$. The normalized linear model for vehicle steering with nonslipping wheels is thus a linear system with only one parameter γ. ∇

Feedback Linearization

Another type of linearization is the use of feedback to convert the dynamics of a nonlinear system into those of a linear one. We illustrate the basic idea with an example.

Example 6.14 Cruise control
Consider again the cruise control system from Example 6.11, whose dynamics are given in equation (6.30):

$$m\frac{dv}{dt} = \alpha_n u T(\alpha_n v) - mgC_\mathrm{r}\,\mathrm{sgn}(v) - \frac{1}{2}\rho C_\mathrm{d} A v|v| - mg\sin\theta.$$

If we choose u as a feedback law of the form

$$u = \frac{1}{\alpha_n T(\alpha_n v)}\left(\tilde{u} + mgC_\mathrm{r}\,\mathrm{sgn}(v) + \frac{1}{2}\rho C_\mathrm{d} A v|v|\right), \tag{6.38}$$

then the resulting dynamics become

$$m\frac{dv}{dt} = \tilde{u} + d, \tag{6.39}$$

where $d(t) = -mg\sin\theta(t)$ is the disturbance force due the slope of the road (which may be changing as we drive). If we now define a feedback law for $\tilde{u}$ (such as a proportional-integral-derivative [PID] controller), we can use equation (6.38) to compute the final input that should be commanded.

Equation (6.39) is a linear differential equation. We have essentially "inverted" the nonlinearity through the use of the feedback law (6.38). This requires that we have an accurate measurement of the vehicle velocity v as well as an accurate model of the torque characteristics of the engine, gear ratios, drag and friction characteristics, and mass of the car. While such a model is not generally available (remembering that the parameter values can change), if we design a good feedback law for $\tilde{u}$, then we can achieve robustness to these uncertainties. ∇

More generally, we say that a system of the form

$$\frac{dx}{dt} = f(x,u), \qquad y = h(x),$$

is *feedback linearizable* if there exists a control law $u = \alpha(x, v)$ such that the resulting closed loop system is input/output linear with input v and output y, as shown in Figure 6.15. To fully characterize such systems is beyond the scope of this text, but we note that in addition to changes in the input, the general theory also allows for (nonlinear) changes in the states that are used to describe the system, keeping only the input and output variables fixed. More details of this process can be found in the textbooks by Isidori [Isi95] and Khalil [Kha01].

One case that comes up relatively frequently, and is hence worth special mention,

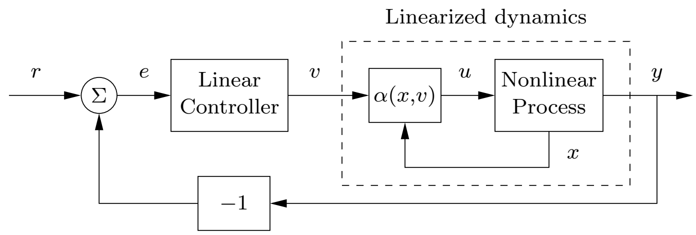


Figure 6.15: Feedback linearization. A nonlinear feedback of the form $u = \alpha(x, v)$ is used to modify the dynamics of a nonlinear process so that the response from the input v to the output y is linear. A linear controller can then be used to regulate the system's dynamics.

is the set of mechanical systems of the form

$$M(q)\ddot{q} + C(q, \dot{q}) = B(q)u.$$

Here $q \in \mathbb{R}^n$ is the configuration of the mechanical system, $M(q) \in \mathbb{R}^{n\times n}$ is the configuration-dependent inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^n$ represents the Coriolis forces and additional nonlinear forces (such as stiffness and friction), and $B(q) \in \mathbb{R}^{n\times p}$ is the input matrix. If $p = n$, then we have the same number of inputs and configuration variables, and if we further have that $B(q)$ is an invertible matrix for all configurations q, then we can choose

$$u = B^{-1}(q)\big(M(q)v + C(q, \dot{q})\big). \tag{6.40}$$

The resulting dynamics become

$$M(q)\ddot{q} = M(q)v \qquad \Longrightarrow \qquad \ddot{q} = v,$$

which is a linear system. We can now use the tools of linear system theory to analyze and design control laws for the linearized system, remembering to apply equation (6.40) to obtain the actual input that will be applied to the system.

This type of control is common in robotics, where it goes by the name of *computed torque*, and in aircraft flight control, where it is called *dynamic inversion*. Some modeling tools like Modelica can generate the code for the inverse model automatically. One caution is that feedback linearization can often cancel out beneficial terms in the natural dynamics, and hence it must be used with care. Extensions that do not require complete cancellation of nonlinearities are discussed in Khalil [Kha01] and Krstić et al. [KKK95].

