Feedback Systems:
Supplemental Exercises and Solutions Manual

Karl Johan Åström
Richard M. Murray

2020-04-04

About this document

This manual contains exercises and solutions for *Feedback Systems* (second edition). In addition to the exercises printed in the text, additional exercises are included here that are of a more specialized nature or too long to easily fit in the book. (Note: some of the solutions for the supplemental exercises are not yet complete and others have not been as carefully checked for errors. Use with caution.)

Text marked in black is included directly from the printed book; text marked in green is not included in the printed book, but may appear in some of the course supplements (including supplements describing running examples). Solutions and instructor notes are marked in blue. Corrections to errors in the printed book have usually been applied, so some problems may appear slighly different than in the print version. All changes are listed on the book errata page, available via the companion website:

http://fbsbook.org

Many of the solutions in the text were contributed by students and colleagues. Primary contributors for problems and solutions are listed in the individual exercises. Problems and solutions without contributors listed are contributions from the primary authors, often based on problems assigned in courses.

This version of the solutions is formatted to be as compact as possible; a "one per page" version of the solutions is also available to allow easier distribution of selected solutions. The authors request that the solutions provided here not be posted to publicly accessible Internet sites or otherwise made available to students outside of the course for which it is being used. Instructors are welcome to contribute additional exercises or improved solutions, which will be included in future editions of the solutions manual at the authors' discretion. Interested instructors should contact the authors for information about formatting and style for contributed exercises and solutions.

Revision history

2.1a [13 Sep 2017] New version (not yet debugged!!)

2.0a [30 Sep 2012] First cut at solution manual for second edition

Contents

Chapter 1 – Introduction

1.1 Identify five feedback systems that you encounter in your everyday environment. For each system, identify the sensing mechanism, actuation mechanism, and control law. Describe the uncertainty with respect to which the feedback system provides robustness and/or the dynamics that are changed through the use of feedback. [B,1ep] intro:fbkexamps

Instructor note: The main point of this problem is to get students to think about what a feedback/control system is and what the different elements are.

1.2 (Balance systems) Balance yourself on one foot with your eyes closed for 15 s. Using Figure 1.4 as a guide, describe the control system responsible for keeping you from falling down. Note that the "controller" will differ from that in the diagram (unless you are an android reading this in the far future). [C,1ep] intro:balance-human

Soln: Soln needs to be updated

1.3 (Eye motion) Perform the following experiment and explain your results: Holding your head still, move one of your hands left and right in front of your face, following it with your eyes. Record how quickly you can move your hand before you begin to lose track of it. Now hold your hand still and shake your head left to right, once again recording how quickly you can move before losing track of your hand. Explain any difference in performance by comparing the control systems used to implement these behaviors. [B,1ep] intro:eyemotion

1.4 (Cruise control) Download the MATLAB code used to produce simulations for the cruise control system in Figure 1.11 from the companion web site. Using trial and error, change the parameters of the control law so that the overshoot in speed is not more than 1 m/s for a vehicle with mass $m = 1200$ kg. Does the same controller work if we set $m = 2000$ kg? [B,1ep] intro:cruise-redesign

1.5 (Integral action) We say that a system with a constant input reaches steady state if all system variables approach constant values as time increases. Show that a controller with integral action, such as those given in equations (1.4) and (1.5), gives zero error if the closed loop system reaches steady state. Notice that there is no saturation in the controller. [A,1ep*] intro:integral

Instructor note: This exercise is worked out in Section 11.1.

1.6 (Combining feedback with logic) Consider a system for cruise control where the overall function is governed by the state machine in Figure 1.16. Assume that the system has a continuous input for vehicle velocity, discrete inputs indicating braking and gear changes, and a PI controller with inputs for the reference and measured velocities and an output for the control signal. Sketch the actions that have to be taken in the states of the finite state machine to handle the system properly. Think about if you have to store some extra variables, and if the PI controller has to be modified. [C,2e] intro:logic-fbk

1.7 Search the web and pick an article in the popular press about a feedback and control system. (On the companion web site under "Popular articles about control" you can find links to some recent articles from the New York Times and other sources.) Describe the feedback system using the terminology given in the [C,1ep] intro:nytexamps article. In particular, identify the control system and describe (a) the underlying process or system being controlled, along with the (b) sensor, (c) actuator, and (d) computational element. If the some of the information is not available in the article, indicate this and take a guess at what might have been used.

*Instructor note:*The goal of this exercise is to have students read a bit about popular descriptions of control systems and relate this to the terminology in Chapter 1.

Supplemental Exercises

[C,1es] intro:milksupply

1.8 Make a schematic picture of the system for supplying milk from a cow to your table. Discuss the impact of refrigerated storage.

[C,1es] intro:cruise-mlintro

1.9 (MATLAB/SIMULINK) Download the file "`cruise_ctrl.mdl`" from the companion web site. It contains a SIMULINK model of a simple cruise controller, similar to the one described in Section 1.5. Figure out how to run the example and plot the vehicle's speed as a function of time.

(a) Leaving the control gains at their default values, plot the response of the system to a step input and measure the time it takes for the system error to settle to within 5% of commanded change in speed (i.e., 0.5 m/s).

(b) By manually tuning the control gains, design a controller that settles at least 50% faster than the default controller. Include the gains you used, a plot of the closed loop response, and describe any undesirable features in the solution you obtain.

All plots should included a title, labeled axes (with units), and reasonable axis limits.

*Instructor note:*The exercise is a variation of Exercise 1.4 above. The purpose of these problem is to give students some familiarity with MATLAB and SIMULINK. The instructor may want to indicate in the problem that students shouldn't worry if they don't yet know how the control law works or why it does what it does.

[C,1es] intro:ballbeam-mlintro

1.10 (MATLAB/SIMULINK) Download the file "ballbeam.mdl" from the companion web site. It contains a SIMULINK model of a "ball and beam" experiment in which you apply a torque to a beam and try to balance a ball that rolls along the beam (see course web page for more documentation).

(a) Run the simulation with default parameters and create a plot of the ball position versus time. Note that the desired action of the system is to move the ball from its initial position at the center of the beam to a new resting point at $r = 0.25$ m.

(b) While keeping the gain on $\dot{\alpha}$ fixed at its default value, vary the gain on α from 75% to three times the default value. Plot the "overshoot" (the maximum amount by which the ball goes past the desired resting point, expressed as a percentage of the commanded position) as a function of this gain for stable cases.

(c) While keeping gain on α fixed at its default value, vary the gain on $\dot{\alpha}$ from zero to twice its default value. Give the numerical range of this gain for which the system is stable. Plot the "settling time" (amount of time required for the system to get within 5% of the desired resting point) as a function of this gain for the stable cases.

All plots should included a title, labeled axes (with units), and reasonable axis limits.

1.11 Search for the term "voltage clamp" on the Internet and explore why it is so advantageous to use feedback to measure the ion current in cells. You may also enjoy reading about the Nobel Prizes received by Hodgkin and Huxley 1963 and Neher and Sakmann (see http://nobelprizes.org). [D,1es] intro:voltageclamp

1.12 Search for the term "force feedback" and explore its use in haptics and sensing. [D,1es] intro:haptics

1.13 Read the April 2007 Detroit News article "Officials mandate anti-rollover rule" (available from the companion web site). This article talks about new regulations that are being proposed to use anti-rollover technology in cars sold in the U.S. beginning in 2012. By reading the article and the companion articles on the National Highway Traffic Safety Administration (NHTSA) web site, identify the sensing and actuation systems that will be used, and summarize how the control algorithm for the system works. [D,1es] intro:rollover

Chapter 2 – Feedback Principles

Comment [RMM, 13 Sep 2018]: Exercise titles are not used at all in this chapter, which is different from other chapters. Should probably fix.

2.1 Let $y \in \mathbb{R}$ and $u \in \mathbb{R}$. Solve the differential equations [B,2e] principles:ode1

$$\frac{dy}{dt} + ay = bu, \qquad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\frac{du}{dt} + u,$$

for $t > 0$. Determine the responses to a unit step $u(t) = 1$ and the exponential signal $u(t) = e^{st}$ when the initial condition is zero. Derive the transfer functions of the systems.

2.2 Let $y_0(t)$ be the response of a system with the transfer function $G_0(s)$ to a given input. The transfer function $G(s) = (1 + sT)G_0(s)$ has the same zero frequency gain but it has an additional zero at $z = -1/T$. Let $y(t)$ be the response of the system with the transfer function $G(s)$ and show that [A,2e] principles:rhp-zero

$$y(t) = y_0(t) + T\frac{dy_0}{dt}, \tag{S2.1}$$

Next consider the system with the transfer function

$$G(s) = \frac{s+a}{a(s^2+2s+1)},$$

which has unit zero-frequency-gain ($G(0) = 1$). Use the result in equation (S2.1) to explore the effect of the zero $s = -1/T$ on the step response of a system

2.3 Consider a closed loop system with process dynamics and a PI controller modeled by [B?,2e] principles:ode-trf

$$\frac{dy}{dt} + ay = bu, \qquad u = k_\mathrm{p}(r - y) + k_\mathrm{i}\int_0^t \bigl(r(\tau) - y(\tau)\bigr)\,d\tau,$$

where r is the reference, u is the control variable, and y is the process output.

(a) Derive a differential equation relating the output y to the reference r by direct manipulation of the equations and compute the transfer function $H_{yr}(s)$. Make the derivations both by direct manipulation of the differential equations and by polynomial algebra.

(b) Draw a block diagram of the system and derive the transfer functions of the process $P(s)$ and the controller $C(s)$.

(c) Use block diagram algebra to compute the transfer function from reference r to output y of the closed loop system and verify that your answer matches your answer in part (a).

2.4 Consider the system described by the differential equation (2.10) and the transfer function (2.16). Determine the zero frequency gain of the system by computing the particular solution of (2.10) for a constant input $u(t) = u_0$. Compare with the $G(0)$. [B,2e] principles:zerofreqgain

[B?,2e] principles:pupilreflex

2.5 (Pupil response) The dynamics of the pupillary reflex is approximated by a linear system with the transfer function

$$P(s) = \frac{0.2(1-0.1s)}{(1+0.1s)^3}.$$

Assume that the nerve system that controls the pupil opening is modeled as a proportional controller with the gain k. Use the Routh–Hurwitz criterion to determine the largest gain that gives a stable closed loop system.

[B,2e] principles:sensitivity

2.6 Consider the feedback system in Figure 2.7. Let the disturbance $v = 0$, $P(s) = 1$ and $C(s) = k_\mathrm{i}/s$. Determine the transfer function G_{yr} from reference r to output y. Also determine how much G_{yr} is changed when the process gain changes by 10%.

[B,2e] principles:piddes

2.7 (PID control design) The calculations in Section 2.3 can be interpreted as a design method for a PI controller for a first-order system. A similar calculation can be made for PID control of a second order system. Let the transfer functions of the process and the controller be

$$P(s) = \frac{b}{s^2 + a_1 s + a_2}, \qquad C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s} + k_\mathrm{d} s.$$

Show that the controller parameters

$$k_\mathrm{p} = \frac{(1+2\alpha\zeta_\mathrm{c})\omega_\mathrm{c}^2 - a_2}{b}, \qquad k_\mathrm{i} = \frac{\alpha\omega_\mathrm{c}^3}{b}, \qquad k_\mathrm{d} = \frac{(\alpha+2\zeta_\mathrm{c})\omega_\mathrm{c} - a_1}{b}.$$

give a closed loop system with the characteristic polynomial

$$(s^2 + 2\zeta_\mathrm{c}\omega_\mathrm{c} s + \omega_\mathrm{c}^2)(s + \alpha\omega_\mathrm{c}).$$

[B,2e] principles:nl-pcon

2.8 Consider an open loop system with the nonlinear input/output relation $y = F(u)$. Assume that the system is closed with the proportional controller $u = k(r-y)$. Show that the input/output relation of the closed loop system is

$$y + \frac{1}{k} F^{-1}(y) = r.$$

Estimate the largest deviation from ideal linear response $y = r$. Illustrate by plotting the input output responses for a) $F(u) = \sqrt{u}$ and b) $F(u) = u^2$ with $0 \le u \le 1$ and $k = 5, 10$ and 100.

[B,2e*] principles:handel

2.9 (Nonlinear distortion) The following MATLAB commands will load and play Handel's Messiah

```
load handel              % Load Handel's Messiah
sound(y, Fs); pause      % Play the original music through speaker
```

Write a MATLAB function that implements nonlinear amplifier with static gain

$$y = 2(z + az(1-z) - 0.5), \qquad z = (x+1)/2,$$

where x is the original signal (assumed to take values between -1 and 1) and a is the amplifier gain. Compare the sound that is obtained when the music is then sent through two amplifiers with the given nonlinearity and gain $a = 1$ versus when the music is sent through the same two amplifiers with feedback $k = 10$.

2.10 Consider the system in Section 2.3 where the controller was designed to give a closed loop system characterized by $\omega_c = 1$ and $\zeta = 0.707$. The transfer functions of the process and the controller are [B,2e] principles:ffandfb

$$P(s) = \frac{2}{s+1}, \qquad C(s) = \frac{0.207s + 0.5}{s}.$$

The response of the closed loop system to step inputs has a settling time (time required to stay within 2% of the final value, see Figure 6.9) of $4/\zeta\omega_c \approx 5.66$. Assume that the attenuation of the load disturbances is satisfactory but that we want a closed loop system system that responds five times faster to reference signals without overshoot. Determine the transfer functions of a controller with the architecture in Figure **??** that gives a response to command signals with a first-order dynamics. Simulate the system in the nominal case of a perfect model and explore the effects of modeling errors by simulation.

2.11 Consider the system in Figure **??**. Let the transfer functions of the process and the feedback controller be $P(s)$ and $C(s)$ and let the feedforward generator be charactrized by the transfer functions $G_{u_{\mathrm{ff}}r}$ and $G_{y_{\mathrm{m}}r}$. Show that the transfer functions that relate output y and control u to reference r, load disturbance v, and measurement noise w are given by [B,2e] principles:twodofgen

$$G_{yr} = P(s)G_{u_{\mathrm{ff}}r}, \qquad G_{ur}.$$

2.12 (Queing systems) Consider a queuing system modeled by [B?,2e] principles:queue-adm-ctrl

$$\frac{dx}{dt} = \lambda - \mu_{\max}\frac{x}{x+1},$$

where λ is the acceptance rate of jobs and x is the length of the queue. The model is nonlinear and the dynamics of the system changes significantly with the queuing length (see Example 3.15 for a more detailed discussion). Investigate the situation when a PI controller is used for admission control. Let r be the rate of arrival of job requests and model the (average) arrival intensity λ as

$$\lambda = k_{\mathrm{p}}(r - x) + k_{\mathrm{i}}\int^t (r(t) - x(t))dt.$$

The controller parameters are determined from the approximate model

$$\frac{dx}{dt} = \lambda.$$

Find controller parameters that give the closed loop characteristic polynomial $s^2 + 2s + 1$ for the approximate model. Investigate the behavior of the control strategy for the full nonlinear model by simulation for the input $r = 5 + 4\sin(0.1t)$.

Supplemental Exercises

[B,2e] principles:posfbk
Comment [RMM, 21 May 2019]: Already covered in the chapter?

2.13† Consider the system in Figure 2.1, where $F(w) = \text{sat}(w)$ with a negative sign in the feedback. Assume that $r = 0$ and $v = 1$. Sketch the input/output relation for $k = -3, -2, -1, 0, 1, 2$.

[B,2e] principles:cruisecon

2.14 (Cruise control)

Comment [RMM, 23 Aug 2019]: This exercise appears as part of Example 11.3 and the results of the example give the solution $\Longrightarrow$ moving this to supplemental exercises.

A simple model for the relation between speed v and throttle u for a car is given by the transfer function

$$G_{vu} = \frac{b}{s+a}$$

where $a = 0.01$ rad/s and $b = 1.32\ \text{m/s}^2$ (see Section 4.1 and Example 6.11 for more details). The control signal is normalized to the range $0 \le u \le 1$. Design a PI controller for the system that gives a closed loop characteristic polynomial

$$a_{\text{cl}}(s) = s^2 + 2\zeta_{\text{c}}\omega_{\text{c}} s + \omega_{\text{c}}^2.$$

What are the consequences of choosing different values of the design parameters ζ_{c} and ω_{c}? Use your judgment to find suitable values. Hint: Investigate maximum acceleration and maximum velocity for step changes in the velocity reference.

[B,2e] principles:ode2

2.15 Let $x \in R$ and $u \in R$. Solve the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2\frac{du}{dt} + u.$$

Determine the responses to a unit step $u(t) = 1$ and the exponential signal $u(t) = e^{st}$ when the initial conditions are zero. Derive the transfer function of the system.

[C,2e] principles:routh-hurwitz

2.16 Prove the Routh–Hurwitz criterion for a third order polynomial with real coefficients.

[C,2e] principles:approx

2.17 Consider a system with the transfer function $G(s) = \frac{s}{1+sT}$. Use MATLAB to investigate the response of the system to the input signal $u(t) = te^{-t}$ for $T = 0.01,\ 0.1,\ 1,\ 10\ 100$. Compare the responses with the low- and high-frequency approximations $G(s) = G_{low} \approx s$ and $G(s) = G_{high} \approx 1/T$.

[C,2e] principles:trf-sim1

2.18 Reproduce the simulation of Figure 2.9.

[B,2e] principles:distred

2.19 Consider the system in Figure 2.10. Derive the transfer function $G_{yr}(s)$ given by equation (2.36). Assume that there is additive measurement noise at the process output. Derive the transfer function from measurement noise to control signal.

[C,2e] principles:pendcart

2.20 (Cart–pendulum system) The equation of motion of a pendulum on a cart is given in equation (**??**). The motion of the pendulum and the cart can then be modeled by the equations

$$J_{\text{t}}\ddot{\theta} - mgl\theta = u, \qquad \ddot{x} = u, \tag{S2.2}$$

where x is the position of the cart. Show that it is possible to stabilize both the pendulum and the cart with the feedback law

$$u = -k_1\dot{\theta} - k_2\theta - k_3\dot{x} - k_4 x.$$

Discuss the signs of the feedback gains and relate them to the physical situations. To simplify the calculations you can assume that $J_t = 1$ and that $mgl = 1$. †

KJA: Solution incomplete

2.21 Consider the closed loop system in Figure **??** where the process and the feedback controller have the transfer functions

[B,2e] principles:ffandfbsimp

$$P(s) = \frac{b}{s+a}, \qquad C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s}.$$

Assume that the desired response to command signals is given by the transfer function

$$F_\mathrm{m}(s) = \frac{a_m}{s+a_m}.$$

Determine the feedforward transfer function $F_\mathrm{ur}(s)$ that gives the desired transfer function F_m. Determine the transfer functions G_{yr} and G_{yv} which tell how the closed loop system responds to reference r and load disturbance v.

2.22 (Cruise control) Consider the cruise control example discussed in Section 1.5, with

[C,2e] principles:cruise-fbkparam

RMM: Nonstandard use of parameters compared to other cruise control examples [RMM, 18 Aug 2019]

$$m\dot{v} = -av + u + w$$

where u is the control input (force applied by engine) and w the disturbance input (force applied by hill, etc.), which will be ignored below ($w = 0$). An *open loop* control strategy to achieve a given reference speed v_ref would be to choose

$$u = \hat{a}v_\mathrm{ref}$$

where $\hat{a}$ is your estimate of a, which may not be accurate.

(a) Compute the steady-state response for both the open loop strategy above, and for the feedback law

$$u = -k_\mathrm{p}(v - v_\mathrm{ref})$$

and compare the steady state (with $w = 0$) as a function of $\beta = a/\hat{a}$ when $k_\mathrm{p} = 10\hat{a}$. (You should solve the problem analytically, and then plot the response $v_\mathrm{ss}/v_\mathrm{ref}$ as a function of β.)

(b) Now consider a proportional-integral (PI) control law

$$u = -k_\mathrm{p}(v - v_\mathrm{ref}) - k_\mathrm{i}\int_0^t (v - v_\mathrm{ref})dt$$

and again compute the steady-state solution (assuming stability) and compare the response with the proportional gain case from above. (Note that if you define $q = \int_0^t (v - v_\mathrm{ref})dt$ then $\dot{q} = v - v_\mathrm{ref}$.)

Chapter 3 – Modeling

3.1 (Chain of integrators form) Consider the linear ordinary differential equation (3.7). Show that by choosing a state space representation with $x_1 = y$, the dynamics can be written as [C,1ep] modeling:chainofint

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & \dots & 0 & 0 \end{pmatrix}.$$

This canonical form is called the *chain of integrators* form.

3.2 (Discrete-time dynamics)Consider the following discrete-time system [B,1ep] modeling:discrete-solutions

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k],$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

In this problem, we will explore some of the properties of this discrete-time system as a function of the parameters, the initial conditions, and the inputs.

(a) For the case when $a_{12} = 0$ and $u = 0$, give a closed form expression for the output of the system.

(b) A discrete system is in *equilibrium* when $x[k+1] = x[k]$ for all k. Let $u = r$ be a constant input and compute the resulting equilibrium point for the system. Show that if $|a_{ii}| < 1$ for all i, all initial conditions give solutions that converge to the equilibrium point.

(c) Write a computer program to plot the output of the system in response to a unit step input, $u[k] = 1$, $k \geq 0$. Plot the response of your system with $x[0] = 0$ and A given by $a_{11} = 0.5$, $a_{12} = 1$, and $a_{22} = 0.25$.

3.3 (Keynesian economics) Keynes' simple model for an economy is given by [B,1ep*] modeling:discrete-keynes

$$Y[k] = C[k] + I[k] + G[k],$$

where Y, C, I, and G are gross national product (GNP), consumption, investment and government expenditure for year k. Consumption and investment are modeled by difference equations of the form

$$C[k+1] = aY[k], \qquad I[k+1] = b(C[k+1] - C[k]),$$

where a and b are parameters. The first equation implies that consumption increases with GNP but that the effect is delayed. The second equation implies that investment is proportional to the rate of change of consumption.

Show that the equilibrium value of the GNP is given by

$$Y_e = \frac{1}{1-a} G_e,$$

where the parameter $1/(1-a)$ is the Keynes multiplier (the gain from G to Y). With $a = 0.75$ an increase of government expenditure will result in a fourfold increase of GNP. Also show that the model can be written as the following discrete-time state model:

$$\begin{pmatrix} C[k+1] \\ I[k+1] \end{pmatrix} = \begin{pmatrix} a & a \\ ab-b & ab \end{pmatrix} \begin{pmatrix} C[k] \\ I[k] \end{pmatrix} + \begin{pmatrix} a \\ ab \end{pmatrix} G[k],$$
$$Y[k] = C[k] + I[k] + G[k].$$

[B,1ep]
modeling:sysid-leastsq

3.4 (Least squares system identification) Consider a nonlinear differential equation that can be written in the form

$$\frac{dx}{dt} = \sum_{i=1}^{M} \alpha_i f_i(x),$$

where $f_i(x)$ are known nonlinear functions and α_i are unknown, but constant, parameters. Suppose that we have measurements (or estimates) of the full state x at time instants $t_1, t_2, \ldots, t_N$, with $N > M$. Show that the parameters α_i can be estimated by finding the least squares solution to a linear equation of the form

$$H\alpha = b,$$

where $\alpha \in \mathbb{R}^M$ is the vector of all parameters and $H \in \mathbb{R}^{N\times M}$ and $b \in \mathbb{R}^N$ are appropriately defined.

[A,1ep*]
modeling:oscillator

3.5 (Normalized oscillator dynamics)Consider a damped spring–mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = F.$$

Let $\omega_0 = \sqrt{k/m}$ be the natural frequency and $\zeta = c/(2\sqrt{km})$ be the damping ratio.

(a) Show that by rescaling the equations, we can write the dynamics in the form

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2 u, \tag{S3.1}$$

where $u = F/k$. This form of the dynamics is that of a linear oscillator with natural frequency ω_0 and damping ratio ζ.

(b) Show that the system can be further normalized and written in the form

$$\frac{dz_1}{d\tau} = z_2, \qquad \frac{dz_2}{d\tau} = -z_1 - 2\zeta z_2 + v. \tag{S3.2}$$

The essential dynamics of the system are governed by a single damping parameter ζ. The *Q-value*, defined as $Q = 1/2\zeta$, is sometimes used instead of ζ.

(c) Show that the solution for the unforced system ($v = 0$) with no damping ($\zeta = 0$) is given by

$$z_1(\tau) = z_1(0)\cos\tau + z_2(0)\sin\tau, \qquad z_2(\tau) = -z_1(0)\sin\tau + z_2(0)\cos\tau.$$

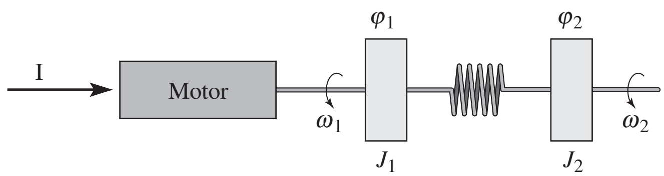
Invert the scaling relations to find the form of the solution $q(t)$ in terms of $q(0)$, $\dot{q}(0)$, and ω_0.

3.6 (Dubins car) Show that the trajectory of a vehicle with reference point chosen as the center of the rear wheels can be modeled by dynamics of the form [B,2e] modeling:steering-dubins

$$\frac{dx}{dt} = v\cos\theta, \qquad \frac{dy}{dt} = v\sin\theta, \qquad \frac{d\theta}{dt} = \frac{v}{b}\tan\delta,$$

where the variables and constants are defined as in Example 3.11.

3.7 (Motor drive) Consider a system consisting of a motor driving two masses that are connected by a torsional spring, as shown in the diagram below. [B,1ep] modeling:dcmotor-modeling



This system can represent a motor with a flexible shaft that drives a load. Assuming that the motor delivers a torque that is proportional to the current I, the dynamics of the system can be described by the equations

$$\begin{aligned} J_1\frac{d^2\varphi_1}{dt^2} + c\Big(\frac{d\varphi_1}{dt} - \frac{d\varphi_2}{dt}\Big) + k(\varphi_1 - \varphi_2) &= k_I I,\\ J_2\frac{d^2\varphi_2}{dt^2} + c\Big(\frac{d\varphi_2}{dt} - \frac{d\varphi_1}{dt}\Big) + k(\varphi_2 - \varphi_1) &= T_{\mathrm{d}}, \end{aligned} \tag{S3.3}$$

where φ_1 and φ_2 are the angles of the two masses, $\omega_i = d\varphi_i/dt$ are their velocities, J_i represents moments of inertia, c is the damping coefficient, k represents the shaft stiffness, k_I is the torque constant for the motor, and T_{d} is the disturbance torque applied at the end of the shaft. Similar equations are obtained for a robot with flexible arms and for the arms of DVD and optical disk drives.

Derive a state space model for the system by introducing the (normalized) state variables $x_1 = \varphi_1$, $x_2 = \varphi_2$, $x_3 = \omega_1/\omega_0$, and $x_4 = \omega_2/\omega_0$, where $\omega_0 = \sqrt{k(J_1+J_2)/(J_1J_2)}$ is the undamped natural frequency of the system when the control signal is zero.

3.8 (Electric generator) An electric generator connected to a strong power grid can be modeled by a momentum balance for the rotor of the generator: [B,1ep] modeling:powergrid

$$J\frac{d^2\varphi}{dt^2} = P_{\mathrm{m}} - P_{\mathrm{e}} = P_{\mathrm{m}} - \frac{EV}{X}\sin\varphi,$$

where J is the effective moment of inertia of the generator, φ is the angle of rotation, P_{m} is the mechanical power that drives the generator, P_{e} is the active electrical power, E is the generator voltage, V is the grid voltage, and X is the reactance of the line. Assuming that the line dynamics are much faster than the rotor dynamics, $P_{\mathrm{e}} = VI = (EV/X)\sin\varphi$, where I is the current component in phase with the voltage E and φ is the phase angle between voltages E and V. Show that the dynamics of the electric generator has a normalized form that is similar to the dynamics of a pendulum with forcing at the pivot.

[B,1ep] modeling:queue-admcontrol

3.9 (Admission control for a queue) Consider the queuing system described in Example 3.15. The long delays created by temporary overloads can be reduced by rejecting requests when the queue gets large. This allows requests that are accepted to be serviced quickly and requests that cannot be accommodated to receive a rejection quickly so that they can try another server. Consider an admission control system described by

$$\frac{dx}{dt} = \lambda u - \mu_{\max}\frac{x}{x+1}, \qquad u = \mathrm{sat}_{(0,1)}(k(r-x)), \tag{S3.4}$$

where the controller is a simple proportional control with saturation ($\mathrm{sat}_{(a,b)}$ is defined by equation (4.10))† and r is the desired (reference) queue length. Use a simulation to show that this controller reduces the rush-hour effect and explain how the choice of r affects the system dynamics.

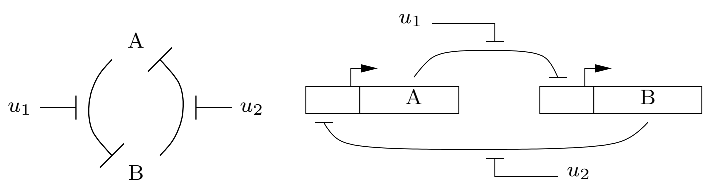
Note: Forward reference; probably OK [RMM, 2 Sep 2016]

You should choose the parameters of your simulation to match those in Example 3.15: $\mu_{\max} = 1$, $\lambda = 0.5$ at time 0, increasing to $\lambda = 4$ at time 20 and returning to $\lambda = 0.5$ at time 25. Test your controller using $r = 2$ and $r = 5$ and explore several different values for k. Your solution should include the MATLAB code that you used plus plots for the final value of k you chose (and the two values of r). Make sure to label your plots and describe how your controller reduces the rush hour effect.

Instructor note: The choice of the gain k is left open to allow students to explore how different gains reduce the rush-hour effect. A simpler version of this problem (focusing only on the use of MATLAB) can be obtained by specifying $k = 1$ (which is what is used in the solution below).

[B,1ep*] modeling:biocircuits-switchmod

3.10 (Biological switch) A genetic switch can be formed by connecting two repressors together in a cycle as shown below.



Using the models from Example 3.18—assuming that the parameters are the same for both genes and that the mRNA concentrations reach steady-state quickly—show that the dynamics can be written in normalized coordinates as

$$\frac{dz_1}{d\tau} = \frac{\mu}{1+z_2^n} - z_1 - v_1, \qquad \frac{dz_2}{d\tau} = \frac{\mu}{1+z_1^n} - z_2 - v_2, \tag{S3.5}$$

where z_1 and z_2 are scaled versions of the protein concentrations and the time scale has also been changed. Show that $\mu \approx 200$ using the parameters in Example 3.18, and use simulations to demonstrate the switch-like behavior of the system.

[B,2e] modeling:fitz-hugh

3.11 (FitzHugh–Nagumo) The second-order FitzHugh–Nagumo equations

$$\frac{dV}{dt} = 10(V - V^3/3 - R + I_{in}), \qquad \frac{dR}{dt} = 0.8(1.25V - R + 1.5),$$

are a simplified version of the Hodgkin–Huxley equations discussed in Example 3.19. The variable V is the voltage across the axon membrane and R is an auxiliary variable that approximates several ion currents that flow across the membrane. Simulate the equations and reproduce the simulation in Figure 3.28. Explore the effect of the input current I_{in}.

Comment [RMM, 26 Dec 2019]: Add more exercisese to this chapter (12 minimum) and fill up blank space

Supplemental Exercises

3.12 (Inverted pendulum)Use the equations of motion for a balance system to derive a dynamical model for the inverted pendulum described in Example 3.3 and verify that the dynamics are given by equation (3.10). [C,1ep] modeling:balance-bal2inv

3.13 (Hodgkin–Huxley equations) The original Hodgkin-Huxley equation has four states. Several approximations have been made to obtain simpler models with similar behavior. One simplification is the second order equations [B,1ep] modeling:hodgkin-huxley

$$
\begin{aligned}
C\,\frac{dV}{dt} &= -(17.81 + 47.71V + 32.63V^2)(V - 0.55) - 26R(V + 0.92) + I\\
\tau\,\frac{dR}{dt} &= -R + 1.35V + 1.03,
\end{aligned}
$$

with $C = 0.8$, $\tau = 1.9$ [ms] and $I = 1$ [mA] (see [9, p. 192]). The variables have been scaled: the time unit is [ms], the voltages have the unit [dV], and the variables have to be multiplied by 100 to obtain their values in millivolts. Simulate the equations, calculate equilibrium points, and try to explain their behavior.

3.14 (Uncertainty lemon) Consider a system where the control variable is a voltage with a maximum value of $u_{\text{max}} = 10$ V, and a noise floor of $u_{\text{min}} = 10$ mV. The actuator has a rate limitation of $v_{\text{max}} = 20$ V/s. Also assume that the electronics driving the actuator has a drift that corresponds to an input signal of $v_{\text{drift}} = 40$ mV/h. Sketch the uncertainty lemon of the system reflected to the input variable. [C,1es] modeling:lemon

3.15 (Frequency response) Consider the spring–mass system given by equation (3.16). Using a simulation, compute the response of the system to a sinusoidal force $u = A \sin \omega t$ for $\omega = \sqrt{k/m}$ and some neighboring frequencies. Determine the amplitude and the phase for the steady-state solution. [C,1es] modeling:freqresp

3.16 (Open loop control versus closed loop control) [Contributed by D. MacMartin, 2011] Consider the cruise-control example discussed in class,† with [C,1es] modeling:cruise-openpi

RMM: Update to refer to Intro? Update parameters

$$m\dot{v} = -av + u + w$$

where u is the control input (force applied by engine) and w the disturbance input (force applied by hill, etc.), which will be ignored below ($w = 0$). An *open loop* control strategy to achieve a given reference speed v_{ref} would be to choose

$$u = \hat{a} v_{\text{ref}}$$

where $\hat{a}$ is your estimate of a, which may not be accurate.

(a) Compute the steady-state response for both the open loop strategy above, and for the feedback law

$$u = -k_{\mathrm{p}}(v - v_{\mathrm{ref}})$$

and compare the steady-state (with $w = 0$) as a function of $\beta = a/\hat{a}$ when $k_{\mathrm{p}} = 10\hat{a}$. (You should solve the problem analytically, and then plot the response $v_{\mathrm{ss}}/v_{\mathrm{ref}}$ as a function of β.)

(b) Now consider a proportional-integral (PI) control law

$$u = -k_{\mathrm{p}}(v - v_{\mathrm{ref}}) - k_{\mathrm{i}} \int_0^t (v - v_{\mathrm{ref}})dt$$

and again compute the steady-state solution (assuming stability) and compare the response with the proportional gain case from above. (Note that if you define $q = \int_0^t (v - v_{\mathrm{ref}})dt$ then $\dot{q} = v - v_{\mathrm{ref}}$.)

(c) Next, simulate the response of the system (using `ode45` in Matlab or similar) with the PI control law above with $m = 1$, $a = 0.1$, $w = 0$, and "input" to the system of $v_{\mathrm{ref}} = \sin(\omega t)$, for ω=0.01, 0.1, 1, and 10 rad/sec. In each case, you should simulate at least 10 cycles; after some initial transient, the response should be periodic. Compute the peak-to-peak amplitude of the final period for the error $v - v_{\mathrm{ref}}$, and plot this as a function of frequency on a log-log scale, for the following control gains:

i. $k_{\mathrm{p}} = 1$, $k_{\mathrm{i}} = 0$

ii. $k_{\mathrm{p}} = 1$, $k_{\mathrm{i}} = 1$

iii. $k_{\mathrm{p}} = 1$, $k_{\mathrm{i}} = 10$

(If you want to see interesting behaviour, simulate the final case at $\omega = 3.3$ rad/sec as well.)

[C,1es] modeling:ocular-blockdiag Exercise 1.3

3.17 (Eye and head motion) Consider Exercise 1.3 in Chapter 1. Consult the web or a book in physiology that describes the system. Construct a block diagram that captures the essence of the experiments and discuss the differences between the two cases qualitatively.

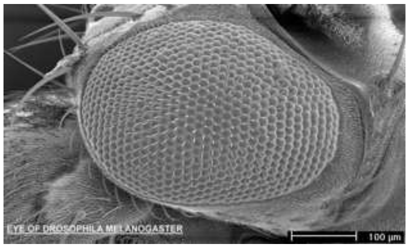
[C,1es] modeling:sysid-logdec

3.18 (Second-order system identification) Verify that equation (3.24) in Example 3.8 is correct and use this formula and the others in the example to compute the parameters corresponding to the step response in Figure 3.16.

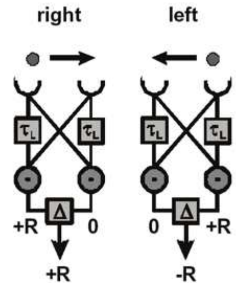
[C,1es] modeling:flyvision

3.19 (Insect vision) [Contributed by Mary J. Dunlop, Sep 06] Insects have compound eyes that are made up of many tiny visual sensors, known as ommatidia, which are arranged in an array. A fruit fly's eye, for example, has 700 ommatidia arranged in a hexagonal pattern. In this problem we consider a simple model for insect vision known as the Reichardt elementary motion detector (EMD).

The elementary motion detector compares the signals on two neighboring ommatidia. The signal on the left ommatidium ($r_i(t)$) is delayed and multiplied by the signal on the right ($r_{i+1}(t)$). The same is done for the right ommatidium and the two signals are subtracted, as shown in the diagram below.



Drosophila compound eye



Elemental motion detector (EMD)

RMM: Get versions of these figures from Michael D and permission to use them as a supplemental exercise. Nonstandard resizing

In this problem we will consider a one dimensional version of the fly eye where there are N ommatidia arranged in a row and we will use a simple model for delay where the signal lags by τ seconds. The output of the visual sensor system then becomes

$$n_i(t) = r_i(t-\tau)r_{i+1}(t) - r_{i+1}(t-\tau)r_i(t),$$

where $n_i(t)$ is the output from the ith EMD and i goes from 1 to $N-1$).

Let $\tau = 1$, $N = 100$, use MATLAB to find the output that the visual system sends to the fly's brain. Attach the MATLAB code that you wrote to solve the problem along with your solution. Be sure to label both x and y axes, include a legend when appropriate and title all graphs.

(a) *Flying down a hallway* The fly is flying down a straight hallway, so it sees walls on both sides, but nothing in front of it:

$$r_i(t) = \begin{cases} 1 & \text{if } i = 1 \text{ and } i = N, \\ 0 & \text{otherwise.} \end{cases}$$

Plot the signal $n(t)$ versus output number (1 to N-1) for $t = 5$.

(b) *Approaching flyswatter* Watch out! A flyswatter comes in from the left, getting darker and larger as it approaches:

$$r_i(t) = \begin{cases} t & \text{if } i \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

Plot the signal $n(t)$ versus output number for $t = 20$, 40, and 60. Plot all three lines on the same graph ($n(20)$ a solid line, $n(40)$ a dashed line, and $n(60)$ a dotted line).

(c) *Control law* In this example the fly's eye is the sensor and the muscles that control its wings are the actuator. The fly's brain computes the control law. Describe a control law that the fly's brain could use to avoid the flyswatter.

3.20 (State space model for balance systems) Show that the dynamics for a balance

[C,1es] modeling:balance-statespace

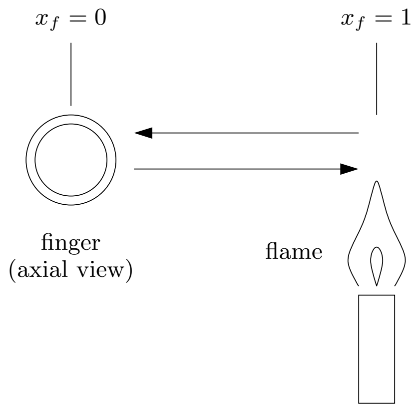
system using normalized coordinates can be written in state space form as

$$\frac{dx}{dt} = \begin{pmatrix} x_3 \\ x_4 \\ \dfrac{-\alpha x_4^2 - \alpha \sin x_2 \cos x_2 + u}{1 - \alpha\beta \cos^2 x_2} \\ \dfrac{-\alpha\beta x_4^2 \cos x_2 - \sin x_2 + \beta u \cos x_2}{1 - \alpha\beta \cos^2 x_2} \end{pmatrix},$$

where $x = (q/l, \theta, \dot{q}/l, \dot{\theta})$.

[D,1es]
modeling:fingerflame

3.21 [Contributed by D. Spanos, 2004; M. Dunlop, 2006] In this problem we will look at how to play with fire without getting burned. The system we want to consider is a finger being moved back and forth across a flame as shown below.



The description of the system is as follows:

- The temperature of a finger is regulated by an internal feedback mechanism. To first order, we will say that heat is convected away by blood flow, at a rate

  $$F_\mathrm{b} = \alpha_\mathrm{b}(T_\mathrm{f} - T_\mathrm{b})$$

  where T_f is the temperature of the fingertip, T_b is the temperature of the blood, and α_b is the convection coefficient (the F signifies the heat flux).

- A flame gives off heat into the ambient air, and we assume a steady-state temperature field around the flame. The ambient air far from the flame is at 25 degrees Celsius.

- The flame is fixed at $x_\mathrm{f} = 1$, and fingertip begins at a position $x_\mathrm{f} = 0$, where the ambient air is precisely at the same temperature as the blood.

- Suppose that the temperature of the air varies exponentially with distance from the flame, so

  $$T_a(x) = 25 + (T_\mathrm{f} - 25)\left(\frac{T_\mathrm{b} - 25}{T_\mathrm{f} - 25}\right)^{(x-1)^2}$$

  where T_f is the flame temperature.

- Heat convects into the finger from the ambient air at a rate

$$F_a = \alpha_a(T_a - T_\mathrm{f}).$$

- The dynamics of the fingertip temperature is given by

$$c_\mathrm{f}\frac{T_\mathrm{f}}{dt} = -F_\mathrm{b} + F_a$$

  where c_f is the fingertip thermal capacity.

- The fingertip is rapidly passed into and out of the flame, according to

$$x_\mathrm{f}(t) = \sin(\omega t).$$

Using the MATLAB `ode45` function (or something similar), build a model for the system and solve the following:

(a) Assume that the finger moves sinusoidally in and out of the flame at frequency $\omega = 1$ rad/s. Plot the temperature of the finger as a function of time and identify the transient and steady-state responses.

(b) Plot the *steady-state* amplitude of the finger temperature as a function of the ω for ω ranging from 1 to 100 rad/s. You should get something similar to the frequency response plot shown in lecture on Monday. You should compute at least 5 points in your graph.

(c) Double the "gain" of the temperature control system by increasing α_b by a factor of 2. Replot the frequency response from part 0b and describe in words how it differs from the original gain (i.e., where is the response bigger, smaller, or unchanged and what is the reason).

You should use the following parameter values in your simulations:

- $T_\mathrm{b} = 37$, $T_\mathrm{f} = 1400$ degrees Celsius.
- $\alpha_a/c_\mathrm{f} = 1\ s^{-1}$
- $\alpha_\mathrm{b}/c_\mathrm{f} = 20, 40\ s^{-1}$

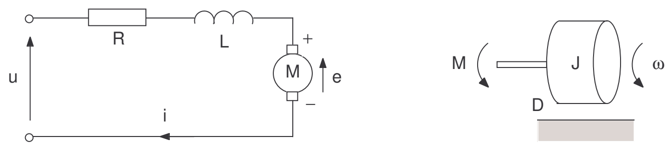
3.22 (Consensus and invariants) Consider the consensus problem described in Example 3.17 with N nodes and a connected graph describing the sensor network. Show that the quantity [C,1es] modeling:compsys-invariants

$$W[k] = \frac{1}{N}\sum_{i=1}^{N} x_i[k]$$

is constant under the consensus protocol and use this fact to show that if the consensus protocol converges, then it converges to the average of the initial values of each node. (In computer programming, qualities such as W are called *invariants,* and the use of invariants is an important technique for verifying the correctness of computer programs.)

[D,1es]
modeling:electricmotor

3.23 (Electric motor) A schematic representation of an electrical motor with a permanent magnet is given below



(a) (b)

RMM: (AT) Voltage on the left should be V, not u

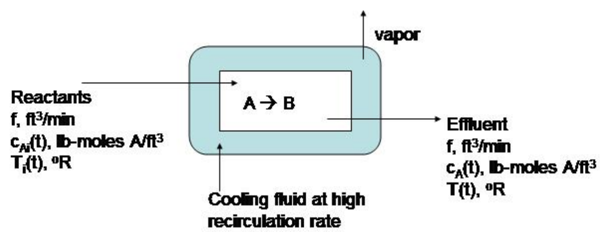
The motor is an electro-mechanical system that is characterized by the following parameters: rotor resistance R, rotor inductance L, moment of inertia J, viscous damping c, torque sensitivity k_I, and back EMF coefficient k_E. The torque created by the rotor current I is $T = k_I I$ and the back EMF caused by the rotation of the rotor is $E = k_E\omega$. Explain why $k_i = k_e$ if proper units are used. Show that the system can be modeled by the equations

$$\frac{d\omega}{dt} = -\frac{c}{J}\omega + \frac{k_I}{J}I \qquad \frac{dI}{dt} = -\frac{c}{J}\omega - \Big(\frac{R}{L} + \frac{k_I^2}{LJ}\Big)I + \frac{1}{L}V. \tag{S3.6}$$

(Hint: Focus on where energy is stored before you start to write the equations.)

[C,1es]
modeling:exothermic

3.24 (Exothermic reaction) (Contributed by Anand Asthagiri, 2004) Consider a chemical reactor in which species A undergoes a first-order, exothermic conversion to species B. To remove the heat of reaction, a jacket surrounds the reactor where a coolant is maintained at 100 ^{o}F. Suppose that such a reactor is performing at steady-state conditions provided in the table below:



Process Information

$V = 13.26$ ft^3	$A = 36$ ft^2
$E = 27{,}820$ Btu/lbmole	$R = 1.987$ Btu/lbmole-°R
$\rho = 55$ lbm/ft^3	$C_p = 0.88$ Btu/lbm-°F
$\Delta H_r = -12{,}020$ Btu/lbmole	$U = 75$ Btu/(h-ft^2-°F)
$k_o = 1.73515 \times 10^{13}$/min	

Steady-State Values

$c_{Ai}(t) = 0.8983$ lbmole/ft^3	$T_i(t) = 578$ °R
$T_c = 560.0$°R	$f = 1.3364$ ft^3/min
$c_A(t) = 0.08023$ lbmole/ft^3	$T(t) = 690.0$°R

Inevitably, under normal process conditions, the reactor will experience disturbances in the inlet temperature ($T_i(t)$) and concentration of species A ($c_{Ai}(t)$) in the input stream. Thus, we would like to know what impact these fluctuations in inlet conditions might have on the concentration of species A ($c_A(t)$) and the temperature ($T(t)$) of the effluent stream. Suggestions: Assume that the reactor contents are well-mixed and that the heat capacity (C_p) and density (ρ) of reactants and products are equal.

(a) Develop a set of equations that could be used to predict temporal changes in effluent temperature and species A concentration ($T(t)$ and $c_A(t)$, respectively).

(b) Since we are interested in *deviations* in process variables, it is useful to reformulate the above equations in terms of deviation variables. A deviation variable (Y') for a process variable (Y) is defined as $Y' \equiv Y - \overline{Y}$ where $\overline{Y}$ is the steady-state value. Reformulate equations in terms of such deviation variables, and solve for $c'_A(t)$ and $T'(t)$.

(c) Plot $c'_A(t)$ and $T'(t)$ versus time for the following cases: (a) $T'_i(t) = -5^o$R and (b) $T'_i(t) = -10^o$R. Explain the observed behavior of the reactor. Does it always return to the same steady-state value? Is the dynamic response "smooth" or oscillatory?

3.25 (Modeling and simulation of a catalyzed reaction) Caffeine and theobromine are both alkaloids known for their stimulating effects; while theobromine (from Greek *theo* "god", *broma* "food") is the primary alkaloid found in cocoa, there are over 60 plants which are source of caffeine. It is possible to synthesize caffeine (Cf) from theobromine (Tb) by the action of certain catalysts. Here we will suppose to synthesize Cf from Tb utilizing a caffeine synthase indicated as CS. [C,1es] modeling:caffeine

This catalyzed reaction can be modeled by the following mass action model, where in squared brackets we indicate concentration of the metabolites:

$$[CS] + [Tb] \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} [CS \cdot Tb] \overset{k_2}{\rightarrow} [CS] + [Cf] \tag{S3.7}$$

Then two differential equations can be derived:

$$\begin{aligned} \frac{d\,[CS \cdot Tb]}{dt} &= k_1[CS][Tb] - k_{-1}[CS \cdot Tb] - k_2[CS \cdot Tb] \\ \frac{d\,[Cf]}{dt} &= k_2[CS \cdot Tb] \end{aligned} \tag{S3.8}$$

(a) Suppose that $[CS \cdot Tb]$ is at steady state and derive an equation expressing the production rate of $[Cf]$ as a function of the concentration of theobromine and of the initial concentration of synthase. (Hint: In the derivation, you will find useful to note that the concentration of the enzyme can be expressed as $[CS] = [CS](0) + [Cs \cdot Tb]$, where $[CS](0)$ is the initial concentration of enzyme.)