6.5 Further Reading

The majority of the material in this chapter is classical and can be found in most books on dynamics and control theory, including early works on control such as James, Nichols, and Phillips [JNP47] and more recent textbooks such as Dorf and Bishop [DB04], Franklin, Powell, and Emami-Naeini [FPEN05], and Ogata [Oga01]. An excellent presentation of linear systems based on the matrix exponential is given in the book by Brockett [Bro70], a more comprehensive treatment is given by

Rugh [Rug95], and an elegant mathematical treatment is given in Sontag [Son98]. Material on feedback linearization can be found in books on nonlinear control theory such as Isidori [Isi95] and Khalil [Kha01]. The idea of characterizing dynamics by considering the responses to step inputs is due to Heaviside, who also introduced an operator calculus to analyze linear systems. The unit step is therefore also called the *Heaviside step function*. Analysis of linear systems was simplified significantly, but Heaviside's work was heavily criticized because of lack of mathematical rigor, as described in the biography by Nahin [Nah88]. The difficulties were cleared up later by the mathematician Laurent Schwartz who developed *distribution theory* in the late 1940s. In engineering, linear systems have traditionally been analyzed using Laplace transforms as described in Gardner and Barnes [GB42]. Use of the matrix exponential started with developments of control theory in the 1960s, strongly stimulated by a textbook by Zadeh and Desoer [ZD63]. Use of matrix techniques expanded rapidly when the powerful methods of numeric linear algebra were packaged in programs like LABVIEW, MATLAB, and Mathematica. The books by Gantmacher [Gan60] are good sources for matrix theory.

Exercises

6.1 (Response to the derivative of a signal) Show that if $y(t)$ is the output of a linear time-invariant system corresponding to input $u(t)$, then the output corresponding to an input $\dot{u}(t)$ is given by $\dot{y}(t)$. (Hint: Use the definition of the derivative: $\dot{z}(t) = \lim_{\epsilon\to 0}\bigl(z(t+\epsilon) - z(t)\bigr)/\epsilon$.)

6.2 (Impulse response and convolution) Show that a signal $u(t)$ can be decomposed in terms of the impulse function $\delta(t)$ as

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau$$

and use this decomposition plus the principle of superposition to show that the response of a linear, time-invariant system to an input $u(t)$ (assuming a zero initial condition) can be written as a convolution equation

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response of the system. (Hint: Use the definition of the Riemann integral.)

6.3 (Pulse response for a compartment model) Consider the compartment model given in Example 6.7. Compute the step response for the system and compare it with Figure 6.10b. Use the principle of superposition to compute the response to the 5 s pulse input shown in Figure 6.10c. Use the parameter values $k_0 = 0.1$, $k_1 = 0.1$, $k_2 = 0.5$, and $b_0 = 1.5$.

6.4 (Matrix exponential for second-order system) Assume that $\zeta < 1$ and let $\omega_\mathrm{d} = \omega_0\sqrt{1-\zeta^2}$. Show that

$$\exp\begin{pmatrix} -\zeta\omega_0 & \omega_\mathrm{d} \\ -\omega_\mathrm{d} & -\zeta\omega_0 \end{pmatrix} t = e^{-\zeta\omega_0 t}\begin{pmatrix} \cos\omega_\mathrm{d} t & \sin\omega_\mathrm{d} t \\ -\sin\omega_\mathrm{d} t & \cos\omega_\mathrm{d} t \end{pmatrix}.$$

Also show that

$$\exp\left(\begin{pmatrix} -\omega_0 & \omega_0 \\ 0 & -\omega_0 \end{pmatrix} t\right) = e^{-\omega_0 t}\begin{pmatrix} 1 & \omega_0 t \\ 0 & 1 \end{pmatrix}.$$

6.5 (Lyapunov function for a linear system) Consider a linear system $\dot{x} = Ax$ with $\operatorname{Re}\lambda_j < 0$ for all eigenvalues λ_j of the matrix A. Show that the matrix

$$P = \int_0^\infty e^{A^T\tau} Q e^{A\tau}\, d\tau$$

defines a Lyapunov function of the form $V(x) = x^T P x$ with $Q \succ 0$ (positive definite).

6.6 (Nondiagonal Jordan form) Consider a linear system with a Jordan form that is non-diagonal.

(a) Prove Proposition 6.3 by showing that if the system contains a real eigenvalue $\lambda = 0$ with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time.

(b) Extend this argument to the case of complex eigenvalues with $\operatorname{Re}\lambda = 0$ by using the block Jordan form

$$J_i = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}.$$

6.7 (Rise time and settling time for a first-order system) Consider a first-order system of the form

$$\tau\frac{dx}{dt} = -x + u, \qquad y = x.$$

We say that the parameter τ is the *time constant* for the system since the zero input system approaches the origin as $e^{-t/\tau}$. For a first-order system of this form, show that the rise time for a step response of the system is approximately 2τ, and that 1%, 2%, and 5% settling times approximately corresponds to 4.6τ, 4τ, and 3τ.