(b) Generate a SIMULINK model for this dynamical system, where the input is the concentration of theobromine: in an experiment, you would supply to the system an initial concentration of Tb that will be consumed by the gradual conversion into Cf. Find a way to simulate such effect by using a step function, to which you need to subtract the consumed input.

Since the dynamics are nonlinear, you should use the *function - Fcn* block that can be found in the *User-defined functions* list. Utilize an initial concentration of enzyme $[CS]$ of $10\mu M$, $k_2 = 100 \quad 1/s$, $k_1 = 2 \quad 1/\mu Ms$, and $k_{-1} = 0.2\, 1/s$. In the step function, set the final value of $[Tb]$ as $200\,\mu < M$

Generate a plot of the Cf and Tb concentrations versus time.

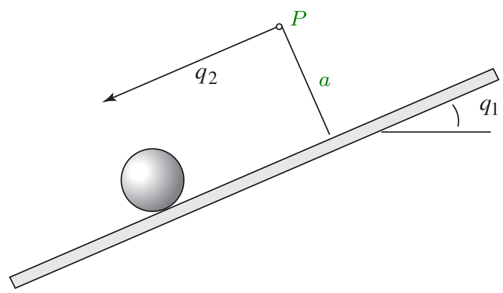
(c) Now assume to add a degradation term to (S3.8):

$$\frac{d[Cf]}{dt} = k_2[CS \cdot Tb] - \beta\,[Cf]$$

where $\beta = .5s^{-1}$. Modify the SIMULINK model and plot again the Tb and Cf time profiles.

[C,1es]
modeling:ballbeam-modeling

3.26 The ball and beam system shown in the figure below is a popular platform for control experiments.



The system consists of a beam that rotates around the pivot P, its angle is controlled by a motor. The ball moves in a grove on the beam. The goal is to control the position of the ball on the beam. The dynamics are similar to the dynamics of the vector thrust vehicle discussed in Example 3.12. Introduce the state variables beam angle q_1 and ball position q_2 as shown in the figure and let the torque from the motor be T_m. Show that the system can be modeled by the following equations

$$\begin{pmatrix} J_{1e} + m_2 q_2^2 & m_{12} \\ m_{12} & m_{2e} \end{pmatrix} \ddot{q} + \begin{pmatrix} 2m_2\dot{q}_1\dot{q}_2 \\ m_2\dot{q}_1^2 q_2 \end{pmatrix} + g \begin{pmatrix} ml_e \sin q_1 - m_2 q_2 \cos q_1 \\ m_2 \sin q_1 \end{pmatrix} = \begin{pmatrix} T_\text{m} \\ 0 \end{pmatrix}$$

where J_e, m_{12}, m_e, and ml_e are constant parameters. The mass matrix is nonlinear because it depends on the position of the ball, the nonlinear damping term represents Coriolis and centripetal forces and the spring term represents the effects of gravity. It is also implicitly assumed that the ball is in contact with the beam at all times.

[C,1es]
modeling:pvtol2invpend

3.27 Consider equation (3.29) describing the motion of a vector thrust vehicle in Example 3.12. Show that the motion in the x, θ plane is the same as that of a pendulum on a cart.

[C,1es]
modeling:flyflight

3.28 Consider the block diagram of the flight control system of a fly shown in Figure 3.15. Using the paper "Vision as a Compensatory Mechanism for Disturbance Rejection in Upwind Flight" by Reiser et al. (available via the course web page), identify the state, input, outputs, and dynamics for each block in the diagram. You may give you answer in words, but be precise as possible. (Hint: Not all of the blocks are "dynamic"; some are static maps.)

3.29 [Contributed by Mary Dunlop, 2006] The motion of an ideal pendulum is described by [C,1es] modeling:pendulum

$$\ddot{\theta} + g \sin\theta = 0,$$

where θ is the angle between the pendulum's position and vertical, and g is the gravitational acceleration.

(a) Using the small angle approximation $\sin\theta \approx \theta$, solve for an expression for $\theta(t)$, written in terms of the initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = \omega_0$ and the parameter g.

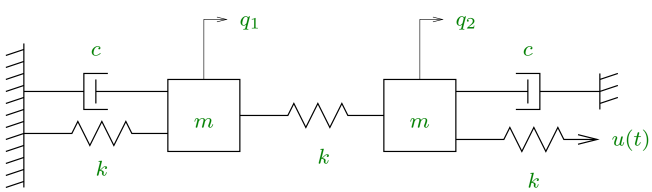
(b) Plot the pendulum's motion in three different environments: Earth ($g = 9.8$ m/s^2), the moon ($g = 1.6$ m/s^2), and on Temple I—the comet that the Deep Impact mission collided with on in July 2005 ($g = 0.00004$ m/s^2). Assume that the pendulum is given a small initial starting angle $\theta(0) = 0.05$ radians (about 3 degrees) and then released with no initial velocity ($\dot{\theta}(0) = 0$). Note that this is an idealized equation of motion and damping is not included, so there are no frictional forces to slow the pendulum down over time.

(c) If the pendulum is pushed with a force $u(t)$, the equation of motion becomes

$$\ddot{\theta} + g \sin\theta = u(t).$$

Apply the small angle approximation and assume $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. Solve for $\theta(t)$ when $u(t) = \sin t$.

3.30 Consider the coupled spring–mass system shown in the figure below: [C,1es] modeling:coupled-modeling



The input to this system is the sinusoidal motion of the end of rightmost spring and the output is the position of each mass, q_1 and q_2.

(a) Write the equations of motion for the system, using the positions and velocities of each mass as states.

(b) Rewrite the dynamics in terms of $z_1 = \frac{1}{2}(q_1+q_2)$ and $z_2 = \frac{1}{2}(q_1-q_2)$. Note that the resulting equations are block diagonal (see discussion on modes in Section 6.2 for some additional insight). Solve these linear ODEs for the step response of $z_1(t)$ and $z_2(t)$ and use this to compute the step response of q_1 and q_2.

(c) Setting $m = 250$, $k = 50$, $b = 10$, plot the motion of the first and second masses in response to an input motion $u = A\sin(\omega t)$ with $\omega = 1$ rad/s and $A = 1$ cm. Determine the amount of time required for the system to reach steady-state oscillations.

(d) Plot the *steady-state* amplitude of the motion of the first and second masses as a function of the input frequency, ω.

*Instructor note:*This problem is considered in Example 6.6 and the solution is essentially given there.

Chapter 4 – Examples

4.1 (Cruise control) Consider the cruise control example described in Section 4.1. Build a simulation that re-creates the response to a hill shown in Figure 4.3b and show the effects of increasing and decreasing the mass of the car by 25%. Redesign the controller (using trial and error is fine) so that it returns to within 1% of the desired speed within 3 s of encountering the beginning of the hill. [A,1ep] examples:cruise-hillsim

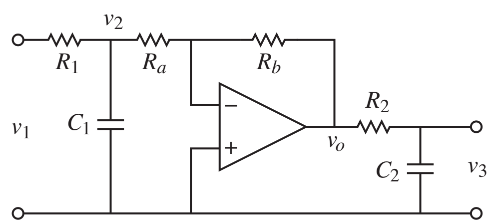
4.2 (Bicycle dynamics) Show that the dynamics of a bicycle frame given by equation (4.5) can be approximated in state space form as [B,1ep] bicycle:statespace

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ mgh/J & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} Dv_0/(bJ) \\ mv_0^2h/(bJ) \end{pmatrix} u,$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

where the input u is the steering angle δ and the output y is the tilt angle φ. What do the states x_1 and x_2 represent?

*Instructor note:*An alternative, but messier, version of this problem is to show that the system can be converted into reachable canonical form.

4.3 (Operational amplifier circuit) Consider the op amp circuit shown below. [C,1ep*] examples:opamp2

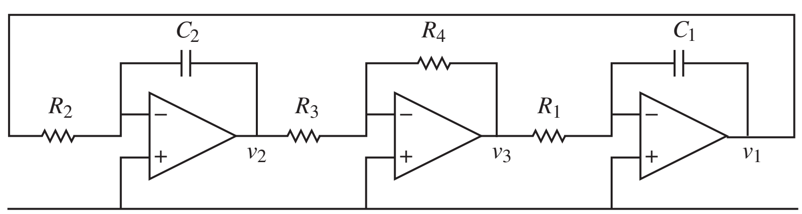


Show that the dynamics can be written in state space form as

$$\frac{dx}{dt} = \begin{pmatrix} -\frac{1}{R_1C_1} - \frac{1}{R_aC_1} & 0 \\ -\frac{R_b}{R_a}\frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{pmatrix} x + \begin{pmatrix} \frac{1}{R_1C_1} \\ 0 \end{pmatrix} u, \quad y = \begin{pmatrix} 0 & 1 \end{pmatrix} x,$$

where $u = v_1$ and $y = v_3$. (Hint: Use v_2 and v_3 as your state variables.)

4.4 (Operational amplifier oscillator) The op amp circuit shown below is an implementation of an oscillator. [A,1ep*] examples:opamposc



Show that the dynamics can be written in state space form as

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \dfrac{R_4}{R_1 R_3 C_1} \\ -\dfrac{1}{R_2 C_2} & 0 \end{pmatrix} x,$$

where the state variables represent the voltages across the capacitors $x_1 = v_1$ and $x_2 = v_2$.

[B,1ep*]
examples:congctrl-red

4.5 (Congestion control using RED [8]) A number of improvements can be made to the congestion control model presented in Section 4.4. To ensure that the router's buffer size remains positive, we can modify the buffer dynamics to satisfy

$$\frac{db_l}{dt} = \begin{cases} s_l - c_l & \text{if } 0 < b_l < b_{l,\max}, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we can model the drop probability of a packet based on how close a filtered estimate of the buffer size is to the buffer limits, a mechanism known as random early detection (RED):

$$p_l = \beta_l(a_l) = \begin{cases} 0 & \text{if } a_l \le b_l^{\text{low}}, \\ \rho_l(a_i - b_l^{\text{low}}) & \text{if } b_l^{\text{low}} < a_l < b_l^{\text{mid}}, \\ \eta_l(a_i - b_l^{\text{mid}}) + \rho_l(b_l^{\text{mid}} - b_l^{\text{low}}) & \text{if } b_l^{\text{mid}} \le a_l < b_l^{\text{max}}, \\ 1 & \text{if } a_l \ge b_l^{\text{max}}, \end{cases}$$

$$\frac{da_l}{dt} = -\alpha_l c_l (a_l - b_l),$$

where α_l, ρ_l, η_l, b_l^{low}, b_l^{mid}, and b_l^{max} are parameters for the RED protocol. The variable a_l is a smoothed version of the buffer size b_l. Using the model above, write a simulation for the system and find a set of parameter values for which there is a stable equilibrium point and a set for which the system exhibits oscillatory solutions. The following sets of parameters should be explored:

$$\begin{aligned} N &= 20, 30, \ldots, 60, & b_l^{\text{low}} &= 40 \text{ pkts}, & \alpha_l &= 10^{-4}, \\ c &= 8, 9, \ldots, 15 \text{ pkts/ms}, & b_l^{\text{mid}} &= 540 \text{ pkts}, & \rho_l &= 0.0002, \\ \tau^{\text{p}} &= 55, 60, \ldots, 100 \text{ ms} & b_l^{\text{max}} &= 1080 \text{ pkts}, & \eta_l &= 0.00167. \end{aligned}$$

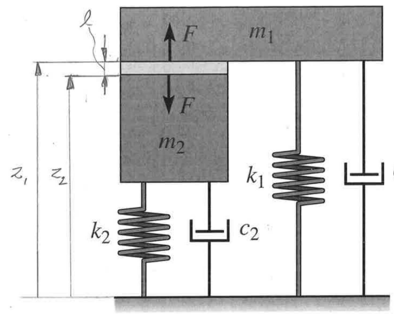
Instructor note: We use a slightly different notation than [8] for the bounds and breakpoints of the buffer size: $b_l^{\text{low}} = b_l^{\text{lower}}$, $b_l^{\text{mid}} = b_l^{\text{upper}}$, $b_l^{\text{max}} = 2b_l^{\text{upper}}$.

[A,1ep*]
examples:afmpreload

4.6 (Atomic force microscope with piezo tube) A schematic diagram of an AFM where the vertical scanner is a piezo tube with preloading is shown below.

PUP: Add annotations to figure. (RMM: once done, remove scaling factor.)

†



Show that the dynamics can be written as

$$(m_1 + m_2)\frac{d^2 z_1}{dt^2} + (c_1 + c_2)\frac{dz_1}{dt} + (k_1 + k_2)z_1 = m_2\frac{d^2 l}{dt^2} + c_2\frac{dl}{dt} + k_2 l,$$

where z_1 is the displacement of the first mass and $l = z_1 - z_2$ is the difference in displacement between the first and second masses. Are there parameter values that make the dynamics particularly simple?

4.7 (Drug administration) The metabolism of alcohol in the body can be modeled by the nonlinear compartment model [A,1ep] examples:drugadmin-alcohol

$$V_\mathrm{b}\frac{dc_\mathrm{b}}{dt} = q(c_\mathrm{l} - c_\mathrm{b}) + q_\mathrm{iv}, \qquad V_\mathrm{l}\frac{dc_\mathrm{l}}{dt} = q(c_\mathrm{b} - c_\mathrm{l}) - q_\mathrm{max}\frac{c_\mathrm{l}}{c_0 + c_\mathrm{l}} + q_\mathrm{gi},$$

where $V_\mathrm{b} = 48$ L and $V_\mathrm{l} = 0.6$ L are the apparent volumes of distribution of body water and liver water, c_b and c_l are the concentrations of alcohol in the compartments, q_iv and q_gi are the injection rates for intravenous and gastrointestinal intake, $q = 1.5$ L/min is the total hepatic blood flow, $q_\mathrm{max} = 2.75$ mmol/min and $c_0 = 0.1$ mmol/L. Simulate the system and compute the concentration in the blood for oral and intravenous doses of 12 g and 40 g of alcohol.

4.8 (Insulin-glucose dynamics) The following model for insulin glucose dynamics by Gaetano and colleagues [4] has three states: glucose concentration in the blood plasma G [mg/dL], insulin concentration in the interstitial fluid I [µUI/ml], and X [min^{-1}] that represents the increased removal rate of glucose due to insulin. The state X is proportional to the concentration of interstitial insulin. The dynamics are: [B,2e] examples:insulin-glucose

$$\begin{aligned}
\frac{dG}{dt} &= -(p_1 + X)G + p_1 G_b + u_\mathrm{G} \\
\frac{dX}{dt} &= -p_2 X + p_3(I - I_\mathrm{b}) \\
\frac{dI}{dt} &= p_4 \max(G - p_5, 0) - p_6(I - I_\mathrm{b}) + u_\mathrm{I}.
\end{aligned}$$

Use the parameters

$$\begin{aligned}
&G_\mathrm{b} = 87, \quad I_\mathrm{b} = 37.9, \quad p_1 = 0.05, \quad p_2 = 0.5, \quad p_3 = 10^{-4}, \\
&p_4 = 10^{-5}, \quad p_5 = 150, \quad p_6 = 0.05, \quad p_7 = 199.
\end{aligned}$$

Simulate the system with the initial conditions $G(0) = 400$, $I(0) = 200$ and $X(0) = 0$. This corresponds to a person having taken a large initial dose of glucose.

[B,1ep] examples:fishery-modeling

4.9 (Fisheries management) Some features of the dynamics of a commercial fishery can be described by the following simple model:

$$\frac{dx}{dt} = f(x) - h(x,u), \qquad y = bh(x,u) - cu,$$

where x is the total biomass, $f(x) = rx(1-x/k)$ is the growth rate, and r and k are constant parameters. The harvesting rate is $h(x,u) = axu$, where a is a constant parameter and u is the fishing effort. The output y is the rate of revenue, where b and c are constants representing the price of fish and the cost of fishing.

Part: eqpt

(a) Find a sustainable equilibrium point where the revenue is as large as possible. Determine the equilibrium value of the biomass and the fishing effort at the equilibrium.

Part: quota

(b) With the parameters $a = 0.1$, $b = 1$, $c = 1$, $k = 100$, and $r = 0.2$ the sustainable equilibrium point corresponds to $x_{\mathrm{e}} = 55$ and $u_{\mathrm{e}} = 0.9$. For an individual fisherman it is profitable to fish as long as the rate of revenue $y = (abx - c)u$ is positive. Explore by simulation what happens if the fishing intensity is much higher than the sustainable fishing rate u_{e}, say $u = 3$. Use the results to discuss the role of having a fishing quota.

*Instructor note:*The second part of this exercise, comparing the situation to regulation to a constant biomass, is a bit open-ended. It may be useful to be more specific about what is required in the answer.

Supplemental Exercises

[C,2e] examples:opamp-dynamics

4.10 The model of the operational amplifier given by equation (4.11) is highly idealized. A more accurate model is given by

$$\frac{dv_{out}}{dt} = -av_{out} + bv$$

Give the equations for the circuits in Figures 4.9a and 4.10 when this model is used instead of equation (4.11).

[D,1es] intro:powertrain

4.11 [MATLAB/SIMULINK] In this problem we will implement the cruise control system described in Section 4.1 of the text. Unless otherwise specified, use the parameter values given in the text.

(a) Using Figure 3.1 as a guide, build a SIMULINK model corresponding to the vehicle dynamics in Section 3.1, but without the human interface block. You should have a separate SIMULINK block for each element in your block diagram. In addition to the equations in the text, you should saturate the input to the actuator so that it lies between the values of 0 and 1. Turn in a printout of your SIMULINK diagram along with a description (equations) of what is in each block.

(b) Using your model, plot the output of the open loop vehicle model (3.3) for a step input of $u = 0.5$, assuming you are in first gear, on flat ground ($\theta = 0$), and using $m = 1000$ kg. What is the rise time for the system (0 to 95% of the final value)?

(c) Implement a PI controller for your system with gains $k_\mathrm{p} = 0.5$ and $k_\mathrm{i} = 0.1$. Plot the response of the system to a change in desired speed from 20 m/s to 30 m/s, assuming you are in third gear and on flat ground. Make sure to implement your simulation so that the system is in steady state at 20 m/s before changing the desired speed.

(d) Now include the effect of a hill on your system. You should model the system so that the car is initially on a flat surface doing 20 m/s and then encounters the hill of 5 degrees. Plot the response of the system (it should look very similar to Figure 3.3b).

4.12 (Bicycle steering) Combine the bicycle model given by equation (4.5) and the model for steering kinematics in Example 3.11 to obtain a model that describes the path of the center of mass of the bicycle. [C,1ep] bicycle:bikesteer

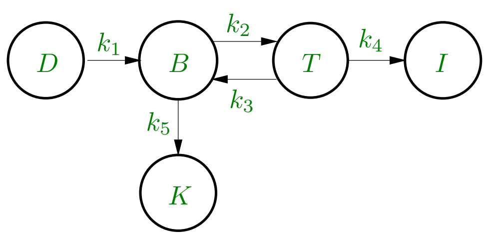
4.13 (Population dynamics) Consider the model for logistic growth given by equation (4.31). Show that the maximum growth rate occurs when the size of the population is half of the steady-state value. [C,1ep] examples:logistic

4.14 (Predator-prey dynamics) The Lotka-Volterra equation [B,2e] examples:lotka-volterra

$$\frac{dx}{dt} = (a - by)x, \qquad \frac{dy}{dx} = (-c + dx)y,$$

where x and y are the numbers of preys and predators, is a model for predator-prey behavior that is simpler than the one given by equation (4.32). Show by scaling all variables x, y, and t that the system is essentially governed by one parameter. Simulate the original equations with the parameters $a = 1.6$, $b = 0.003$, $c = 0.6$, and $d = 0.001$ and the initial conditions $x(0) = 50$, $y(0) = 200$.

4.15 Show that the model represented by the schematic diagram in Figure 4.17 can be represented by the compartment model shown below: [C,2e] intro:drugadmin-teorell



RMM: Nonstandard resizing commands

where compartment D represents the issue where the drug is injected, compartment B represents the blood, compartment T represents tissue where the drug should be active, compartment K the kidney where the drug is eliminated, and I a part of the body where the drug is inactive. Write a simulation for the system and explore how the amount of the drug in the different compartments develops over time. Relate your observations to your physical intuition and the schematic diagram above. Modify your program so that you can investigate what happens if the drug is injected directly to the bloodstream, compartment B, instead of in compartment D.

Chapter 5 – Dynamic Behavior

5.1 (Time-invariant systems) Show that if we have a solution of the differential equation (5.1) given by $x(t)$ with initial condition $x(t_0) = x_0$, then $\tilde{x}(\tau) = x(t - t_0)$ is a solution of the differential equation

[A,1ep*] dynamics:timeshift

$$\frac{d\tilde{x}}{d\tau} = F(\tilde{x})$$

with initial condition $\tilde{x}(0) = x_0$, where $\tau = t - t_0$.

5.2 (Flow in a tank) Consider a cylindrical tank with cross sectional area A m^2, effective outlet area a m^2, and inflow q_{in} m^3/s. An energy balance shows that the outlet velocity is $v = \sqrt{2gh}$ m/s, where g m/s^2 is the acceleration of gravity and h is the distance between the outlet and the water level in the tank (in meters). Show that the system can be modeled by

[B,1ep] dynamics:tank-modeling

$$\frac{dh}{dt} = -\frac{a}{A}\sqrt{2gh} + \frac{1}{A}q_{\text{in}}, \qquad q_{\text{out}} = a\sqrt{2gh}.$$

Use the parameters $A = 0.2$, $a = 0.01$. Simulate the system when the inflow is zero and the initial level is $h = 0.2$. Do you expect any difficulties in the simulation?

5.3 (Lyapunov functions) Consider the second-order system

[B,1ep] dynamics:lyap-secord

$$\frac{dx_1}{dt} = -ax_1, \qquad \frac{dx_2}{dt} = -bx_1 - cx_2,$$

where $a, b, c > 0$. Investigate whether the functions

$$V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2, \qquad V_2(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \frac{b}{c-a}x_1)^2$$

are Lyapunov functions for the system and give any conditions that must hold.

5.4 (Damped spring–mass system) Consider a damped spring–mass system with dynamics

[B,1ep] dynamics:lyap-oscillator

$$m\ddot{q} + c\dot{q} + kq = 0.$$

A natural candidate for a Lyapunov function is the total energy of the system, given by

$$V = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2.$$

Use the Krasovski–Lasalle theorem to show that the system is asymptotically stable.

5.5 (Electric generator) The following simple model for an electric generator connected to a strong power grid was given in Exercise 3.8:

[B,1ep] dynamics:powergrid Exercise 3.8

$$J\frac{d^2\varphi}{dt^2} = P_{\text{m}} - P_{\text{e}} = P_{\text{m}} - \frac{EV}{X}\sin\varphi.$$

The parameter

$$a = \frac{P_{\max}}{P_{\text{m}}} = \frac{EV}{XP_{\text{m}}}$$

is the ratio between the maximum deliverable power $P_{\max} = EV/X$ and the mechanical power P_{m}.

(a) Consider a as a bifurcation parameter and discuss how the equilibrium points depend on a.

(b) For $a > 1$, show that there is a center at $\varphi_0 = \arcsin(1/a)$ and a saddle at $\varphi = \pi - \varphi_0$.

(c) Assume $a > 1$ and show that there is a solution through the saddle that satisfies

$$\frac{J}{2}\Big(\frac{d\varphi}{dt}\Big)^2 - P_{\text{m}}(\varphi - \varphi_0) - \frac{EV}{X}(\cos\varphi - \cos\varphi_0) = 0. \tag{S5.1}$$

Set $J/P_{\text{m}} = 1$ and use simulation to show that the stability region is the interior of the area enclosed by this solution. Investigate what happens if the system is in equilibrium with a value of a that is slightly larger than 1 and a suddenly decreases, corresponding to the reactance of the line suddenly increasing.

[B,1ep] dynamics:lyapeqn

5.6 (Lyapunov equation) Show that Lyapunov equation (5.17) always has a solution if all of the eigenvalues of A are in the left half-plane. (Hint: Use the fact that the Lyapunov equation is linear in P and start with the case where A has distinct eigenvalues.)

*Instructor note:*Depending on the background of the class, it may be more appropriate to simply ask the students to prove the case where A has n independent eigenvectors or (simpler still) distinct eigenvalues.

[B,2e] dynamics:invpend:changebehavior

5.7 (Shaping behavior by feedback) An inverted pendulum can be modeled by the differential equation

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = \sin x_1 + u\cos x_1,$$

where x_1 is the angle of the pendulum clockwise), and x_2 is its angular velocity (see Example 5.14). Qualitatively discuss the behavior of the open loop system and how the behavior changes when the feedback $u = -2\sin(x)$ is introduced. (Hint: use phase portraits.)

[B,1ep] dynamics:invpend-swingup

5.8 (Swinging up a pendulum) Consider the inverted pendulum, discussed in Example 5.4, that is described by

$$\ddot\theta = \sin\theta + u\cos\theta,$$

where θ is the angle between the pendulum and the vertical and the control signal u is the acceleration of the pivot. Using the energy function

$$V(\theta,\dot\theta) = \cos\theta - 1 + \frac{1}{2}\dot\theta^2,$$

show that the state feedback $u = k(V_0 - V)\dot\theta\cos\theta$ causes the pendulum to "swing up" to the upright position.

[B,1ep] dynamics:rootlocus

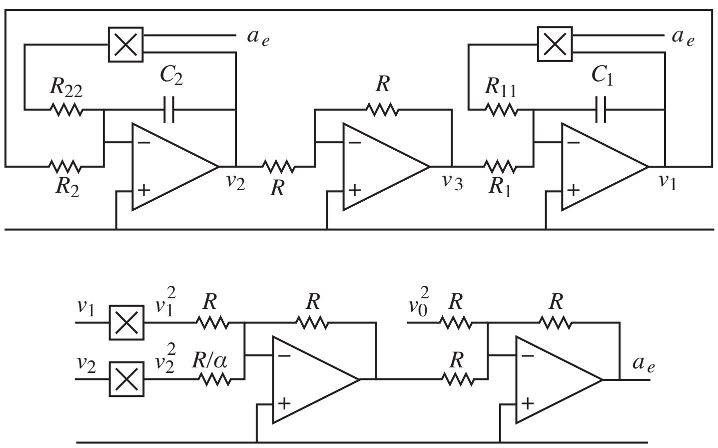
5.9 (Root locus diagram) Consider the linear system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 4 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

with the feedback $u = -ky$. Plot the location of the eigenvalues as a function the parameter k. Identify the approximate gains at which the system becomes unstable and label these on your plot. (To create your plot, you should compute the eigenvalues at multiple values of k and plot these on the complex plane. Label the locations of the eigenvalues for $k = 0$ with '×' and the locations for $k \to \infty$ with an '∘' if they converge to a finite value. Choose the units on your graph so that key features are visible and use arrows on your plot to indicate which direction corresponds to increasing gain, similar to Figure 5.19b in the text.)

5.10 (Discrete-time Lyapunov function) Consider a nonlinear discrete-time system with dynamics $x[k{+}1] = f(x[k])$ and equilibrium point $x_e = 0$. Suppose there exists a smooth, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(f(x)) - V(x) < 0$ for $x \neq 0$ and $V(0) = 0$. Show that $x_e = 0$ is (locally) asymptotically stable. [B,1ep] dynamics:discrete-lyapunov

5.11 (Operational amplifier oscillator) An op amp circuit for an oscillator was shown in Exercise 4.4. The oscillatory solution for that linear circuit was stable but not asymptotically stable. A schematic of a modified circuit that has nonlinear elements is shown in the figure below. [A,1ep*] dynamicbehavior:opamp-oscill Exercise 4.4



The modification is obtained by making a feedback around each of the operational amplifiers that has capacitors and making use of multipliers. The signal $a_e = v_1^2 + \alpha v_2^2 - v_0^2$ is the amplitude error. Show that the system is modeled by

$$\frac{dv_1}{dt} = \frac{1}{R_1C_1}v_2 + \frac{1}{R_{11}C_1}v_1(v_0^2 - v_1^2 - \alpha v_2^2),$$
$$\frac{dv_2}{dt} = -\frac{1}{R_2C_2}v_1 + \frac{1}{R_{22}C_2}v_2(v_0^2 - v_1^2 - \alpha v_2^2).$$

Determine α so that the circuit gives an oscillation with a stable limit cycle with amplitude v_0. (Hint: Use the results of Example 5.9.)

*Instructor note:*The oscillation generated by this circuit can get a bit complicated unless some specific values are given for the resistances and capacitances. To simplify the problem, $R_1C_1 = R_2C_2$, and $R_{11}C_1 = R_{22}C_2$ or use numerical values that have these relationships.

[B,1ep]
dynamics:congctrl-lyapstab

5.12 (Congestion control) Consider the congestion control problem described in Section 4.4. Confirm that the equilibrium point for the system is given by equation (4.22) and compute the stability of this equilibrium point using a linear approximation.

[A,1ep]
dynamics:biocircuits-posfbk

5.13 (Self-activating genetic circuit) Consider the dynamics of a genetic circuit that implements *self-activation*: the protein produced by the gene is an activator for the protein, thus stimulating its own production through positive feedback. Using the models presented in Example 3.18, the dynamics for the system can be written as

$$\frac{dm}{dt} = \frac{\alpha p^2}{1 + kp^2} + \alpha_0 - \delta m, \qquad \frac{dp}{dt} = \kappa m - \gamma p, \tag{S5.2}$$

for $p, m \geq 0$. Find the equilibrium points for the system and analyze the local stability of each using Lyapunov analysis.

Instructor note: As written, this exercise leads to an analysis that provides conditions for determining stability but does not actually compute the stability of the equilibrium points (since this requires knowledge of the specific parameters). If desired, the parameters from Example 3.18 can be used to allow the stability to be determined explicitly.

[A,1ep*]
dynamics:modal-form

5.14 (Diagonal systems) Let $A \in \mathbb{R}^{n\times n}$ be a square matrix with real eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$. Assume that the eigenvalues are distinct ($\lambda_i \neq \lambda_j$ for $i \neq j$).

(a) Show that $v_i \neq v_j$ for $i \neq j$.

(b) Show that the eigenvectors form a basis for $\mathbb{R}^n$ so that any vector x can be written as $x = \sum \alpha_i v_i$ for $\alpha_i \in \mathbb{R}$.

(c) Let $T = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}$ and show that $T^{-1}AT$ is a diagonal matrix of the form (5.10).

(d) Show that if some of the λ_i are complex numbers, then A can be written as

$$A = \begin{pmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_k \end{pmatrix}, \quad \text{where} \quad \Lambda_i = \lambda \in \mathbb{R} \quad \text{or} \quad \Lambda_i = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}.$$

in an appropriate set of coordinates.

This form of the dynamics of a linear system is often referred to as *block diagonal form*.

Supplemental Exercises

[C,2e]
dynamics:lyap-scalarnl

5.15 (Scalar nonlinear system) Analyze the stability of nonlinear system in Example 5.10 at the equilibrium point $x = -2$.

5.16 We say that an equilibrium point $x^* = 0$† is an *exponentially stable* equilibrium point of (5.2) if there exist constants m, $\alpha > 0$, and $\epsilon > 0$ such that

$$\|x(t)\| \leq m e^{-\alpha(t-t_0)} \|x(t_0)\| \tag{S5.3}$$

for all $\|x(t_0)\| \leq \epsilon$ and $t \geq t_0$. Prove that an equilibrium point is exponentially stable if and only if there exists an $\epsilon > 0$ and a function $V(x,t)$ that satisfy

$$\alpha_1 \|x\|^2 \leq V(x,t) \leq \alpha_2 \|x\|^2, \qquad \left.\frac{dV}{dt}\right|_{\dot x = f(x,t)} \leq -\alpha_3 \|x\|^2, \qquad \|\frac{\partial V}{\partial x}(x,t)\| \leq \alpha_4 \|x\|,$$

for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\|x\| \leq \epsilon$.

[D,1es] RMM: x_e? dynamics:expstab

5.17 (Cruise control) Consider the cruise control system described in Section 4.1. Generate a phase portrait for the closed loop system on flat ground ($\theta = 0$), in fourth gear, using a PI controller (with $k_p = 0.5$ and $k_i = 0.1$), $m = 1600$ kg, and desired speed 20 m/s. Your system model should include the effects of saturating the input between 0 and 1. (Hint: Keep in mind that when modeling feedback control, additional states can arise that do not appear in the original dynamics. You should include the MATLAB code used to generate your phase portrait.)

[C,1ep] dynamics:cruise-phaseplot

5.18 Consider the nonlinear model of population dynamics given by equation (4.31). Determine all equilibrium points, their stability and the linearizations around the equilibrium points.

[N,1es] dynamics:popdyn-logistic

5.19 Consider the predator–prey example described in Section 4.7. Identify the equilibrium points and limit cycles for this system and determine the stability of each of these solutions (using numerical simulation).

[D,1es] dynamics:predprey-limit

5.20 (Windup protection by conditional integration) Closed loop systems where the controller has integral action and the actuator saturates may encounter a phenomenon called integrator windup. One method that has been suggested to avoid the difficulty is to update the integral only when the error is sufficiently small. The drawback of introducing *ad hoc* nonlinearities is illustrated by the following simple case. Consider a system with PI control described by

$$\frac{dx_1}{dt} = u, \qquad u = \operatorname{sat}(k_p e + k_i x_2, -u_0, u_0), \qquad \frac{dx_2}{dt} = \begin{cases} e & \text{if } |e| < 1, \\ 0 & \text{if } |e| \geq e_0, \end{cases}$$

where $e = r - x$. Plot the phase portrait of the system for the parameter values $k_p = 1$, $k_i = 1$, $u_0 = 1$, and $e_0 = 1$ and discuss the properties of the system. (Integral windup and other ways to avoid it are discussed in Section 11.4.)

*Instructor note:*This exercise appears in Chapter 11 as Exercise 11.10. It can also be used at this point, especially in courses where students have already been exposed to windup.

[B,1es] dynamics:windup-condint

5.21 Consider a nonlinear control system with gain scheduled feedback

$$\dot e = f(e, v) \qquad v = k(\mu) e,$$

[C,?] dynamics:gainsched-stability

where $\mu(t) \in \mathbb{R}$ is an externally specified parameter (e.g., the desired trajectory) and $k(\mu)$ is chosen such that the linearization of the closed loop system around the origin is stable for each fixed μ.

Show that if $|\dot{\mu}|$ is sufficiently small then the equilibrium point is locally asymptotically stable for the full nonlinear, time-varying system. (Hint: find a Lyapunov function of the form $V = x^T P(\mu) x$ based on the linearization of the system dynamics for fixed μ and then show this is a Lyapunov function for the full system.)

[B,1es] dynamics:fitzhugh-nagumo **KJA**: Consider rewriting this example to be consistent with work by Rodolphe Sepulchre. See 24 Sep 2016 e-mail from Karl. KJA: Nov 8, I have added a bit to the text in the Modeling Chapter and added exercises.

5.22 (FitzHugh-Nagumo model for spike generation in neurons) The second-order FitzHugh–Nagumo equations

$$\frac{dV}{dt} = 10(V - V^3/3 - R + I_{in}), \qquad \frac{dR}{dt} = 0.8(-R + 1.25V + 1.5),$$

are a simplified version of the Hodgkin–Huxley equations discussed in Example 3.19. The variable V is the voltage across the axon membrane and R is an auxiliary variable that approximates the total effect of ion currents flowing across the membrane. Explore the effect of the input current I_{in}. Determine the equilibrium points and their stability, plot the phase portrait of the equations and simulate a few trajectories. (Hint: Try $I_{\text{in}} = 0.5$ and 1.5. See Section 8.3 in [9].)

[D,1es] dynamics:compsys-consensus

Supplement 3.22

5.23 Consider the consensus protocol introduced in Example 3.17. Show that if the graph of the sensor network is connected and balanced (in-degree equals out-degree at each node), then we can find a gain γ such that the agent states converge to the average value of the measured quantity. (Hint: Use Exercise **??** and the facts that the row and column sums of the Laplacian are zero, the Laplacian is positive semi-definite, and it has one zero eigenvalue for each connected component of the system.)

[C,2e] dynamics:queuing-model

5.24 Consider the queuing model given by equation (3.33) discussed in Example 3.15:

$$\frac{dx}{dt} = \lambda - \mu_{\max}\frac{x}{x+1}.$$

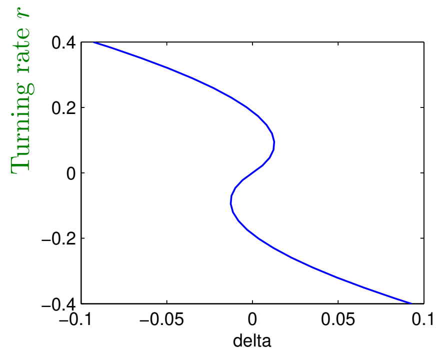
Let the arrival rate $\lambda > 0$ be constant, show that there is a unique positive steady-state solution for any $\lambda < \mu_{\max}$ and that the steady-state solution is stable.

[C,1es] dynamics:tanker-dynamics

5.25 (Steering dynamics of a ship) The normalized steering dynamics of a large ship can be described by the equations

$$\frac{dv}{dt} = a_1 v + a_2 r + \alpha v|v| + b_1\delta, \qquad \frac{dr}{dt} = a_3 v + a_4 r + b_2\delta,$$

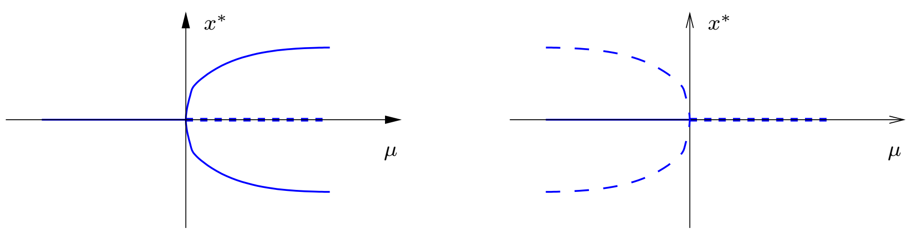
where v is the normalized sway velocity, i.e., the component of the velocity vector that is orthogonal to the long ship direction, r is the turning rate and δ is the rudder angle. (The normalization is made by using the ship length l as length unit and the time to travel one ship length as the time unit. The mass is normalized by $\rho l^3/2$, where ρ is the density of water.) Consider the following parameters: $a_1 = -0.6$, $a_2 = -0.3$, $a_3 = -5$, $a_4 = -2$, $\alpha = -2$, $b_1 = 0.1$, and $b_2 = -0.8$. Determine all of the steady-state solutions that are obtained when the rudder is fixed in the midship position ($\delta = 0$). For each equilibrium point, linearize the system about the equilibrium point, determine the stability of the equilibrium point and describe how the ship will behave near this equilibrium point. (Hint: The derivative of the function $v|v|$ is $2v$ if v is positive and $-2v$ if v is negative, this can be written as $2|v|$. Hint. The figure below shows the turning rate as a function of the rudder angle.)



5.26 (Pitchfork bifurcation) Consider the scalar dynamical system [C,1es] dynamics:pitchfork

$$\frac{dx}{dt} = \mu x - \rho x^3.$$

Show that the equilibrium values of x have the form shown below, with solid lines representing stable equilibrium points and dashed lines representing unstable equilibrium points:



(a) supercritical pitchfork (b) subcritical pitchfork

RMM: Check font sizes

Label each branch according to the signs of μ and ρ that correspond to the equilibrium point.

5.27 For each of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable) or unstable. To determine stability, you can either use a phase portrait (if appropriate), analyze the linearization or simulate the system using multiple nearby initial conditions to determine how the state evolves. [C,?] dynamics:stabexmp

*Instructor note:*5pts max per system. 1pt for determining equilibrium points. 4 pts for stability analysis. If eigenvalue stability was used, 2pts for correct linearization and 2 pts for stability explanation. If phase curve was used, 3 pts for the phase curve, 1pt for stability discussion. (A lot of people who chose to use phase plots loss that point).

(a) *Nonlinear spring mass.* Consider a nonlinear spring mass system with dynamics

$$m\ddot{q} = -k(q - aq^3) - c\dot{q},$$

where $m = 1000$ kg is the mass, $k = 250$ kg/s^2 is the nominal spring constant, $a = 0.01$ represents the nonlinear "softening" coefficient of the spring and $c = 100$ kg/s is the damping coefficient. Note that this is very similar to the spring mass system we have studied in Section 3.2, except for the nonlinearity.

(b) *Predator prey ODE.* Consider the following alternative model for a predator-prey system, known as the *Lotka–Volterra equations*:

$$\frac{dx_1}{dt} = b_\mathrm{h} x_1 - a x_1 x_2, \qquad \frac{dx_2}{dt} = b x_1 x_2 - d_\mathrm{l} x_2.$$

Here x_1 represents the population of the prey and x_2 represents the population of the predator. Use the parameters $b_\mathrm{h} = 0.7$, $d_\mathrm{l} = 0.5$, $a = 0.007$, and $b = 0.0005$.

Exercise 3.10

(c) *Genetic toggle switch.* Consider the dynamics of two repressors connected together in a cycle. It can be shown (Exercise 3.10) that the normalized dynamics of the system can be written as

$$\frac{dz_1}{d\tau} = \frac{\mu}{1 + z_2^n} - z_1 - v_1, \qquad \frac{dz_2}{d\tau} = \frac{\mu}{1 + z_1^n} - z_2 - v_2.$$

where z_1 and z_2 represent scaled versions of the protein concentrations, v_1 and v_2 represent external inputs and the time scale has been changed. Let $\mu = 2.16$, $n = 2$, and $v_1 = v_2 = 0$.

(d) *Congestion control of the Internet.* A simplified model for congestion control between N computers connected by a router is given by the differential equation

$$\frac{dx_i}{dt} = -b\frac{x_i^2}{2} + (b_\mathrm{max} - b), \qquad \frac{db}{dt} = \Big(\sum_{i=1}^{N} x_i\Big) - c,$$

where $x_i \in \mathbb{R}$, $i = 1, \dots, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_\mathrm{max} > 0$ is the maximum buffer size, and $c > 0$ is the capacity of the link connecting the router to the computers. The $\dot{x}_i$ equation represents the control law that the individual computers use to determine how fast to send data across the network and the $\dot{b}$ equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1, x_2 and b) and take $b_\mathrm{max} = 1$ Mb and $c = 2$ Mb/s.

(e) *Inverted pendulum.* The equations of motion for a single inverted pendulum are given by

$$ml^2\ddot{\theta} = -c\dot{\theta} - mgl\sin(\theta),$$

where θ is the angle of the pendulum ($\theta = 0$ rad corresponds to pointing down), $m = 1$ kg is the mass of the pendulum (assumed concentrated at the end), $l = 0.5$ m is the length of the pendulum, $c = 0.25$ N m s is the damping coefficient and $g = 9.8$ m/s^2 is the gravitational constant.

(f) *Moore–Greitzer model.* The Moore-Greitzer equations model rotating stall and surge in gas turbine engines:

$$
\begin{aligned}
\frac{d\psi}{dt} &= \frac{1}{4B^2 l_c}\left(\varphi - \Phi_T(\psi)\right),\\
\frac{d\varphi}{dt} &= \frac{1}{l_c}\left(\Phi_c(\varphi) - \psi + \frac{J}{8}\frac{\partial^2 \Psi_c}{\partial \varphi^2}\right),\\
\frac{dJ}{dt} &= \frac{2}{\mu + m}\left(\frac{\partial \Phi_c}{\partial \varphi} + \frac{J}{8}\frac{\partial^3 \Phi_c}{\partial \varphi^3}\right) J,
\end{aligned}
$$

where

$$
\begin{aligned}
B &= 0.2, & \Phi_T(\psi) &= \sqrt{\psi},\\
l_c &= 6, & \Psi_c(\varphi) &= 1 + 1.5\varphi - 0.5\varphi^3,\\
\mu &= 1.256, & m &= 2.
\end{aligned}
$$

This is a model for the dynamics of the compression system (first part of a jet engine) with ψ representing the pressure rise across the compressor, φ representing the mass flow through the compressor and J representing the amplitude squared of the first modal flow perturbation (corresponding to a rotating stall disturbance). (Hint: There is more than one equilibrium point and not all of them are stable.)

Add listing of python code? **RMM**

5.28 Find a Lyapunov function for the cruise control system in Exercise 5.3, showing that the system is locally asymptotically stable at the desired speed. If you like, you can use the specific parameters listed above, although it is also possible to solve the problem leaving parameter values unspecified (with some assumptions, which you should state). [C,?] dynamics:cruise-lyapunov Exercise 5.3

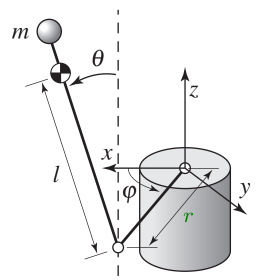
5.29 For each of the systems in the table below, defined in more detail in Exercise 5.27, determine if there exists a Lyapunov function of the given form that proves that the indicated equilibrium point is asymptotically stable. The parameter γ should be taken as a free (scalar) parameter and used as needed to satisfy the conditions of the Lyapunov theorem. You should try to solve the problem for general parameter values if possible, but if you can't find a general solution in a reasonable amount of time, then you should use the numerical values from Problem 5.27. [C,?] dynamics:lyapexmp

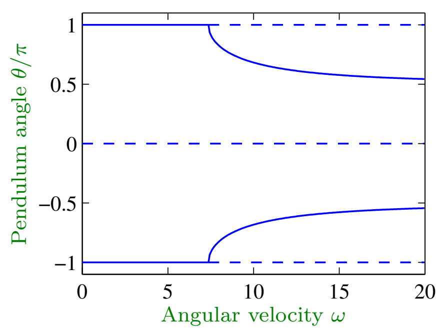
Part	System/equilibrium point	Parameters	Lyapunov function candidate
(a)	Nonlinear spring mass, $x_\mathrm{e} = (0,0)$	$m, k, a, c > 0$	$V = kx^2 + \gamma x\dot{x} + m\dot{x}^2$
(b)	Predator prey ODE, $x_\mathrm{e} \neq (0,0)$	$a, b, b_\mathrm{h}, d_\mathrm{l} > 0$	$V = (x_1 - x_{e,1})^2 + \gamma(x_2 - x_{e,2})^2$
(c)	Congestion control, $x_\mathrm{e}, b_\mathrm{e} \neq 0$	$N, b_\mathrm{max}, c > 0$	$V = \gamma \sum (x_i - x_{e,i})^2 + (b - b_\mathrm{e})^2$

Note: for some of these systems, the equilibrium point may be asymptotically stable but the Lyapunov function candidate may not allow you to prove stability. This is one of the limitations of Lyapunov stability: you have to *find* a Lyapunov function that proves stability of the system. *Optional:* If you are not able to find a Lyapunov function of the given form for an equilibrium point that you showed in problem 1 is asymptotically stable, try to find a Lyapunov function of a more general form that works. *Optional:* For those systems in which you are able to find a Lyapunov function, determine whether the given function can also be used to prove whether the system is exponentially stable and whether the system is globally asymptotically stable.

[B,1ep] dynamics:furuta

5.30 (Furuta pendulum) The Furuta pendulum, an inverted pendulum on a rotating arm, is shown to the left in the figure below.





Consider the situation when the pendulum arm is spinning with constant rate. The system has multiple equilibrium points that depend on the angular velocity ω, as shown in the bifurcation diagram on the right.

The equations of motion for the system are given by

$$J_\mathrm{p}\ddot{\theta} - J_\mathrm{p}\omega_0^2 \sin\theta\cos\theta - m_\mathrm{p} g l \sin\theta = 0,$$

where J_p is the moment of inertia of the pendulum with respect to its pivot, m_p is the pendulum mass, l is the distance between the pivot and the center of mass of the pendulum, and ω_0 is the the rate of rotation of the arm.

(a) Determine the equilibrium points for the system and the condition(s) for stability of each equilibrium point (in terms of ω_0).

(b) Consider the angular velocity as a bifurcation parameter and verify the bifurcation diagram given above. This is an example of a *pitchfork bifurcation*.

Chapter 6 – Linear Systems

6.1 (Response to the derivative of a signal) Show that if $y(t)$ is the output of a linear time-invariant system corresponding to input $u(t)$, then the output corresponding to an input $\dot{u}(t)$ is given by $\dot{y}(t)$. (Hint: Use the definition of the derivative: $\dot{z}(t) = \lim_{\epsilon\to 0}\bigl(z(t+\epsilon) - z(t)\bigr)/\epsilon$.) [B,1ep*] linsys:dersys

6.2 (Impulse response and convolution) Show that a signal $u(t)$ can be decomposed in terms of the impulse function $\delta(t)$ as [C,1ep*] linsys:impulse-response

$$u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau$$

and use this decomposition plus the principle of superposition to show that the response of a linear, time-invariant system to an input $u(t)$ (assuming a zero initial condition) can be written as a convolution equation

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response of the system. (Hint: Use the definition of the Riemann integral.)