6.8 (Discrete-time systems) Consider a linear discrete-time system of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k] + Du[k].$$

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1}Bu[j] + Du[k].$$

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of A have a magnitude strictly less than 1.

(c) Show that a discrete-time linear system is unstable if any of the eigenvalues of A have magnitude greater than 1.

(d) Derive conditions for stability of a discrete-time linear system having one or more eigenvalues with magnitude identically equal to 1. (Hint: Use Jordan form.)

6.9 (Keynesian economics) Consider the following simple Keynesian macroeconomic model in the form of a linear discrete-time system discussed in Exercise 6.8:

$$\begin{pmatrix} C[t+1] \\ I[t+1] \end{pmatrix} = \begin{pmatrix} a & a \\ ab-b & ab \end{pmatrix} \begin{pmatrix} C[t] \\ I[t] \end{pmatrix} + \begin{pmatrix} a \\ ab \end{pmatrix} G[t],$$
$$Y[t] = C[t] + I[t] + G[t].$$

Determine the eigenvalues of the dynamics matrix. When are the magnitudes of the eigenvalues less than 1? Assume that the system is in equilibrium with constant values capital spending C, investment I, and government expenditure G. Explore what happens when government expenditure increases by 10%. Use the values $a = 0.25$ and $b = 0.5$.

6.10 (Keynes model in continuous time) A continuous version of the Keynes model is given by the equations

$$Y = C + I + G, \qquad T\frac{dC}{dt} + C = ay, \qquad T\frac{dI}{dt} + I = b\frac{dc}{dt}.$$

Write the equations in state space form, and give the conditions for stability.

6.11 (State variables in compartment models) Consider the compartment model described by equation (4.28). Let x_1 and x_2 be the total mass of the drug in the compartments. Show that the system can be described by the equation

$$\frac{dx}{dt} = \begin{pmatrix} -k_0 - k_1 & k_2 \\ k_1 & -k_2 \end{pmatrix} x + \begin{pmatrix} c_0 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1/V_2 \end{pmatrix} x. \tag{6.41}$$

Compare the this equation with equation (4.28), where the state variables were concentrations. Mass is called an *extensive variable*, and concentration is called an *intensive variable*.

6.12 (Time responses from frequency responses) Consider the following MATLAB program, which computes the approximate step response from the frequency response. Explain how it works and explore the effects of the parameter `tmax`.

```
P = '1./(s+1).^2';        % process dynamics
tmax = 20;                % simulation time
N = 2^(12);               % number of points for simulation
dt = tmax/N;              % time interval
dw = 2*pi/tmax;           % frequency interval

% Compute the time and frequency vectors
t = dt*(0:N-1);
```

```
omega = -pi/dt:dw:(pi/dt-dw);
s = i*omega;

% Evaluate the frequency response
pv=eval(P);

% Compute the input and output signals using the frequency response
u = [ones(1,N/2) zeros(1,N/2)]; U = fft(u);
y = ifft(fftshift(pv) .* U); y = real(y);

%  Analytic solution in the time domain
ye = 1 - exp(-t) - t .* exp(-t);

% Plot analytic and approximate step responses
subplot(211); plot(t, y, 'b-', t, ye, 'r--');

% Zoom in on the first half of the response
tp = t(1:N/2); yp = y(1:N/2); ye = 1-exp(-t) - t .* exp(-t);
subplot(212); plot(tp, yp, 'b-', t, ye, 'r--');
```

6.13 Consider a scalar system

$$\frac{dx}{dt} = 1 - x^3 + u.$$

Compute the equilibrium points for the unforced system ($u = 0$) and use a Taylor series expansion around the equilibrium point to compute the linearization. Verify that this agrees with the linearization in equation (6.34).

6.14 Consider the model for queuing dynamics in Example 3.15. Let the admission rate λ be the control variable. Linearize the system around an equilibrium point, compute the time constant of the system and determine how it depends on the queue length.

6.15 (Transcriptional regulation) Consider the dynamics of a genetic circuit that implements *self-repression*: the protein produced by a gene is a repressor for that gene, thus restricting its own production. Using the models presented in Example 3.18, the dynamics for the system can be written as

$$\frac{dm}{dt} = \frac{\alpha}{1 + kp^2} + \alpha_0 - \delta m - u, \qquad \frac{dp}{dt} = \kappa m - \gamma p, \tag{6.42}$$

where u is a disturbance term that affects RNA transcription and $m, p \geq 0$. Find the equilibrium points for the system and use the linearized dynamics around each equilibrium point to determine the local stability of the equilibrium point and the step response of the system to a disturbance.

6.16 (Monotone step response) Consider a stable linear system with monotone step response $S(t)$. Let the input signal be bounded: $|u(t)| \leq u_{\max}$. Assuming that the initial conditions are zero, show that $|y(t)| \leq S(\infty)u_{\max}$. (Hint: Use the convolution integral.)