6.3 (Pulse response for a compartment model) Consider the compartment model given in Example 6.7. Compute the step response for the system and compare it with Figure 6.10b. Use the principle of superposition to compute the response to the 5 s pulse input shown in Figure 6.10c. Use the parameter values $k_0 = 0.1$, $k_1 = 0.1$, $k_2 = 0.5$, and $b_0 = 1.5$. [B,1ep*] linsys:compartment-pulse

6.4 (Matrix exponential for second-order system) Assume that $\zeta < 1$ and let $\omega_\mathrm{d} = \omega_0\sqrt{1-\zeta^2}$. Show that [A,1ep*] linsys:matrixexp

$$\exp\begin{pmatrix} -\zeta\omega_0 & \omega_\mathrm{d} \\ -\omega_\mathrm{d} & -\zeta\omega_0 \end{pmatrix} t = e^{-\zeta\omega_0 t}\begin{pmatrix} \cos\omega_\mathrm{d} t & \sin\omega_\mathrm{d} t \\ -\sin\omega_\mathrm{d} t & \cos\omega_\mathrm{d} t \end{pmatrix}.$$

Also show that

$$\exp\left(\begin{pmatrix} -\omega_0 & \omega_0 \\ 0 & -\omega_0 \end{pmatrix} t\right) = e^{-\omega_0 t}\begin{pmatrix} 1 & \omega_0 t \\ 0 & 1 \end{pmatrix}.$$

Use the results of this problem and the convolution equation to compute the unit step response for a spring mass system

$$m\ddot{q} + c\dot{q} + kq = F$$

with initial condition $x(0)$.

Instructor note: The optional part of this problem is a messy computational problem and the solution is not yet complete.

[B,1ep]
linsys:lyap-linear

6.5 (Lyapunov function for a linear system) Consider a linear system $\dot{x} = Ax$ with $\operatorname{Re}\lambda_j < 0$ for all eigenvalues λ_j of the matrix A. Show that the matrix

$$P = \int_0^\infty e^{A^T\tau} Q e^{A\tau}\, d\tau$$

defines a Lyapunov function of the form $V(x) = x^T P x$ with $Q \succ 0$ (positive definite).

[B,1ep*]
linsys:jordan-nontrivial

6.6 (Nondiagonal Jordan form) Consider a linear system with a Jordan form that is non-diagonal.

(a) Prove Proposition 6.3 in *Feedback Systems* by showing that if the system contains a real eigenvalue $\lambda = 0$ with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time.

(b) Extend this argument to the case of complex eigenvalues with $\operatorname{Re}\lambda = 0$ by using the block Jordan form

$$J_i = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}.$$

[B,1ep*]
linsys:risetime-firstord

6.7 (Rise time and settling time for a first-order system) Consider a first-order system of the form

$$\tau \frac{dx}{dt} = -x + u, \qquad y = x.$$

We say that the parameter τ is the *time constant* for the system since the zero input system approaches the origin as $e^{-t/\tau}$. For a first-order system of this form, show that the rise time for a step response of the system is approximately 2τ, and that 1%, 2%, and 5% settling times approximately corresponds to 4.6τ, 4τ, and 3τ.

[A,1ep]
linsys:discrete-linsys

6.8 (Discrete-time systems) Consider a linear discrete-time system of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] \;\; = Cx[k] + Du[k].$$

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j] + Du[k].$$

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of A have a magnitude strictly less than 1.

(c) Show that a discrete-time linear system is unstable if any of the eigenvalues of A have magnitude greater than 1.

(d) Derive conditions for stability of a discrete-time linear system having one or more eigenvalues with magnitude identically equal to 1. (Hint: use Jordan form.)

(e) Let $u[k] = \sin(\omega k)$ represent an oscillatory input with frequency $\omega < \pi$ (to avoid "aliasing"). Show that the steady-state component of the response has gain M and phase θ, where

$$Me^{i\theta} = C(e^{i\omega}I - A)^{-1}B + D.$$

(f) Show that if we have a nonlinear discrete-time system

$$\begin{aligned} x[k+1] &= f(x[k], u[k]), \qquad & x[k] \in \mathbb{R}^n,\ u \in \mathbb{R}, \\ y[k] &= h(x[k], u[k]), \qquad & y \in \mathbb{R}, \end{aligned}$$

then we can linearize the system around an equilibrium point $(x_{\mathrm{e}}, u_{\mathrm{e}})$ by defining the matrices A, B, C, and D as in equation (6.35).

*Instructor note:*Tags:

- discrete-linsys-partc: Frequency response of a discrete-time system
- discrete-linsys-partd: Linearization of a discrete time system

*Instructor note:*To simplify this problem, consider focusing just on the diagonal case or the case where A has a full basis of eigenvectors.

6.9 (Keynesian economics) Consider the following simple Keynesian macroeconomic model in the form of a linear discrete-time system discussed in Exercise 6.8: [B,1ep] linsys:discrete-keynes Exercise 6.8

$$\begin{pmatrix} C[t+1] \\ I[t+1] \end{pmatrix} = \begin{pmatrix} a & a \\ ab-b & ab \end{pmatrix} \begin{pmatrix} C[t] \\ I[t] \end{pmatrix} + \begin{pmatrix} a \\ ab \end{pmatrix} G[t],$$
$$Y[t] = C[t] + I[t] + G[t].$$

Determine the eigenvalues of the dynamics matrix. When are the magnitudes of the eigenvalues less than 1? Assume that the system is in equilibrium with constant values capital spending C, investment I, and government expenditure G. Explore what happens when government expenditure increases by 10%. Use the values $a = 0.25$ and $b = 0.5$.

6.10 (Keynes model in continuous time) A continuous version of Keynes model is given by the equations [B,?] linsys:keynes-ode

$$Y = C + I + G, \qquad T\frac{dC}{dt} + C = ay, \qquad T\frac{dI}{dt} + I = b\frac{dc}{dt}.$$

Write the equations in state space form, and give the conditions for stability.

[B,1es]
linsys:compartment-states

6.11 (State variables in compartment models) Consider the compartment model described by equation (4.28). Let x_1 and x_2 be the total mass of the drug in the compartments. Show that the system can be described by the equation

$$\frac{dx}{dt} = \begin{pmatrix} -k_0 - k_1 & k_2 \\ k_1 & -k_2 \end{pmatrix} x + \begin{pmatrix} c_0 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1/V_2 \end{pmatrix} x. \tag{S6.1}$$

Compare the this equation with equation (4.28), where the state variables were concentrations. Mass is called an *extensive variable*, and concentration is called an *intensive variable*.

[?,2e*]
linsys:timeresp-fft

6.12 (Time responses from frequency responses) Consider the following MATLAB program, which computes the approximate step response from the frequency response. Explain how it works and explore the effects of the parameter `tmax`.

```
P = '1./(s+1).^2';       % process dynamics
tmax = 20;               % simulation time
N = 2^(12);              % number of points for simulation
dt = tmax/N;             % time interval
dw = 2*pi/tmax;          % frequency interval

% Compute the time and frequency vectors
t = dt*(0:N-1);
omega = -pi/dt:dw:(pi/dt-dw);
s = i*omega;

% Evaluate the frequency response
pv=eval(P);

% Compute the input and output signals using the frequency response
u = [ones(1,N/2) zeros(1,N/2)]; U = fft(u);
y = ifft(fftshift(pv) .* U); y = real(y);

%  Analytic solution in the time domain
ye = 1 - exp(-t) - t .* exp(-t);

% Plot analytic and approximate step responses
subplot(211); plot(t, y, 'b-', t, ye, 'r--');

% Zoom in on the first half of the response
tp = t(1:N/2); yp = y(1:N/2); ye = 1-exp(-t) - t .* exp(-t);
subplot(212); plot(tp, yp, 'b-', t, ye, 'r--');
```

Comment [RMM, 2018]: This exercise is a very different style compared with all others. Need to discuss if we decide to include it in the printed text.

Response [KJA, 16 Aug 2018]: I think it is cool that we can compute time responses for irrational transfer functions, should we introduce other examples of similar type or is there any ohter way of handling it?

Response [KJA, 15 Jul 2019]: Maybe that we could make this to somehting clickable. I added a bit of text.

6.13 Consider a scalar system [B,1ep] linsys:linearization-scalar

$$\frac{dx}{dt} = 1 - x^3 + u.$$

Compute the equilibrium points for the unforced system ($u = 0$) and use a Taylor series expansion around the equilibrium point to compute the linearization. Verify that this agrees with the linearization in equation (6.34).

6.14 Consider the model for queuing dynamics in Example 3.15. Let the admission rate λ be the control variable. Linearize the system around an equilibrium point, compute the time constant of the system and determine how it depends on the queue length. [N,1es] linsys:queue-linsys

6.15 (Transcriptional regulation) Consider the dynamics of a genetic circuit that implements *self-repression*: the protein produced by a gene is a repressor for that gene, thus restricting its own production. Using the models presented in Example 3.18, the dynamics for the system can be written as [A,1ep] linsys:biocircuits-negfbk

$$\frac{dm}{dt} = \frac{\alpha}{1 + kp^2} + \alpha_0 - \delta m - u, \qquad \frac{dp}{dt} = \kappa m - \gamma p, \tag{S6.2}$$

where u is a disturbance term that affects RNA transcription and $m, p \geq 0$. Find the equilibrium points for the system and use the linearized dynamics around each equilibrium point to determine the local stability of the equilibrium point and the step response of the system to a disturbance.

6.16 (Monotone step response) Consider a stable linear system with monotone step response $S(t)$. Let the input signal be bounded: $|u(t)| \leq u_{\max}$. Assuming that the initial conditions are zero, show that $|y(t)| \leq S(\infty)u_{\max}$. (Hint: Use the convolution integral.) [B,1es] linsys:conveq-monotone

Supplemental Exercises

6.17 (Normalized coordinates for second-order system) Assume that $\zeta < 1$ and let $\omega_\mathrm{d} = \omega_0\sqrt{1-\zeta^2}$. Show that the systems [C,1es] linsys:transform-oscillator

$$\frac{dx}{dt} = \begin{pmatrix} -\zeta\omega_0 & \omega_\mathrm{d} \\ -\omega_\mathrm{d} & -\zeta\omega_0 \end{pmatrix} x, \qquad \frac{dz}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} z$$

are related through $z = Tx$ where

$$T = \begin{pmatrix} \zeta + \sqrt{1-\zeta^2} & \zeta - \sqrt{1-\zeta^2} \\ 1 & 1 \end{pmatrix}.$$

6.18 State feedback for block diagram in reachable canonical form Consider a linear system in reachable canonical form, whose block diagram is shown in Figure 7.4. Show directly in the block diagram that any characteristic polynomial can be obtained by state feedback. [C,2e] linsys:reachable-block-sfb

[C,1es] linsys:matexp-jordan3x3

6.19 (Matrix exponential for Jordan form) Using the computation for the matrix exponential, show that equation (6.11) in *Feedback Systems* holds for the case of a 3×3 Jordan block. (Hint: Decompose the matrix into the form $S + N$, where S is a diagonal matrix.)

[C,1es] linsys:conveq-secord

6.20 (Solution of a second-order system) Using the convolution equation, write down the complete solution for a second-order linear system with sinusoidal input:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = A\sin\omega t.$$

Make sure to describe any cases in which the form of the solution changes.

[C,2e] linsys:coord-trans

6.21 (Coordinate transformations) Consider the linear system with matrices

$$A = \begin{pmatrix} -0.5 & 0.5 \\ -1.5 & -2.5 \end{pmatrix}, \qquad B = \begin{pmatrix} -0.5 \\ 2.5 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 2 \end{pmatrix}$$

Determine the systems obtained after coordinate changes $z = Tx$ with the matrices

$$T_1 = \begin{pmatrix} -2.0 & 0 \\ 2.5 & 0.5 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 4 & 2 \\ 7 & 3 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Explore the transformed systems and explore if you can find some patterns.

[C,1es] linsys:purecosine

6.22 Consider the linear system (6.3) and let the input be $u = \cos\omega t$. Show that if the initial condition is chosen as

$$x(0) = -A(\omega^2 I + A^2)^{-1}B,$$

then there is no transient and that the output is a cosine function. Pick a system and verify the result by simulation.

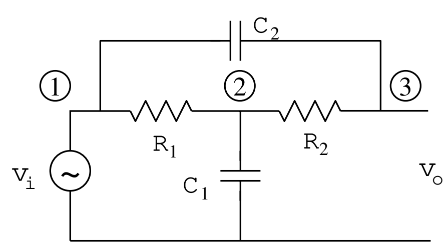
[D,1es] linsys:linsys-diffeq

6.23 Show that the differential equation (6.3) is a linear input/output system using the definitions in Section 6.1.

6.24 For each of the following linear systems, determine whether the origin is asymptotically stable and, if so, plot the step response and frequency response for the system. If there are multiple inputs or outputs, plot the response for each pair of inputs and outputs.

(a) *Coupled spring mass system.* Consider the coupled mass spring system from Example 6.6 with $m = 250$, $k = 50$, and $c = 10$. The input $u(t)$ is the force applied to the right-most spring and the outputs are the positions q_1 and q_2.

(b) *Bridged Tee Circuit.* Consider the following electrical circuit, with input v_i and output $y = v_o$.



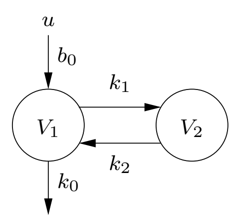
The dynamics are given by

$$\frac{d}{dt}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{pmatrix}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} + \begin{pmatrix} \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \\ \frac{1}{C_2 R_2} \end{pmatrix} v_i,$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} + v_i,$$

where v_{c1} and v_{c2} are the voltages across the two capacitors. Assume that $R_1 = 100\ \Omega$, $R_2 = 100\ \Omega$, and $C_1 = C_2 = 1 \times 10^{-6}$ F.

(c) *Compartment model.* Consider the two-compartment model described in Section 4.6 and shown below.



The dynamics for this system can be written as

$$\frac{dc}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} c.$$

Use the parameter values $k_0 = 0.1$, $k_1 = 0.1$, $k_2 = 0.5$, and $b_0 = 1.5$.

6.25 Consider the balance system described in Example 2.1 of the text, using the following parameters: [C,?] linsys:cartpend

$$\begin{aligned} M &= 10 \text{ kg}, & m &= 80 \text{ kg}, & J &= 100 \text{ kg m}^2, \\ c &= 0.1 \text{ N/m/sec}, & l &= 1 \text{ m}, & \gamma &= 0.01 \text{ Nms}, \end{aligned} \qquad g = 9.8 \text{ m/s}^2.$$

This system has been modeled in SIMULINK in the file `balance_simple.mdl`, available from the course web page. (Note: in the SIMULINK model, the output has been set to include all of the states ($y = x$). You will need this for part (c) below.)

(a) Use the MATLAB `linmod` command to numerically compute the linearization of the original nonlinear system at the equilibrium point $(q, \theta, \dot{q}, \dot{\theta}) = (0, 0, 0, 0)$. Compare the eigenvalues of the analytical linearization (from the text) to those of the one you obtained with `linmod` and verify they agree. (Make sure to look at the errata sheet for the text; there are some small glitches in the equations listed in both Example 2.1 and Example 6.7.)

(b) We can design a stabilizing control law for this system using "state feedback", which is a control law of the form $u = -Kx$ (we will learn about this more next week). The closed loop system under state feedback has the form

$$\frac{dz}{dt} = (A - BK)z.$$

Show that the following state feedback stabilizes the linearization of the inverted pendulum on a cart: $K = [-15.3\ 1730\ \ -50\ 443]$.

(c) Now build a simulation for the closed loop, *nonlinear* system in SIMULINK. Use the file `balance_simple.mdl` for the nonlinear equations of motion in it (you should look in the file and try to understand how it works). Simulate several different initial conditions and show that the controller *locally* asymptotically stabilizes the system to x_e from these initial conditions. Include plots of a representative simulation for an initial condition that is in the region of attraction of the controller and one that is outside the region of attraction.

6.26 Consider a stable linear time-invariant system. Assume that the system is initially at rest and let the input be $u = \sin \omega t$, where ω is much larger than the magnitudes of the eigenvalues of the dynamics matrix. Show that the output is approximately given by

$$y(t) \approx |G(i\omega)| \sin\big(\omega t + \arg G(i\omega)\big) + \frac{1}{\omega} h(t),$$

where $G(s)$ is the frequency response of the system and $h(t)$ its impulse response.

*Instructor note:*This exercise makes use of the notation of a transfer function and is appropriate for students who have some prior background in frequency domain modeling.

6.27 Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

which is stable but not asymptotically stable. Show that if the system is driven by the bounded input $u = \cos t$ then the output is unbounded.

Chapter 7 – State Feedback

7.1 (Double integrator) Consider the double integrator. Find a piecewise constant control strategy that drives the system from the origin to the state $x = (1, 1)$. [N,1ep] statefbk:reachable-doubleint

7.2 (Reachability from nonzero initial state) Extend the argument in Section 7.1 in *Feedback Systems* to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state. [B,1ep] statefbk:reachable-nonzeroic

7.3 (Cayley–Hamilton theorem) Let $A \in \mathbb{R}^{n\times n}$ be a matrix with characteristic polynomial $\lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. Show that the matrix A satisfies [C,1ep*] statefbk:cayley-hamilton

$$\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0,$$

where the zero on the right hand side represents a matrix of elements with all zeros. Use this result to show that A^n can be written in terms of lower order powers of A and hence any matrix polynomial in A can be rewritten using terms of order at most $n - 1$.

*Instructor note:*Use tag ‘diagonal‘ for a version of the problem in which the matrix A is assumed to be diagonal.

7.4 (Unreachable systems) Consider a system with the state x and z described by the equations [C,1ep*] statefbk:unreachable

$$\frac{dx}{dt} = Ax + Bu, \qquad \frac{dz}{dt} = Az + Bu.$$

If $x(0) = z(0)$ it follows that $x(t) = z(t)$ for all t regardless of the input that is applied. Show that this violates the definition of reachability and further show that the reachability matrix W_r is not full rank. What is the rank of the reachability matrix?

7.5 (Rear-steered bicycle) A simple model for a bicycle was given by equation (4.5) in Section 4.2. A model for a bicycle with rear-wheel steering is obtained by reversing the sign of the velocity in the model. Determine the conditions under which this systems is reachable and explain any situations in which the system is not reachable. [B,1ep] statefbk:bicycle-rearsteer

7.6 (Characteristic polynomial for reachable canonical form) Show that the characteristic polynomial for a system in reachable canonical form is given by equation (7.7) and that [B,1ep] statefbk:reachable-form

$$\frac{d^n z_k}{dt^n} + a_1 \frac{d^{n-1} z_k}{dt^{n-1}} + \cdots + a_{n-1}\frac{dz_k}{dt} + a_n z_k = \frac{d^{n-k}u}{dt^{n-k}},$$

where z_k is the kth state.

[N,1ep] statefbk:inverse-reachmat

7.7 (Reachability matrix for reachable canonical form) Consider a system in reachable canonical form. Show that the inverse of the reachability matrix is given by

$$\tilde{W}_{\mathrm{r}}^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ & 1 & a_1 & \cdots & a_{n-2} \\ & & 1 & \ddots & \vdots \\ & 0 & & \ddots & a_1 \\ & & & & 1 \end{pmatrix}.$$

[B,1ep] statefbk:pendcart-noref

7.8 (Non-maintainable equilibrium points) Consider the normalized model of a pendulum on a cart

$$\frac{d^2x}{dt^2} = u, \qquad \frac{d^2\theta}{dt^2} = -\theta + u,$$

where x is cart position and θ is pendulum angle. Can the angle $\theta = \theta_0$ for $\theta_0 \neq 0$ be maintained?

[N,1ep] statefbk:robust-statespace

7.9 (Eigenvalue assignment) Consider the system

$$\frac{dx}{dt} = Ax + Bu = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} a-1 \\ 1 \end{pmatrix} u,$$

with $a = 1.25$. Design a state feedback that gives $\det(sI - BK) = s^2 + 2\zeta_{\mathrm{c}}\omega_{\mathrm{c}} s + \omega_{\mathrm{c}}^2$, where $\omega_{\mathrm{c}} = 5$, and $\zeta_{\mathrm{c}} = 0.6$.

[B,1ep] statefbk:unreachable-assign

7.10 (Eigenvalue assignment for unreachable system) Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_{\mathrm{f}} r.$$

Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

[N,1ep*] statefbk:dcmotor-statefbk Exercise 3.7

7.11 (Motor drive) Consider the normalized model of the motor drive in Exercise 3.7. Using the following normalized parameters,

$$J_1 = 10/9, \qquad J_2 = 10, \qquad c = 0.1, \qquad k = 1, \qquad k_I = 1,$$

verify that the eigenvalues of the open loop system are $0, 0, -0.05 \pm i$. Design a state feedback that gives a closed loop system with eigenvalues -2, -1, and $-1 \pm i$. This choice implies that the oscillatory eigenvalues will be well damped and that the eigenvalues at the origin are replaced by eigenvalues on the negative real axis. Simulate the responses of the closed loop system to step changes in the reference signal for θ_2 and a step change in a disturbance torque on the second rotor.

7.12 (Whipple bicycle model) Consider the Whipple bicycle model given by equation (4.8) in Section 4.2. Using the parameters from the companion web site, the model is unstable at the velocity $v_0 = 5$ m/s and the open loop eigenvalues are -1.84, -14.29, and $1.30 \pm 4.60i$. Find the gains of a controller that stabilizes the bicycle and gives closed loop eigenvalues at -2, -10, and $-1 \pm i$. Simulate the response of the system to a step change in the steering reference of 0.002 rad.

[N,1ep] statefbk:bicycle-whipple

Next, find the controller gains corresponding to choosing the final pair of complex poles at $-2 \pm 2i$ and $-5 \pm 5i$.† In addition to calculating the state feedback gains, make sure to solve for the feedforward gain k_f as well. For each case, simulate the response to a step change in the steering reference of 0.002 rad and plot both the steering angle and the torque command.

RMM: Wording confusing; perhaps given the entire list of poles for the two additional cases?

7.13 (Performance specifications and transfer functions) Find the transfer function of a second order system that satisfies the following closed loop specifications: zero steady-state error, 2% settling time less than 2 s, rise time less than 0.8 s, and overshoot less than 3%.

[?,2e] statefbk:req-to-trf

7.14 (Dominant eigenvalues) Consider the following two linear systems:

[?,1ep] statefbk:dominant-pairs

$$\Sigma_1: \begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -1.1 & -0.1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} 1.01 & 0.11 \end{pmatrix} x, \end{aligned} \qquad \Sigma_2: \begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -1.1 & -0.1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} 1.1 & 1.01 \end{pmatrix} x. \end{aligned}$$

Show that although both systems have the same eigenvalues, the step responses of the two systems are dominated by different sets of eigenvalues.

7.15 Consider the second-order system

[B,1ep] statefbk:secord-zero

$$\frac{d^2y}{dt^2} + 0.5\frac{dy}{dt} + y = a\frac{du}{dt} + u.$$

Let the initial conditions be zero.

(a) Show that the initial slope of the unit step response is a. Discuss what it means when $a < 0$.

(b) Show that there are points on the unit step response that are invariant with a. Discuss qualitatively the effect of the parameter a on the solution.

(c) Simulate the system and explore the effect of a on the rise time and overshoot.

7.16 (Integral feedback for rejecting constant disturbances) Consider a linear system of the form

[B,1ep*] statefbk:integral-noiserej

$$\frac{dx}{dt} = Ax + Bu + Fd, \qquad y = Cx$$

where u is a scalar and v is a disturbance that enters the system through a disturbance vector $F \in \mathbb{R}^n$. Assume that the matrix A is invertible and the zero frequency gain $CA^{-1}B$ is nonzero. Show that integral feedback can be used to compensate for a constant disturbance by giving zero steady-state output error even when $d \neq 0$.

*Instructor note:*The solution for this exercise makes use of the Cayley–Hamilton theorem (Exercise 7.3), which it might make sense to assign (or at least introduce) first.

[?,1ep*] statefbk:brysons-rule

7.17 (Bryson's rule) Bryson and Ho [3] have suggested the following method for choosing the matrices Q_x and Q_u in equation (7.29). Start by choosing Q_x and Q_u as diagonal matrices whose elements are the inverses of the squares of the maxima of the corresponding variables. Then modify the elements to obtain a compromise among response time, damping, and control effort. Apply this method to the motor drive in Exercise 7.11. Assume that the largest values of the φ_1 and φ_2 are 1, the largest values of $\dot\varphi_1$ and $\dot\varphi_2$ are 2 and the largest control signal is 10. Simulate the closed loop system for $\varphi_2(0) = 1$ and all other states are initialized to 0. Explore the effects of different values of the diagonal elements for Q_x and Q_u.

Exercise 7.11

[?,2e] statefbk:lqr-proof

RMM: New exercise please check

7.18 (LQR proof) Use the Riccati equation (7.31) and the relation

$$x^T(t_\mathrm{f})Q_\mathrm{f}x(t_\mathrm{f}) - x^T(0)S(0)x(0) = \int_0^{t_\mathrm{f}} \Big(\dot x^T(t)S(t)x(t) + x^T\dot S(t)x(t) + x^T(t)S(t)\dot x(t)\Big)dt.$$

to show that the cost function for the linear quadratic regulator problem can be written as

$$\int_0^{t_\mathrm{f}} \Big(x^T(t)Q_x x(t) + u^T(t)Q_u u(t)\Big)dt + x^T(t_\mathrm{f})Q_\mathrm{f}x(t_\mathrm{f})$$
$$= x^T(0)S(0)x(0) + \int_0^{t_\mathrm{f}} \Big(u(t) + Q_u^{-1}B^TS(t)x(t)\Big)^T Q_u \Big(u(t) + Q_u^{-1}B^TS(t)x(t)\Big)dt,$$

from which it follows that the control law $u(t) = -Kx(t) = -Q_u^{-1}B^TS(t)x(t)$ is optimal. Does the proof hold when all matrices depend on time?

Supplemental Exercises Comment [RMM, 27 Aug 2008]: The following three exercises were removed based on conversations with Karl. (It would be nice to indicate why they were removed.) The first of these was referenced in the text, so we might want to replace it with something?

[C,1es?] statefbk:unreachable-pendula

7.19 (Unreachable systems (variant)) Consider the system shown in Figure 7.3. Write the dynamics of the two systems as

$$\frac{dx}{dt} = Ax + Bu, \qquad \frac{dz}{dt} = Az + Bu.$$

If $x(0) = z(0)$ it follows that $x(t) = z(t)$ for all t regardless of the input that is applied. Show that this violates the definition of reachability and further show that the reachability matrix W_r is not full rank. What is the rank of the reachability matrix?

*Instructor note:*This problem can be confusing. The figure shows the two pendula rigidly attached to a beam =¿ there are really only three degrees of freedom (6 states) not four DOF (8 states). Also, Figure 7.3 is drawn such that it appears that $\theta_1 = \theta_2$ when the pendula are at opposite angles, rather than the same angle.

7.20 Consider a control system with integral feedback, as given in equations (7.26) and (7.27). Show that the proper value of k_f to achieve $y = r$ with no contribution from the integrator is given by [C,1es] statefbk:integral-refgain

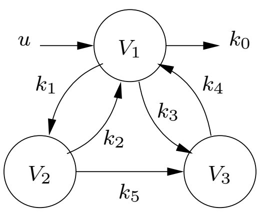
$$k_\text{f} = -1/\bigl(C(A - BK)^{-1}B\bigr),$$

where A, B, and C describe the dynamics of the original system (without the integrator added) and K is the corresponding gain matrix for the process states. Specialize to the inverted pendulum and explore the effects of k_f on the step response from reference to output.

This is not a great exercise as written. If we include this exercise, add additional material to describe how k_f trades off with the integral term. Note from KJA: I agree, tried a simple addition above, but I still think it is too weak **RMM**

7.21 Prove that there exists a transformation matrix T that transforms (A, B) to reachable canonical form if and only if (A, B) is reachable. [?,2e] statefbk:reachable-conditions

7.22 (Unreachable compartment model) Consider the compartment model below with parameters $k_0 = 0.1$, $k_1 = 0.1$, $k_2 = 0.5$, $k_3 = 0.4$, $k_4 = 0.6$, and $k_5 = 0.08$. [D,1es] statefbk:compartment-nonreach



RMM: Add photo showing a relevant physical example?

Assume that a drug is injected in compartment V_1 and that we wish to obtain a given drug concentration in compartment V_3. Determine whether system is reachable.

7.23 Equation (7.13) gives the gain required to maintain a given reference value for a system with no direct term. Compute the feedforward gain in the case where $D \neq 0$. [D,1es] statefbk:refgain-direct

7.24 Build a simulation for the speed controller designed in Example 7.8 and show that even with $k_\text{f} = 0$, the system still achieves zero steady-state error. [D,1es] statefbk:cruise-norefgain

7.25 (Atomic force microscope) Consider the model of an AFM in contact mode given in Example 6.9: [C,1ep] statefbk:afm-scalelqr

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_2/(m_1+m_2) & -c_2/(m_1+m_2) & 1/m_2 & 0 \\ 0 & 0 & 0 & \omega_3 \\ 0 & 0 & -\omega_3 & -2\zeta_3\omega_3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_3 \end{pmatrix} u,$$

$$y = \frac{m_2}{m_1+m_2} \begin{pmatrix} \dfrac{m_1 k_2}{m_1+m_2} & \dfrac{m_1 c_2}{m_1+m_2} & 1 & 0 \end{pmatrix} x.$$

RMM: Put reference to file on web site

Use the MATLAB script `afm_data.m` from the companion web site to generate the system matrices.†

(a) Compute the reachability matrix of the system and numerically determine its rank. Scale the model by using milliseconds instead of seconds as time units. Repeat the calculation of the reachability matrix and its rank.

(b) Find a state feedback controller that gives a closed loop system with complex poles having damping ratio 0.707. Use the scaled model for the computations.

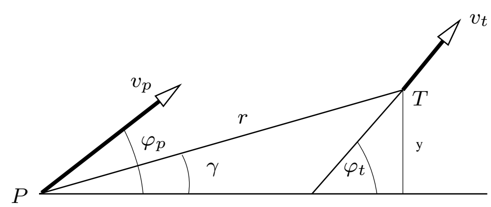
(c) Compute state feedback gains using linear quadratic control. Experiment by using different weights. Compute the gains for $q_1 = q_2 = 0, q_3 = q_4 = 1$ and $\rho_1 = 0.1$ and explain the result. Choose $q_1 = q_2 = q_3 = q_4 = 1$ and explore what happens to the feedback gains and closed loop eigenvalues when you change ρ_1. Use the scaled system for this computation.

*Instructor note:*Part (c) can be made more concrete by refining the specifications for the LQR weights.

[B,2e] statefbk:prop-nav

RMM: New exercise please check

7.26 (Proportional navigation) The figure below is a schematic representation of a two-dimensional pursuit problem. The pursuer is attempting to reach the target as



quickly as possible. Develop a linear model and use linear optimal control theory to derive a control law. You can use the model $\dot{\varphi}_\mathrm{p} = u$ as a model for the pursuer. Express the control law in terms of the line of sight $\gamma \approx y/r$ to the pursuer.

[B,1es] statefbk:dint-sfb

7.27 (State feedback for double integrator) The double integrator is described by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx$$

Determine a state feedback that gives a closed loop system with unit static gain and the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

[B,1es] statefbk:dint-lqg

7.28 (LQR control for double integrator) Consider the double integrator

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu.$$

Find a state feedback that minimizes the quadratic cost function

$$J = \int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_u u^2\right) dt$$

where $q_1 \geq 0$ is the penalty on position, $q_2 \geq 0$ is the penalty on velocity, and $q_u > 0$ is the penalty on control actions. Analyze the coefficients of the closed loop characteristic polynomial and explore how they depend on the penalties.

7.29 (State feedback for inverted pendulum) Consider the normalized, linearized inverted pendulum which is described by [B,1es] statefbk:invpend-sfb

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}u = Ax + Bu, \qquad y = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = Cx$$

Determine a state feedback and feedforward gain $u = -Kx + k_\mathrm{f} r$ that gives a closed loop system with unit static gain (steady-state output $y = r$) and with the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

7.30 (LQR control for inverted pendulum) Consider the normalized, linearized inverted pendulum which is described by [B,1es] statefbk:invpend-lqg

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}u = Ax + Bu.$$

Find a state feedback that minimizes the quadratic cost function

$$J = \int_0^\infty \big(q_1 x_1^2 + q_2 x_2^2 + q_u u^2\big)\,dt$$

where $q_1 \geq 0$ is the penalty on position, $q_2 \geq 0$ is the penalty on velocity, and $q_u > 0$ is the penalty on control actions. Show that the closed loop characteristic polynomial has the form $s^2 + 2\zeta_0\omega_0\, s + \omega_0^2$ and that $\omega_0 \geq 1$ and $\zeta_0 \geq 2/\sqrt{2}$.

7.31 (LQR with integral feedback for a web server) Design a controller for the web server from Example 7.10 that includes integral feedback on the desired processor load and the memory usage. [C,2e] webserver:lqr-integral

7.32 (State feedack for Keynes economic model) Keynes' model for a national economy, discussed in Exercise 3.3, is a simple discrete model described by [B,1es] statefbk:keynes-sfb Exercise 3.3

$$\begin{pmatrix}C[k+1]\\I[k+1]\end{pmatrix} = \begin{pmatrix}a & a\\ab-b & ab\end{pmatrix}\begin{pmatrix}C[k]\\I[k]\end{pmatrix} + \begin{pmatrix}a\\ab\end{pmatrix}G[k],$$
$$Y[k] = C[k] + I[k] + G[k],$$

where C denotes consumption, I investmend, G government expenditure and Y the GNP. Let the time increment be a quarter year and let the parameters be $a = 0.8$ and $b = 1.25$. Show that the system is marginally stable and simulate the effect of a unit increase in government expenditure. Then design a state feedback such that the closed loop system has two eigenvalues at $\lambda = 0.5$. Simulate the response of the closed loop system to a unit step in government spending and compare with the open loop results.

7.33 (Linear quadratic regulator) Consider the first-order system [?,2e] statefbk:lqr-first-order **RMM**: New exercise please check

$$\frac{dx}{dt} = ax + bu, \qquad x(0) = x_0,$$

where all variables are scalar. Find a control law that minimizes the criterion

$$J(x_0) = \min\Big(q_\mathrm{f} x^2(t_\mathrm{f}) + \int_0^{t_\mathrm{f}} \big(q_x x^2(t) + q_u u^2(t)\big)\,dt\Big),$$

where q_f, q_x, and q_u are all positive.

[?,2e] statefbk:ric-euler

KJA: Euler equation not mentioned in exercise?

7.34 (Riccati and Euler equations) Consider the Riccati equation

$$-\frac{dS}{dt} = A^T S + SA - SBQ_u^{-1}B^T S + Q_x, \qquad S(t_f) = Q_\mathrm{f},$$

which is quadratic in S. Show that the solution is

$$S(t) = [\Psi_{21}(t) + \Psi_{22}(t)Q_\mathrm{f}][\Psi_{11}(t) + \Psi_{12}(t)Q_\mathrm{f}]^{-1},$$

where the matrix Ψ satisfies the (linear) differential equation

$$\frac{d\Psi}{dt} = \frac{d}{dt}\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix} = \begin{pmatrix}A & -BQ_u^{-1}B^T\\ -Q_x & -A^T\end{pmatrix}\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix},$$

with final conditions

$$\Psi(t_f) = \begin{pmatrix}\Psi_{11}(t_f) & \Psi_{12}(t_f)\\ \Psi_{21}(t_f) & \Psi_{22}(t_f)\end{pmatrix} = \begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}.$$

Comment [RMM, 11 Aug 2019]: Add an exercise on reachability Grammian?

Chapter 8 – Output Feedback

8.1 (Observability) Consider the system given by [B,2e] outputfbk:observability

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$. Show that the states can be determined from the input u and the output y and their derivatives if the observability matrix W_o given by equation (8.4) has n independent rows.

8.2 (Coordinate transformations) Consider a system under a coordinate transformation $z = Tx$, where $T \in \mathbb{R}^{n \times n}$ is an invertible matrix. Show that the observability matrix for the transformed system is given by $\widetilde{W}_\mathrm{o} = W_\mathrm{o} T^{-1}$ and hence observability is independent of the choice of coordinates. [B,1ep] outputfbk:coordxform

8.3 Show that the system depicted in Figure 8.2 is not observable. [C,1ep*] outputfbk:unobservable

8.4 (Observable canonical form) Show that if a system is observable, then there exists a change of coordinates $z = Tx$ that puts the transformed system into observable canonical form. [A,1ep] outputfbk:obsform-transform

8.5 (Bicycle dynamics) The linearized model for a bicycle is given in equation (4.5), which has the form [A,1ep] outputfbk:bicycle-observability

$$J\frac{d^2\varphi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mv_0^2 h}{b}\delta,$$

where φ is the tilt of the bicycle and δ is the steering angle. Give conditions under which the system is observable and explain any special situations where it loses observability.

8.6 (Observer design by eigenvalue assignment) Consider the system [A,1ep] outputfbk:robust-statespace

$$\frac{dx}{dt} = Ax = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} a-1 \\ 1 \end{pmatrix} u, \qquad y = Cx = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

Design an observer such that $\det(sI - LC) = s^2 + 2\zeta_\mathrm{o}\omega_\mathrm{o} s + \omega_\mathrm{o}^2$ with values $\omega_\mathrm{o} = 10$ and $\zeta_\mathrm{o} = 0.6$.

8.7 (Vectored thrust aircraft) The lateral dynamics of the vectored thrust aircraft example described in Example 7.9 can be obtained by considering the motion described by the states $z = (x, \theta, \dot{x}, \dot{\theta})$. Construct an estimator for these dynamics by setting the eigenvalues of the observer into a *Butterworth pattern* with $\lambda_\mathrm{bw} = -3.83 \pm 9.24i$, $-9.24 \pm 3.83i$. Using this estimator combined with the state space controller computed in Example 7.9, plot the step response of the closed loop system. [B,1ep] outputfbk:pvtol-ssctrl

8.8 (Observers using differentiation) Consider the linear system (8.2), and assume that the observability matrix W_o is invertible. Show that [A,1ep] outputfbk:diff

$$\hat{x} = W_\mathrm{o}^{-1} \begin{pmatrix} y & \dot{y} & \ddot{y} & \cdots & y^{(n-1)} \end{pmatrix}^T$$

is an observer. Show that it has the advantage of giving the state instantaneously but that it also has some severe practical drawbacks.

Comment [KJA, 15 Jul 2019]: I suggest that we move this earlier; for example after 8.4

[C,1ep] outputfbk:compartment-teorellobs

8.9 (Observer for Teorell's compartment model) Teorell's compartment model, shown in Figure 4.17, has the following state space representation:

$$\frac{dx}{dt} = \begin{pmatrix} -k_1 & 0 & 0 & 0 & 0 \\ k_1 & -k_2-k_4 & 0 & k_3 & 0 \\ 0 & k_4 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_3-k_5 & 0 \\ 0 & 0 & 0 & k_5 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u,$$

where representative parameters are $k_1 = 0.02$, $k_2 = 0.1$, $k_3 = 0.05$, $k_4 = k_5 = 0.005$. The concentration of a drug that is active in compartment 5 is measured in the bloodstream (compartment 2). Determine the compartments that are observable from measurement of concentration in the bloodstream and design an estimator for these concentrations base on eigenvalue assignment. Choose the closed loop eigenvalues -0.03, -0.05, and -0.1. Simulate the system when the input is a pulse injection.

[C,1ep] outputfbk:bicycle-whipple Exercise 7.12

8.10 (Whipple bicycle model) Consider the Whipple bicycle model given by equation (4.8) in Section 4.2. A state feedback for the system was designed in Exercise 7.12. Design an observer and an output feedback for the system.

[B,1ep] outputfbk:kalman-decomp-2x2

8.11 (Kalman decomposition) Consider a linear system characterized by the matrices

$$A = \begin{pmatrix} -2 & 1 & -1 & 2 \\ 1 & -3 & 0 & 2 \\ 1 & 1 & -4 & 2 \\ 0 & 1 & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}, \quad D = 0.$$

Construct a Kalman decomposition for the system. (Hint: Try to diagonalize.)

[C,2e] outputfbk:kalman-firstorder

8.12 (Kalman filtering for a first-order system) Consider the system

$$\frac{dx}{dt} = ax + v, \qquad y = cx + w$$

where all variables are scalar. The signals v and w are uncorrelated white noise disturbances with zero mean values and covariance functions

$$\mathbb{E}(v(s)v^T(t)) = r_v \delta(t-s), \quad \mathbb{E}(w(s)w^T(t)) = r_w \delta(t-s).$$

The initial condition is Gaussian with mean value x_0 and covariance P_0. Determine the Kalman filter for the system and analyze what happens for large t.

[B,2e] outputfbk:vert-align

8.13 (Vertical alignment) In navigation systems it is important to align a system to the vertical. This can be accomplished by measuring the vertical acceleration and controlling the platform so that the measured acceleration is zero. A simplified one-dimensional version of the problem can be modeled by

$$\frac{d\varphi}{dt} = u, \quad u = -ky, \quad y = \varphi + w,$$

where φ is the alignment error, u the control signal, y the measured signal, and w the measurement noise, which is assumed to be white noise with zero mean and covariance function $\mathbb{E}(w(s)w^T(t)) = r_w\delta(t-s)$. The initial misalignment is assumed to be a random variable with zero mean and the covariance P_0. Determine a time-varying gain $k(t)$ such that the error goes to zero as fast as possible. Compare this with a constant gain.

Supplemental Exercises

8.14 (Transformation to observable form) Consider the system [C,2e] observable-form

$$\frac{dx}{dt} = \begin{pmatrix} -4 & 1 \\ -6 & 1 \end{pmatrix} x + \begin{pmatrix} 3 \\ 7 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & -1 \end{pmatrix} x.$$

Transform the system to observable canonical form.

8.15 Consider a control system having state space dynamics [B,1es] outputfbk:secord-observer

$$\frac{dx}{dt} = \begin{bmatrix} -\alpha-\beta & 1 \\ -\alpha\beta & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k \end{bmatrix} u, \qquad y \quad = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(a) Construct an observer for the system and find expressions for the observer gain $L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^T$ such that the observer has natural frequency ω_0 and damping ratio ζ.

(b) Suppose that we choose a different output

$$\tilde{y} = \begin{bmatrix} 1 & \gamma \end{bmatrix} x.$$

Are there any values of γ for which the system is *not* observable? If so, provide an example of an initial condition and output where it is not possible to uniquely determine the state of the system by observing its inputs and outputs.

8.16 (Balance system) Consider the linearized model of a pendulum on a cart given in Example 3.10. Is the system is observable from the cart position? What happens if the ratio m/M goes to zero? Discuss qualitatively the effect of friction on the cart. [A,1es] outputfbk:balance-observable

*Instructor note:*Note: Need to be more concrete about what exactly the answer should contain.

8.17 (Uniqueness of observers) Show that the design of an observer by eigenvalue assignment is unique for single-output systems. Construct examples that show that the problem is not necessarily unique for systems with many outputs. [D,1ep] outputfbk:observer-assign

8.18 (Observer design for motor drive) Consider the normalized model of the motor drive in Exercise 3.7 where the open loop system has the eigenvalues $0, 0, -0.05 \pm i$. A state feedback that gave a closed loop system with eigenvalues in -2, -1, and $-1 \pm i$ was designed in Exercise 7.11. Design an observer for the system that has eigenvalues -4, -2, and $-2 \pm 2i$. Combine the observer with the state feedback from Exercise 7.11 to obtain an output feedback and simulate the complete system. [D,1ep] outputfbk:dcmotor-observer

Exercise 3.7
Exercise 7.11
Exercise 7.11

[D,1ep] outputfbk:dcmotor-feedforward Exercise 3.7

8.19 (Feedforward design for motor drive) Consider the normalized model of the motor drive in Exercise 3.7. Design the dynamics of the block labeled “trajectory generation” in Figure 8.11 so that the dynamics relating the output η to the reference signal r has the dynamics

$$\frac{d^3 y_\mathrm{m}}{dt^3} + a_{m1}\frac{d^2 y_\mathrm{m}}{dt^2} + a_{m2}\frac{dy_\mathrm{m}}{dt} + a_{m3}y_\mathrm{m} = a_{m3}r, \tag{S8.1}$$

with parameters $a_{m1} = 2.5\omega_\mathrm{m}$, $a_{m2} = 2.5\omega_\mathrm{m}^2$, and $a_{m3} = \omega_\mathrm{m}^3$. Discuss how the largest value of the feedforward signal for a unit step in the reference signal depends on ω_m.

[D,1ep] outputfbk:kalman-randomwalk

8.20 (Discrete-time random walk) Suppose that we wish to estimate the position of a particle that is undergoing a random walk in one dimension (i.e., along a line). We model the position of the particle as

$$x[k+1] = x[k] + u[k],$$

where x is the position of the particle and u is a white noise processes with $E\{u[i]\} = 0$ and $E\{u[i]\,u[j]\} = R_u\delta(i-j)$. We assume that we can measure x subject to additive, zero-mean, Gaussian white noise with covariance 1.

(a) Compute the expected value and covariance of the particle as a function of k.

(b) Construct a Kalman filter to estimate the position of the particle given the noisy measurements of its position. Compute the steady-state expected value and covariance of the error of your estimate.

(c) Suppose that $E\{u[0]\} = \mu \neq 0$ but is otherwise unchanged. How would your answers to parts (a) and (b) change?

[D,1es] outputfbk:balance-statectrl

8.21 (Balance system with biased measurement) A normalized model of a pendulum on a cart is described by the equations

$$\ddot{x} = u, \qquad \ddot{\theta} = \theta + u,$$

where we have assumed that the cart is very heavy (see Example 3.10). Assume that the cart position q and the pendulum angle θ are measured. It is often difficult to exactly calibrate an angle sensor and thus there may be a constant error θ_0 in the measurement. This can be modeled by introducing a new state θ_0 whose dynamics are given by $\dot{\theta}_0 = 0$. Show that the augmented system is observable from $y_1 = q$ and $y_2 = \theta + \theta_0$ and use this to design a controller that compensates for the bias.

[D,1es] outputfbk:observer-duality

8.22 (Duality) Show that the following MATLAB function computes the gain L of an observer for the system $\dot{x} = Ax$, $y = Cx$ that gives an observer whose eigenvalues are the elements of the vector p.

```
function L=observer(A,C,p)
L=place(A',C',p);L=L';
```

Test the program on some examples where you have calculated the result by hand.

8.23 (Balance system) Design an observer for the pendulum on a cart. Combine the observer with the state feedback developed in Example 7.7 to obtain an output feedback. Simulate the system and investigate the effect of a bias error in the angle sensor.

[B,1es] outputfbk:cartpend-observer

*Instructor note:*Note: Need to be more concrete about what exactly the answer should contain.

8.24 (Trajectory generation) Consider the trajectory generation problem for the system

[B,1es] outputfbk:trajgen-scalar

$$\frac{dx}{dt} = -ax^3 + bu,$$

where $x \in \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants.

(a) Show that the system is differentially flat with appropriate choice of output(s) and compute the state and input as a function of the flat output(s).

(b) Using the polynomial basis $\{t^k,\ k = 1, \dots, M\}$ with an appropriate choice of M, solve for the (non-optimal) trajectory between $x(t_0)$ and $x(t_\mathrm{f})$. Your answer should specify the explicit input $u_\mathrm{d}(t)$ and state $x_\mathrm{d}(t)$ in terms of t_0, t_f, $x(t_0)$, $x(t_\mathrm{f})$, and t.

8.25 (Selection of eigenvalues) Pick up the program for simulating Figure 8.4 from the `wiki`. Read the program and make sure that you understand it. Explore the behavior of the estimates for different choices of eigenvalues.

[C,1es] **RMM**: Move to wiki outputfbk:compartment-poleplace

8.26 (Kalman filter for scalar ODE) Consider a scalar control system

[C,?] outputfbk:kalman-scalar

$$\frac{dx}{dt} = \lambda x + u + \sigma_v v, \qquad y = x + \sigma_w w,$$

where v and w are zero-mean, Gaussian white noise processes with covariance 1 and $\sigma_v, \sigma_w > 0$. Assume that the initial value of x is modeled as a Gaussian with mean x_0 and variance $\sigma_{x_0}^2$.

(a) Write down the Kalman filter for the optimal estimate of the state x and compute the steady-state value(s) of the mean and covariance of the estimation error.

(b) Assume that we initialize our filter such that the initial covariance starts near a steady-state value p^*. Given conditions on λ such that error covariance is locally stable about this solution.

8.27 (Double integrator) Consider the normalized double integrator described by

[C,2e] outputfbk:dint-kf

RMM: New exercise

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx$$

Determine an observer and find the observer gain that gives dynamics characterized by the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

[C,2e]
outputfbk:invpend-kf
RMM: New exercise

8.28 (Inverted pendulum) Consider the normalized inverted pendulum described by

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}u = Ax + Bu, \qquad y = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = Cx$$

Determine an observer and find the observer gain that gives dynamics characterized by the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

[C,2e]
outputfbk:invpend-kf2
RMM: New exercise

8.29 (Inverted pendulum with rate sensor) Consider the normalized inverted pendulum, where the angular velocity is the measured output. The system is described by

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}u = Ax + Bu, \qquad y = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = Cx$$

Determine an observer and find the observer gain that gives dynamics characterized by the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

[C,2e]
outputfbk:steering-lanechange

8.30 Consider the vehicle steering problem described in Example 8.8. Construct a trajectory that executes a lange change and sets of the initial and final steering angle to zero.

[C,2e]
outputfbk:lqg-firstorder

8.31 (LQG control for a first-order system) Consider the system

$$\frac{dx}{dt} = ax + bu + v, \qquad y = cx + w$$

where all variables are scalar and w are uncorrelated white noise disturbances with zero mean values and covariance functions

$$\mathbb{E}(v(s)v^T(t)) = r_v\delta(t-s), \quad \mathbb{E}(w(s)w^T(t)) = r_w\delta(t-s).$$

The initial condition is Gaussian with mean value x_0 and covariance P_0. Determine a controller that minimizes the cost function

$$J = \min\Big(q_0 x^2(t_\mathrm{f}) + \int_0^{t_\mathrm{f}}\big(q_x x^2(t) + q_u u^2(t)\big)dt\Big),$$

where q_0, q_x, and q_u are all positive. Explore the different contributions to the minimal loss and Investigate what happens when t_f goes to infinity.

RMM Additional exercises:

- Copy pred-prey
- AFM design
- Observability Grammian? Balanced realization/Hankel singular values?

Chapter 9 – Transfer Functions

9.1 Consider the system [A,1ep*] xferfcns:modalinput

$$\frac{dx}{dt} = ax + u.$$

Compute the exponential response of the system and use this to derive the transfer function from u to x. Show that when $s = a$, a pole of the transfer function, the response to the exponential input $u(t) = e^{st}$ is $x(t) = e^{at}x(0) + te^{at}$.

9.2 Let $G(s)$ be the transfer function for a linear system. Show that if we apply an input $u(t) = A\sin(\omega t)$, then the steady-state output is given by $y(t) = |G(i\omega)|A\sin(\omega t + \arg G(i\omega))$. (Hint: start by showing that the real part of a complex number is a linear operation and then use this fact.) [B,1ep] xferfcns:sininput

9.3 (Inverted pendulum) A model for an inverted pendulum was introduced in Example 3.3. Neglecting damping and linearizing the pendulum around the upright position gives a linear system characterized by the matrices [B,1ep*] xferfcns:invpend-xferfcn

$$A = \begin{pmatrix} 0 & 1 \\ mgl/J_\mathrm{t} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1/J_\mathrm{t} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0.$$

Determine the transfer function of the system.

9.4 (Operational amplifier) Consider the operational amplifier described in Åström and Murray, Section 4.3 and analyzed in Example 9.2. An analog implementation of a PI controller can be constructed using an op amp by replacing the resistor R_2 with a resistor and capacitor in series, as shown in Figure 4.10. The resulting transfer function of the circuit is given by [B,1ep] xferfcns:opamp-pictrl

$$H(s) = -\left(R_2 + \frac{1}{Cs}\right) \cdot \left(\frac{kCs}{\big((k+1)R_1C + R_2C\big)s + 1}\right),$$

where k is the gain of the op amp, R_1 and R_2 are the resistances in the compensation network and C is the capacitance.

(a) Sketch the Bode plot for the system under the assumption that $k \gg R_2 > R_1$. You should label the key features in your plot, including the gain and phase at low frequency, the slopes of the gain curve, the frequencies at which the gain changes slope, etc.

(b) Suppose now that we include some dynamics in the amplifier, as outlined in Example 9.2. This would involve replacing the gain k with the transfer function

$$G(s) = \frac{ak}{s+a}.$$

Compute the resulting transfer function for the system (i.e., replace k with $G(s)$) and find the poles and zeros assuming the following parameter values

$$\frac{R_2}{R_1} = 100, \qquad k = 10^6, \qquad R_2C = 1, \qquad a = 100.$$

(c) Sketch the Bode plot for the transfer function in part (b) using straight line approximations and compare this to the exact plot of the transfer function (using MATLAB). Make sure to label the important features in your plot.

Note: it is not important that you understand the details of the circuit dynamics to complete this problem; you can simply work with the transfer functions that are given.

[B,2e]
xferfcns:ode-with-delay

9.5 (Delay differential equation) Consider a system described by

$$\frac{dx}{dt} = -x(t) + u(t-\tau)$$

Derive the transfer function for the system.

[A,1ep*]
xferfcns:congctrl-xferfcns

9.6 (Congestion control) Consider the congestion control model described in Section 4.4. Let w represent the individual window size for a set of N identical sources, q represent the end-to-end probability of a dropped packet, b represent the number of packets in the router's buffer, and p represent the probability that a packet is dropped by the router. We write $\bar{w} = Nw$ to represent the total number of packets being received from all N sources. Show that the linearized model can be described by the transfer functions

$$G_{b\bar{w}}(s) = \frac{e^{-\tau^{\mathrm{f}} s}}{\tau_{\mathrm{e}}^{\mathrm{p}} s + e^{-\tau^{\mathrm{f}} s}}, \qquad G_{\bar{w}q}(s) = \frac{N}{q_{\mathrm{e}}(\tau_{\mathrm{e}}^{\mathrm{p}} s + q_{\mathrm{e}} w_{\mathrm{e}})}, k$$
$$G_{qp}(s) = e^{-\tau^{\mathrm{b}} s}, \qquad G_{pb}(s) = \rho e^{-\tau_{\mathrm{e}}^{\mathrm{p}} s},$$

where $(w_{\mathrm{e}}, b_{\mathrm{e}})$ is the equilibrium point for the system, $\tau_{\mathrm{e}}^{\mathrm{p}}$ is the router processing time, and τ^{f} and τ^{b} are the forward and backward propagation times.

*Instructor note:*This exercise is fairly intricate and getting exactly the given expressions requires following the assumptions made by Low *et al.* [7]. It might be worth assigning [7] as reading for this problem. The results of this problem are used in Exercise 10.7.

[B,1ep]
xferfcns:ssxferfcn

9.7 (Transfer function for state space system) Consider the linear state space system

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

(a) Show that the transfer function is

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

where the coefficients for the numerator polynomial are linear combinations of the *Markov parameters* CA^iB, $i = 0, \dots, n-1$:

$$b_1 = CB, \quad b_2 = CAB + a_1 CB, \quad \dots, \quad b_n = CA^{n-1}B + a_1 CA^{n-2}B + \cdots + a_{n-1} CB$$

and $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic polynomial for A.

(b) Compute the transfer function for a linear system in reachable canonical form and show that it matches the transfer function given above.

9.8 Consider linear time-invariant systems with the control matrices [B,2e*] xferfcns:ss2trf

$$
\begin{aligned}
&(a) \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad D = 0,\\
&(b) \quad A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0,\\
&(c) \quad A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 \end{pmatrix}, \quad D = 0.
\end{aligned}
$$

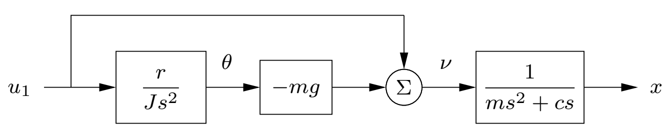
Show that all systems have the transfer function $G(s) = \dfrac{s+3}{(s+1)(s+2)}$.

9.9 (Kalman decomposition) Show that the transfer function of a system depends only on the dynamics in the reachable and observable subspace of the Kalman decomposition. (Hint: Consider the representation given by equation (8.20).) [B,1ep] xferfcns:kalmandecomp

9.10 Using block diagram algebra, show that the transfer functions from v to y and w to y in Figure 9.6 are given by [B,1ep*] xferfcns:ctrlxferfcns

$$
G_{yv} = \frac{P}{1+PC} \qquad G_{yw} = \frac{1}{1+PC}.
$$

9.11 (Vectored thrust aircraft) Consider the lateral dynamics of a vectored thrust aircraft as described in Example 3.12. Show that the dynamics can be described using the following block diagram: [C,1ep*] xferfcns:pvtol-lateraltf

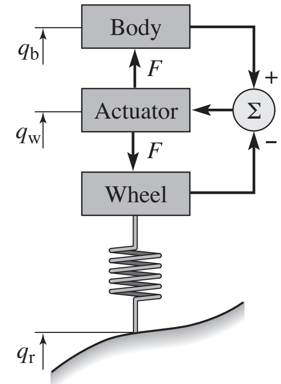


Use this block diagram to compute the transfer functions from u_1 to θ and x and show that they satisfy

$$
H_{\theta u_1} = \frac{r}{Js^2}, \qquad H_{xu_1} = \frac{Js^2 - mgr}{Js^2(ms^2+cs)}.
$$

9.12 (Vehicle suspension [5]) † Active and passive damping are used in cars to give a smooth ride on a bumpy road. A schematic diagram of a car with a damping system in shown in the figure below. [C,1ep] xferfcns:vehicle-suspension Comment [RMM, 12 Aug 2019]: Changed x to q throughout

(Porter Class I race car driven by Todd Cuffaro)



This model is called a *quarter car model*, and the car is approximated with two masses, one representing one fourth of the car body and the other a wheel. The actuator exerts a force F between the wheel and the body based on feedback from the distance between the body and the center of the wheel (the *rattle space*).

Let q_b, q_w and q_r represent the heights of body, wheel, and road measured from their equilibrium points. A simple model of the system is given by Newton's equations for the body and the wheel,

$$m_\mathrm{b}\ddot{q}_\mathrm{b} = F, \qquad m_\mathrm{w}\ddot{q}_\mathrm{w} = -F + k_t(q_\mathrm{r} - q_\mathrm{w}),$$

where m_b is a quarter of the body mass, m_w is the effective mass of the wheel including brakes and part of the suspension system (the *unsprung mass*) and k_t is the tire stiffness. For a conventional damper consisting of a spring and a damper, we have $F = k(q_\mathrm{w} - q_\mathrm{b}) + c(\dot{q}_\mathrm{w} - \dot{q}_\mathrm{b})$. For an active damper the force F can be more general and can also depend on riding conditions. Rider comfort can be characterized by the transfer function G_{aq_r} from road height q_r to body acceleration $a = \ddot{q}_\mathrm{b}$. Show that this transfer function has the property $G_{aq_\mathrm{r}}(i\omega_t) = k_t/m_\mathrm{b}$, where $\omega_t = \sqrt{k_t/m_\mathrm{w}}$ (the *tire hop frequency*). The equation implies that there are fundamental limits to the comfort that can be achieved with any damper.

[B,1ep] xferfcns:pzsolns

9.13 (Solutions corresponding to poles and zeros) Consider the differential equation

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_1\frac{d^{n-1}u}{dt^{n-1}} + b_2\frac{d^{n-2}u}{dt^{n-2}} + \cdots + b_n u.$$

(a) Let λ be a root of the characteristic equation

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0.$$

Show that if $u(t) = 0$, the differential equation has the solution $y(t) = e^{\lambda t}$.

(b) Let κ be a zero of the polynomial

$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n.$$

Show that if the input is $u(t) = e^{\kappa t}$, then there is a solution to the differential equation that is identically zero.

9.14 (Pole/zero cancellation) Consider a closed loop system of the form of Figure 9.6, with $F = 1$ and P and C having a pole/zero cancellation. Show that if each system is written in state space form, the resulting closed loop system is not reachable and not observable. [A,1ep*] xferfcns:pzcancel

9.15 (Inverted pendulum with PD control) Consider the normalized inverted pendulum system, whose transfer function is given by $P(s) = 1/(s^2 - 1)$ (Exercise 9.3). A proportional-derivative control law for this system has transfer function $C(s) = k_\mathrm{p} + k_\mathrm{d} s$ (see Table 9.1). Suppose that we choose $C(s) = \alpha(s-1)$. Compute the closed loop dynamics and show that the system has good tracking of reference signals but does not have good disturbance rejection properties. [B,1ep] xferfcns:invpend-pzcancel Exercise 9.3

Supplemental Exercises

9.16 (Water heater) Consider the water heater in Example 3.13, which is modeled by [C,2e*] xferfcns:waterheater

$$\frac{dm}{dt} = q_\mathrm{in} - q_\mathrm{out}, \qquad \frac{dT}{dt} = \frac{q_\mathrm{in}}{m}(T_\mathrm{in} - T) + \frac{1}{m\,C}P,$$

(see equation (3.31)). Linearize the equations and derive the transfer functions from the inflow q_in and the heating power P to the level h and the temperature T of the tank.

9.17 (Bicycle dynamics) The linearized model for a bicycle is given in equation (4.5), which has the form [C,2e] xferfcns:bicycle-trf

$$J\frac{d^2\varphi}{dt^2} - \frac{Dv_0}{b}\frac{d\delta}{dt} = mgh\varphi + \frac{mv_0^2 h}{b}\delta,$$

where φ is the tilt of the bicycle and δ is the steering angle. Derive the transfer function of the system using the approximation $J = mh^2$ and $D = mah$. Show that the system has poles at $s = \pm\sqrt{g/h}$ and a zero at $s = -v_0/a$.

9.18 (Bode plot for a simple zero) Show that the Bode plot for transfer function $G(s) = (s + a)/a$ can be approximated by [C,1ep] xferfcns:bode-zero

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega < a, \\ \log\omega - \log a & \text{if } \omega > a, \end{cases}$$

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10, \\ 45 + 45(\log\omega - \log a) & \text{if } a/10 < \omega < 10a, \\ 90 & \text{if } \omega > 10a. \end{cases}$$

9.19 [Contributed by D. MacMartin, Nov 2011] For each of the following systems, sketch the frequency response (magnitude and phase) *by hand* (you may use MATLAB first with specific parameter values if you wish). Make sure that the gain at zero frequency is marked, the slope indicated, and key frequencies, but don't worry about the details. All parameters (a, b, ω, ζ) below are positive. [B,2e] xferfcns:freqresp-sketch

(a) $G_1(s) = \dfrac{1}{(s+a)(s+b)}$, with $b = 10a$

(b) $G_2(s) = \dfrac{s+a}{s+b}$, with $b = 2a$

(c) $G_3(s) = \dfrac{s+a}{s^4}$

(d) $G_4(s) = \dfrac{1+s/a}{s+b}$, with $b > 10a$

(f) $G_5(s) = \dfrac{s+a}{s-a}$

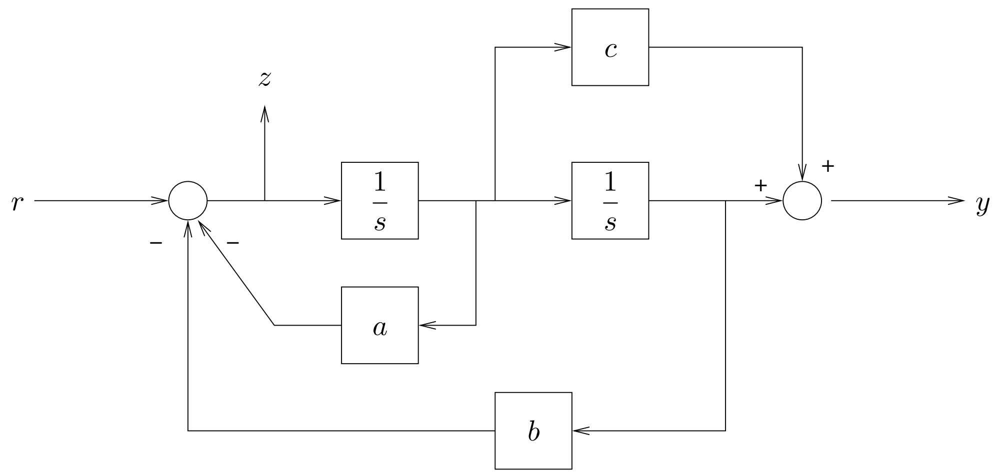
(e) $G_6(s) = \dfrac{s-a}{s+a}$

(g) $G_7(s) = \dfrac{1}{s^2+2\zeta\omega s+\omega^2}$, with $\zeta \ll 1$

(h) $G_8(s) = \dfrac{a}{s^2-2\zeta\omega s+\omega^2}$, with $\zeta \ll 1$

[C,2e]
xferfcns:secord-blk2tf

9.20 Consider the block diagram for the following second-order system



(a) Compute the transfer function H_{yr} between the input r and the output y.

(b) Show that the following state space system has the same transfer function, with the appropriate choice of parameters:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

$$y = \begin{pmatrix} b_2 & b_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + dr$$

Give the values of a_i, b_i, and d that correspond to the transfer function you computed in (a).

(c) Compute the transfer function H_{zr} between the input r and the output z. (Hint: It is *not* $H_{zr} = 1$.)

[B,1es]
xferfcns:freqexamps

9.21 For the control systems below, determine the steady-state error, the maximum frequency for which the closed loop system can track with less than 5% error and the approximate closed loop bandwidth of the system.

(a) Disk drive read head positioning system, using "lead" compensator:

$$P(s) = \frac{1}{s^3+10s^2+3s+10} \qquad C(s) = 1000\frac{s+1}{s+10}$$

(b) Second-order system with PD compensator:

$$P(s) = \frac{100}{(100s+1)(s+1)} \qquad C(s) = s + 10$$

9.22 A nice property of Bode plots is that they can be used to simplify a complicated model in given frequency regimes. Consider for example the model of the AFM whose Bode, plot is shown in Figure 9.17. The transfer function is close to the transfer function $G(s) = k$ for frequencies less than 1 kHz. Demonstrate that it may indeed be possible to use this approximation by designing an integral controller ($C(s) = k_{\mathrm{i}}/s$) for the simplified model that gives a closed loop system with the transfer function $G_{\mathrm{cl}}(s) = a/(s+a)$, with $a = 5000$. Apply the controller to the complex model, simulate the system and explore if the controller works.

[B,1es] xferfcns:model-simplification

9.23 Consider the following simplified equations of motion for a cruise control system (these are a linearization of the equations from Section 3.1 in Åström and Murray):

[B,1es] xferfcns:cruise-pictrl

RMM: Parameters don't match running

$$m\,\frac{dv}{dt} = -cv + b\tau + F_{\mathrm{hill}},$$

where $m = 1000\dagger$ kg is the mass of the vehicle, $c = 50$ N s/m is the viscous damping coefficient, and $b = 25$ is the conversion factor between engine torque and the force applied to the vehicle. We model the engine using a simple first-order equation

RMM: 1600?

$$\frac{d\tau}{dt} = a(-\tau + Tu),$$

where $a = 0.2$ is the lag coefficient and $T = 200$ is the conversion factor between the throttle input and the steady-state torque.

The simplest controller for this system is a proportional control, $u = k_{\mathrm{p}}e$, where $e = (r - v)$ (r is the reference speed).

(a) Draw a block diagram for the system, with the engine dynamics and the vehicle dynamics in separate blocks and represented by transfer functions. Label the reference input to the closed loop system as r, the disturbance due to the hill as d, and the output as y ($= v$).

(b) (MATLAB) Construct the transfer functions H_{er} and H_{yd} for the closed loop system and use MATLAB to generate the step response (`step`) and frequency response (`bode`) for the each. Assume that $k_{\mathrm{p}} = 0.5$. Make sure to use the transfer function computation.

(c) Consider a more sophisticated control law of the form

$$\frac{dx_{\mathrm{c}}}{dt} = r - v, \qquad u \;= k_{\mathrm{p}}e + k_{\mathrm{i}}x_{\mathrm{c}}.$$

This control law contains an "integral" term, which uses the controller state x_{c} to integrate the error. Compute the transfer functions for this control law and redraw your block diagram from part (a) with the default controller replaced by this one.

(d) (MATLAB) Using the gains $k_\mathrm{p} = 0.5$ and $k_\mathrm{i} = 0.1$, use MATLAB to compute the transfer function from r to y and plot the step response and frequency response for the system.

[B,1es]
xferfcns:dcmotor-xferfcn

9.24 Consider the electrical motor in Exercise **??**. Show that the transfer function from voltage to angular velocity is given by Supplement 3.23

$$G_{\omega V} = \frac{k_I}{(Js+c)(Ls+R)+k_E k_I}.$$

In addition, show that if the damping c is negligible and the inductance L is small, the transfer function can be approximated by

$$G_{\omega V} = \frac{1}{k_E}\frac{1}{(1+fsT_\mathrm{m})(1+sT_\mathrm{e})},$$

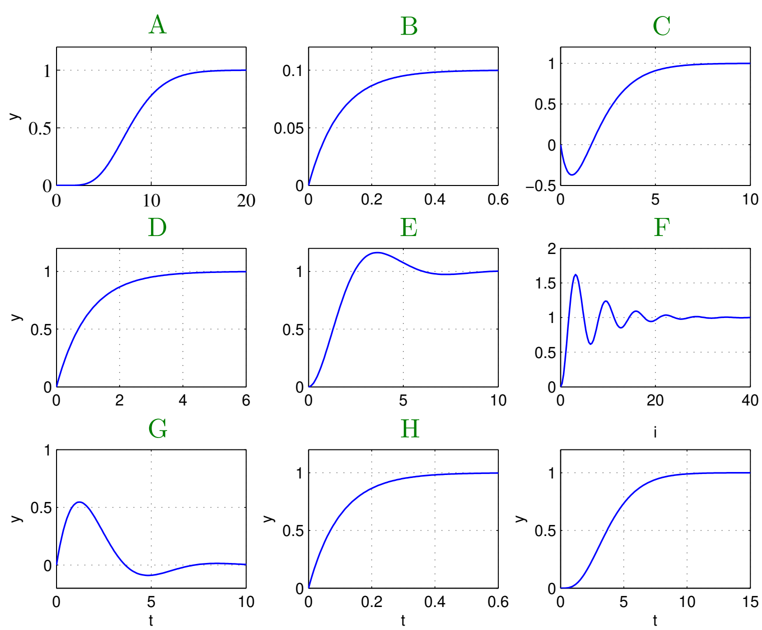
where $T_\mathrm{m} = \dfrac{RJ}{k_E k_I}$ is the mechanical time constant and $T_\mathrm{e} = \dfrac{L}{R}$ is the electrical time constant.

[B,1es]
xferfcns:xferfcn-match

9.25 Match the transfer functions

$$Y_1(s) = \frac{s}{s^2+s+1} \qquad Y_2(s) = \frac{1}{s+1} \qquad Y_3(s) = \frac{-1.5s+1}{s^2+2s+1}$$
$$Y_4(s) = \frac{1}{s+10} \qquad Y_5(s) = \frac{10}{s+10} \qquad Y_6(s) = \frac{1}{(s+1)^4}$$
$$Y_7(s) = \frac{1}{s^2+0.3s+1} \qquad Y_8(s) = \frac{1}{(s+1)^8} \qquad Y_9(s) = \frac{1}{s^2+s+1}$$

with the step responses given below.



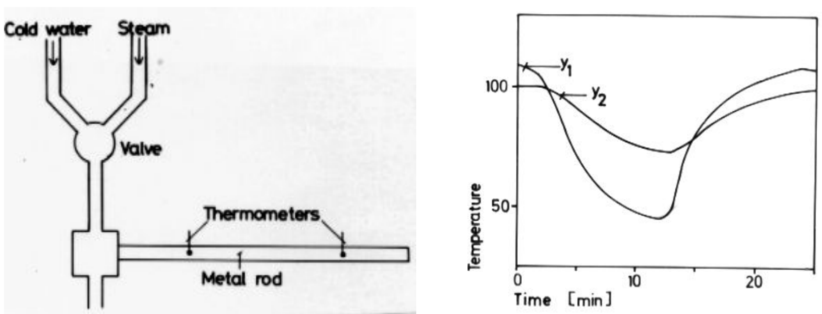
RMM: Nonstandard resizing commands

(Hint: Think first and use computer tools to check your results!)

9.26 [C,1es] xferfcns:heat-propagation

Comment [KJA, 12 Sep 2018]: Edited to make it more of an exercise

The physicist Ångström, who is associated with the length unit Å, used frequency response to determine the thermal diffusivity of metals [1]. A long metal rod with small cross section was used. A heat wave is generated by periodically varying the temperature at one end of the sample. Thermal diffusivity is then determined by analyzing the attenuation and phase shift of the heat wave. A schematic diagram of Ångström's apparatus along with some sample data is shown below:



The copper rod had length 0.57 m, diameter 23.75 mm, and holes were drilled at distances of 0.05 m. The input was generated by switching from steam to cold water periodically. Switching was done manually because of the low frequencies used.

Heat propagation in a metal rod is described by the partial differential equation

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} - \mu T,$$

where $a = \lambda/(\rho C)$ is the thermal diffusivity and the last term represents thermal loss to the environment. †

RMM: Make sure notation is consistent with other thermal diffusivity examples.

Show that the transfer function relating temperatures at points with the distance l is

$$G(s) = e^{-l\sqrt{(s+\mu)/a}},$$

and prove Ångström's formula

$$\log|G(i\omega)| \ \arg G(i\omega) = \frac{l^2\omega}{2a}, \tag{S9.1}$$

which shows that diffusivity can be determined by propagation of temperature in the material. The formula was the key in Ångström's method for determining thermal diffusivity. Notice that the parameter μ representing the thermal losses does not appear in the formula.

9.27 Consider the simple queue model [B,1es] xferfcns:queue-xferfcn

$$\frac{dx}{dt} = \lambda - \mu\frac{x}{x+1}$$

based on continuous approximation, where λ is the arrival rate and μ is the service rate. Linearize the system around the equilibrium point obtained with $\lambda = \lambda_e$ and $\mu = \mu_e$. The queue can be controlled by influencing the admission rate $\lambda = u\lambda_e$ or the service rate $\mu = u\mu_e$. Compute the transfer functions for these two control inputs and give the gains and the time constants of the system. Discuss the particular case when the ratio $r = \lambda_e/\mu_e$ goes to 1.

[B,2e] xferfcns:invpend-ratesens

9.28 (Inverted pendulum with rate sensor) Consider a normalized inverted pendulum with a rate sensor described by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx$$

Design a controller based on state feedback and an observer such that the matrices $A-BK$ and $A-LC$ have the characteristic polynomials $s^2+a_1s+a_2$ and $s^2+b_1s+b_2$ with all coefficients positive. Show that the controller transfer function always has a pole in the right half-plane.

[C,2e] xferfcns:opamp-dynamics

RMM: Drop this exercise? This is included as an example

9.29 The static model of the operational amplifier given by equation (4.11) can be improved by introducing dynamics. A more realistic model is obtained by representing the linear behavior of the amplifier by the transfer function

$$G(s) = \frac{b}{s+a}$$

The parameter b is called the gain-bandwidth-product and is typically in the range 10^7–10^{10} rad/s, but it can also be several order of magnitudes larger. The time constant a is typically around 0.1. Designers of operational amplifiers are making great efforts to obtain this characteristics. There are however typically additional poles at frequencies around b. Use this model to derive the transfer function of the systems in Figures 4.9a and 4.10.

[B,2e] xferfcns:opamp-filter

9.30 It is necessary to filter signals before using them in a feedback loop. Design a circuit with operational amplifiers that gives a second-order filter with damping ratio $\zeta = 0.707$. Show that the filter can deliver filtered versions of the signal *and its derivative.*

[C,1es] xferfcns:lq-scalar

9.31 Consider a process P with dynamics

$$\frac{dx}{dt} = ku + d, \qquad y \quad = x + n.$$

We wish to design a controller $C(s)$ using an observer-based optimal control law.

(a) Design a state feedback controller $u = -Kx$ for the system that minimizes the cost function

$$J = \int_0^\infty y^2 + u^2\,dt$$

(assume that the full state is available for feedback and ignore the disturbances and noise).

(b) Design an observer for this system that places the closed loop pole for the observer at $s = -1$.

(c) Letting $\alpha = K$ represent the state feedback gain in part 0a and $\beta = L$ the observer feedback gain in part 0b, compute the controller transfer function resulting from applying the optimal state feedback gain to the estimated state. Under what conditions is the closed loop system stable?

(d) Design an observer for this system that minimizes the steady state, mean square of the observation error under the assumption that the process disturbance v is Gaussian white noise with covariance 1 and the sensor noise w is Gaussian white noise with covariance 0.1.

(e) Letting $\alpha = K$ represent the state feedback gain in part 0a and $\beta = L$ the observer feedback gain in part 0d, compute the controller transfer function resulting from applying the optimal state feedback gain to the estimated state. Under what conditions is the closed loop system stable?

9.32 [Dullerud and Paganini, 2.19] Consider a transfer function $G(s)$ and let (A, B, C, D) be a realization: $G(s) = C(sI - A)^{-1}B + D$. [C,2e?] xferfcns:minimal-eigenvalues

(a) Prove that if (A, B, C, D) is a minimal realization for $G(s)$ then every eigenvalue of A must be a pole of $G(s)$.

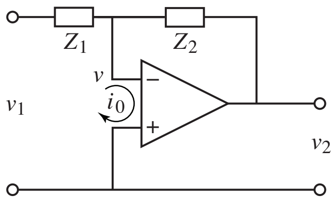
(b) Prove that if (A, B, C, D) is a non-minimal realization, then every eigenvalue of A is either a pole of $G(s)$ or corresponds to an eigenvector that is in the unreachable or unobservable subspace.

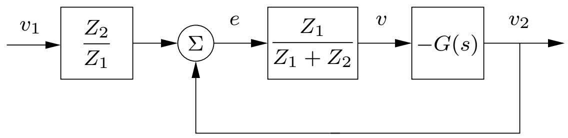
Chapter 10 – Frequency Domain Analysis

10.1 (Operational amplifier loop transfer function) Consider the operational amplifier circuit shown below, where Z_1 and Z_2 are generalized impedances and the open loop amplifier is modeled by the transfer function $G(s)$.

[?,2e*] loopanal:opamp-loop

Comment [RMM, 11 Nov 2018]: This exercise used to be an example. It is referenced in Chapter 13.





Show that the system can be modeled as the block diagram on the right, with loop transfer function $L = Z_1G/(Z_1 + Z_2)$ and feedforward transfer function $F = Z_1/(Z_1 + Z_2)$.

10.2 (Atomic force microscope) The dynamics of the tapping mode of an atomic force microscope are dominated by the damping of the cantilever vibrations and the system that averages the vibrations. Modeling the cantilever as a spring–mass system with low damping, we find that the amplitude of the vibrations decays as $\exp(-\zeta\omega_0 t)$, where ζ is the damping ratio and ω_0 is the undamped natural frequency of the cantilever. The cantilever dynamics can thus be modeled by the transfer function

[B,1ep*] loopanal:afm-tapping

$$G(s) = \frac{a}{s+a},$$

where $a = \zeta\omega_0$. The averaging process can be modeled by the input/output relation

$$y(t) = \frac{1}{\tau}\int_{t-\tau}^{t} u(v)dv,$$

where the averaging time is a multiple n of the period of the oscillation $2\pi/\omega$. The dynamics of the piezo scanner can be neglected in the first approximation because they are typically much faster than a. A simple model for the complete system is thus given by the transfer function

$$P(s) = \frac{a(1 - e^{-s\tau})}{s\tau(s+a)}.$$

Plot the Nyquist curve of the system and determine the gain of a proportional controller that brings the system to the boundary of stability.

10.3 (Congestion control in overload conditions) A strongly simplified flow model of a TCP loop under overload conditions is given by the loop transfer function

[B,1ep*] loopanal:congctrl-overload

$$L(s) = \frac{k}{s}e^{-s\tau},$$

where the queuing dynamics are modeled by an integrator, the TCP window control is a time delay τ, and the controller is simply a proportional controller. A major difficulty is that the time delay may change significantly during the operation of the system. Show that if we can measure the time delay, it is possible to choose a gain that gives a stability margin of $s_{\mathrm{m}} \geq 0.6$ for all time delays τ.

[B,1ep] loopanal:heatcond-nyquist

10.4 (Heat conduction) A simple model for heat conduction in a solid is given by the transfer function

$$P(s) = ke^{-\sqrt{s}}.$$

Sketch the Nyquist plot of the system. Determine the frequency where the phase of the process is $-180°$ and the gain at that frequency. Show that the gain required to bring the system to the stability boundary is $k = e^{\pi}$.

[B,1ep] loopanal:secord-margins

10.5 (Stability margins for second-order systems) A process whose dynamics is described by a double integrator is controlled by an ideal PD controller with the transfer function $C(s) = k_d s + k_p$, where the gains are $k_d = 2\zeta\omega_0$ and $k_p = \omega_0^2$. Calculate and plot the gain, phase, and stability margins as a function ζ.

[A,1ep] loopanal:opamp-unitygain

10.6 (Unity gain operational amplifier) Consider an op amp circuit with $Z_1 = Z_2$ that gives a closed loop system with nominally unit gain. Let the transfer function of the operational amplifier be

$$G(s) = \frac{ka_1a_2}{(s+a)(s+a_1)(s+a_2)},$$

where $a_1, a_2 \gg a$. Show that the condition for oscillation is $k < a_1 + a_2$ and compute the gain margin of the system. Hint: Assume $a = 0$.

[B,1ep] loopanal:steering-margins

10.7 (Vehicle steering) Consider the linearized model for vehicle steering with a controller based on state feedback discussed in Example 8.4. The transfer functions for the process and controller are given by

$$P(s) = \frac{\gamma s + 1}{s^2}, \quad C(s) = \frac{s(k_1l_1 + k_2l_2) + k_1l_2}{s^2 + s(\gamma k_1 + k_2 + l_1) + k_1 + l_2 + k_2l_1 - \gamma k_2 l_2},$$

as computed in Example 9.10. Let the process parameter be $\gamma = 0.5$ and assume that the state feedback gains are $k_1 = 0.5$ and $k_2 = 0.75$ and that the observer gains are $l_1 = 1.4$ and $l_2 = 1$. Compute the stability margins numerically.

[C,1ep] loopanal:pvtol-looptf

10.8 (Vectored thrust aircraft) Consider the state space controller designed for the vectored thrust aircraft in Examples 7.9 and 8.7. The controller consists of two components: an optimal estimator to compute the state of the system from the output and a state feedback compensator that computes the input given the (estimated) state. Compute the loop transfer function for the system and determine the gain, phase, and stability margins for the closed loop dynamics.

[B,2e] loopanal:lqr-kalman-ineq

10.9 (Kalman's inequality) Consider the linear system (7.20). Let $u = -Kx$ be a state feedback control law obtained by solving the linear quadratic regulator problem. Prove the inequality

$$\big(I + L(-i\omega)\big)^T Q_u \big(I + L(i\omega)\big) \geq Q_u,$$

where

$$K = Q_u^{-1}B^TS, \qquad L(s) = K(sI - A)^{-1}B.$$

(Hint: Use the Riccati equation (7.33), add and subtract the terms sS, multiply with $B^T(sI + A)^{-T}$ from the left and $(sI - A)^{-1}B$ from the right.)

For single-input single-output systems this result implies that the Nyquist plot of the loop transfer function has the property $|1 + L(i\omega)| \geq 1$, from which it follows that the phase margin for a linear quadratic regulator is always greater than 60°.

[C,1ep] loopanal:bode-weight

10.10 (Bode's formula) Consider Bode's formula (10.9) for the relation between gain and phase for a transfer function that has all its singularities in the left half-plane. Plot the weighting function and make an assessment of the frequencies where the approximation $\arg G \approx (\pi/2) d \log |G| / d \log \omega$ is valid.

[B,1ep*] loopanal:pade-approx

10.11 (Padé approximation to a time delay) Consider the transfer functions

$$G(s) = e^{-s\tau}, \qquad G_1(s) = \frac{1 - s\tau/2}{1 + s\tau/2}. \tag{S10.1}$$

Show that the minimum phase properties of the transfer functions are similar for frequencies $\omega < 1/\tau$. A long time delay τ is thus equivalent to a small right half-plane zero. The approximation $G_1(s)$ in equation (S10.1) is called a first-order *Padé approximation*.

[B,1ep] loopanal:inverse-response

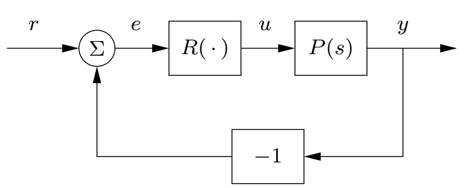
10.12 (Inverse response) Consider a system whose input/output response is modeled by $G(s) = 6(-s+1)/(s^2 + 5s + 6)$, which has a zero in the right half-plane. Compute the step response for the system, and show that the output goes in the wrong direction initially, which is also referred to as an *inverse response*. Compare the response to a minimum phase system by replacing the zero at $s = 1$ with a zero at $s = -1$.

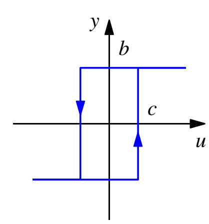
[B,2e] loopanal:circlecrit-ex

10.13 (Circle criterion) Consider the system in Figure 10.17, where H_1 is a linear system with the transfer function $H(s)$ and H_2 is a static nonlinearity $F(x)$ with the property $xF(x) \geq 0$. Use the circle criterion to prove that the closed loop system is stable if $H(s)$ is strictly passive.

[B,1ep] loopanal:desfcn-hystrelay

10.14 (Describing function analysis) Consider the system with the block diagram shown on the left below.





The block R is a relay with hysteresis whose input/output response is shown on the right and the process transfer function is $P(s) = e^{-s\tau}/s$. Use describing function analysis to determine frequency and amplitude of possible limit cycles. Simulate the system and compare with the results of the describing function analysis.

[B,1ep] loopanal:desfcn-saturation

10.15 (Describing functions) Consider the saturation function

$$y = \operatorname{sat}(x) = \begin{cases} -1 & \text{if } x \leq 1, \\ x & \text{if } -1 < x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Show that the describing function is

$$N(a) = \begin{cases} x & \text{if } |x| \leq 1, \\ \dfrac{2}{\pi}\left(\arcsin\dfrac{1}{x} + \dfrac{1}{x}\sqrt{1 - \dfrac{1}{x^2}}\right) & \text{if } |x| > 1. \end{cases}$$

RMM **Supplemental Exercises** Redefine `exerlabel` to flag any supplementary exercises that are cited in the main text.

[B,1es] loopanal:nyquist-rhp

10.16 (Right half-plane pole) Consider a system with the loop transfer function

$$L(s) = \frac{k}{s(s-1)(s+5)}.$$

This transfer function has a pole at $s = 1$, which is inside the Nyquist contour. Draw the Nyquist plot for this system and determine if the closed loop system is stable.

[D,1es] loopanal:nyquist-thirdorder

10.17 (Third-order system) Consider a closed loop system with the loop transfer function

$$L(s) = \frac{k}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}.$$

(a) Assuming that $a \ll \omega_0$ and $\zeta = 1$, sketch the Bode and Nyquist plots for the system, labeling the key features (in terms of k, a and ω_0).

(b) For each of the following parameter sets, use the Nyquist criterion to determine if the closed loop system is stable and, if so, what the gain, phase, and stability margins are:

i. $k = 200$, $a = 1$, $\zeta = 1$, $\omega_0 = 10$

ii. $k = 100$, $a = 1$, $\zeta = 0.1$, $\omega_0 = 10$

iii. $k = 100$, $a = 0$, $\zeta = 1$, $\omega_0 = 10$

iv. $k = 80$, $a = -1$, $\zeta = 1$, $\omega_0 = 10$

Be sure to show the Nyquist plot for each case and show the gain and phase margins on the Nyquist plots.

[?,?] loopanal:bodenyq-examples

10.18 Plot the (open loop) Nyquist and Bode plots for the following systems and compute the gain and phase margin of each. You should annotate your plots to show the gain and phase margin computations. For the Nyquist plot, mark the branches corresponding to the following sections of the Nyquist "D" contour: negative imaginary axis, positive imaginary axis, semi-circle at infinity (the curved part of the "D").

(a) Disk drive read head positioning system, using a lead compensator (described in Chapter 12):

$$P(s) = \frac{1}{s^3 + 10s^2 + 3s + 10}, \qquad C(s) = 1000\frac{s+1}{s+10}.$$

(b) Second-order system with PD compensator:

$$P(s) = \frac{100}{(100s+1)(s+1)}, \qquad C(s) = s + 10.$$

Note: you may find it easier to sketch the Nyquist plot from the Bode plot (taking some liberties with the scale) rather than relying on MATLAB. Figures S10.1 and S10.2 are the required plots. The Bode plots were generated using MATLAB's `margin` command, which calculates the gain and phase margin and displays them on the Bode plot.

(a) The gain margin of this system is 1.6 (4.11 dB) and the phase margin is 13 degrees. The upper branch of the Nyquist plot corresponds to the negative imaginary axis, the lower branch to the positive imaginary axis, and the origin to the semi-circle at infinity. There are no encirclements of the -1 point, indicating that the closed loop system is stable (because the open loop system has no RHP poles).

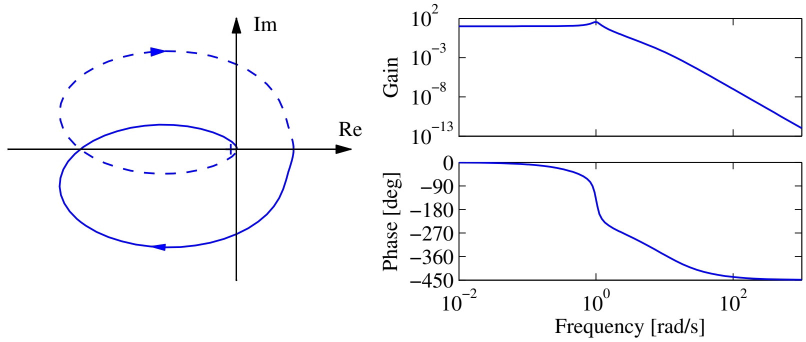


Figure S10.1: Disk drive read head positioning system. (a) Bode plot for $P(s)C(s)$ with margins shown. (b) Nyquist plot.

(b) The gain margin of this system is infinite and the phase margin is 35.3 degrees. The upper branch of the Nyquist plot corresponds to the negative imaginary axis, the lower branch to the positive imaginary axis, and the origin to the semi-circle at infinity. There are no encirclements of the -1 point, indicating that the closed loop system is stable (because the open loop system has no RHP poles).

*Instructor note:*10pts max, 2 for each Bode plot, 3 for each Nyquist plot. Bode plot: Features looking for: slope, limiting value for very large and very small frequency. Each wrong feature incurs 0.5 pt off. Nyquist plot: The plot itself worth 1 pt. Features looking for: Value for zero and infinite frequency, intermediate "shape" The gain and phase margin worth 1 pt, 0.5 for each. The annotation worth 1 pt. Each wrong annotation incurs 0.5 pt off.

Comment [RMM, 2017?]: I commented out problem about pendulum Lyapunov function being an ellipse

Comment [RMM, 20 May 2017]: Uncommented this exercise so that we can see it, but not sure it is correct.

Comment [RMM, 14 Sep 2018]: Sort out whether the exercise is correct and decide whether to keep. Need to look at solution.

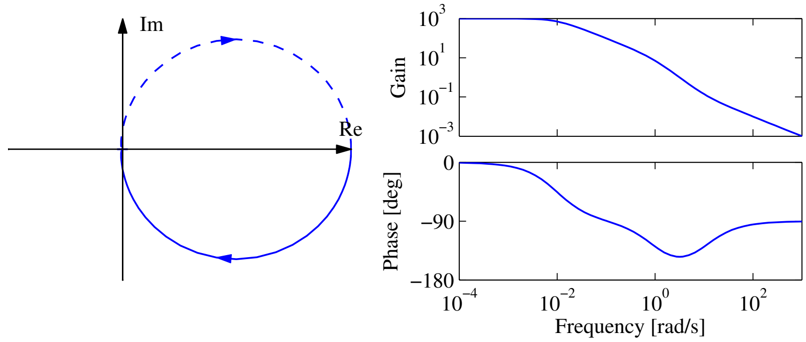


Figure S10.2: Second-order system with PD compensator. (a) Bode plot for $P(s)C(s)$ with margins shown. (b) Nyquist plot.

[D,1es] loopanal:invpend-nyquist

10.19 (Inverted pendulum) Consider the inverted pendulum in Example 10.5. Show that the Nyquist curve is the ellipse

$$\Big(\mathrm{Re}\big(L(i\omega)+k\big)\Big)^2 + 4\Big(\mathrm{Im}\big(L(i\omega)+k\big)\Big)^2 = k^2.$$

[B,2e] loopanal:pupil-stability

10.20 (Pupillary light reflex dynamics) Consider the pupillary light reflex dynamics discussed in Example 9.18. The system has a feedback loop which maintains a constant light intensity at the retina by changing the pupil area using feedback. To study the system dynamics, Stark focused a light narrow beam in the middle of the pupil. If the beam is sufficiently narrow the pupil area does not change, the feedback loop is broken and the loop transfer function $L(s)$ can be measured. In one experiment Stark found

$$L(s) = \frac{0.17}{1+0.08s}e^{-0.2s}.$$

Plot the Bode and Nyquist plots for this loop transfer function and determine whether the closed loop system is stable. What are the gain, phase, and stability margins for the system?

[B,1es] loopanal:congctrl-margins

10.21 (Congestion control using TCP/Reno) A linearized model of the dynamics for a congestion control mechanism on the Internet is given in Example 10.4, following [7] and [6]. A linearized version of the model is represented by the transfer function

$$L(s) = \rho \cdot \frac{N}{\tau_\mathrm{e} s + e^{-\tau^f s}} \cdot \frac{c^3\tau_\mathrm{e}^3}{2N^3(c\tau_\mathrm{e}^2 s + 2N^2)}e^{-\tau_\mathrm{e} s},$$

where c is the link capacity in packets/ms, N is the load factor (number of TCP sessions), ρ is the drop probability factor and τ is the round-trip time in seconds. Consider the situation with the parameters $N = 80$, $c = 4$, $\rho = 10^{-2}$, and $\tau_\mathrm{e} = 0.25$. Find the stability margin of the system and also determine the stability margin if the time delay becomes $\tau_\mathrm{e} = 0.5$.

10.22 (Stability margins for the linear quadratic regulator) Consider a single-input, single-output (SISO) linear system with constant parameters and a state feedback obtained by solving the algebraic Riccati equation (7.33). Show that the system has the loop transfer function

[B,2e]
loopanal:lqr-robust

$$L(s) = K(sI - A)^{-1}B.$$

Use Kalman's inequality (Exercise 10.9) to show that the system has infinite gain margin and a phase margin of 60°.

10.23 An operational amplifier has a transfer function that for high frequencies can be modeled by

[A,1es]
loopanal:opamp-routh

$$G(s) = \frac{k_v}{s(1 + sT_1)(1 + sT_2)},$$

where k_v is the gain-bandwidth product. A unit gain amplifier is constructed by the standard feedback configuration in Figure 4.9b† with $R_1 = R_2$. Show that the feedback amplifier is stable if $k_v T < 4$ where $T = 2T_1T_2/(T_1 + T_2)$ is the harmonic mean of the time constants T_1 and T_2.

RMM: Is this the best reference. Should probably use Figure 9a with $Z_1 = R_1$, $Z_2 = R_2$

*Instructor note:*This problem is similar to Exercise 10.6.

Action: Uncommented this exercise, but it was not included in the first edition. [RMM, 20 May 2017]

RMM

10.24 Consider the speed control system described in Section 4.1 and analyzed using state space techniques in Example 7.8. Using a modified PI controller

[B,?]
loopanal:cruise-nyquist

RMM: Need to provide the controller parameters

$$C(s) = G_{ue}(s) = k_p + \frac{k_i}{s + \beta} = \frac{k_p s + k_i + k_p\beta}{s + \beta}$$

plot the Nyquist curve for the system and determine the gain, phase, and stability margins.†

RMM: Add engine dynamics to make it interesting?

The next two exercises should be combined into one

RMM

10.25 (Limits for an inverted pendulum on a cart) Consider an inverted pendulum on a cart. It was shown in Example **??** that the pendulum can be stabilized in the upright position. Discuss the difficulties of stabilizing the pendulum at an arbitrary angle using physical arguments. Then show how the difficulties appear by using a simple normalized linear model with the normalized transfer function $G(s) = s^2/(s^2 - 1)$.† The input is the cart position and the output is the pendulum angle.

[C,1es]
loopanal:invpend-limits

RMM: Too vague

10.26 (Limits for unstable system with time delay) Illustrate the difficulties of controlling an unstable system with a time delay in the measurement by considering an inverted pendulum where the angle is measured with a delay. Use a simple linear model for your analysis.

[C,1es]
loopanal:invpend-delay

[B,1es] loopanal:delayde

10.27 (Delay differential equations) Differential equations with delay appear in models of congestion control for communication networks, such as in Example 10.4. A simple example is the system

$$\frac{dx}{dt} = -ax(t-\tau) + bu$$

which has the transfer function $P(s) = b/(s + ae^{-s\tau})$.

(a) Explore how the shape of the Nyquist plot of $P(s)$ depend on the parameter τ for $a = b = 1$.

(b) The poles of the system are the zeros of the function $s + ae^{-s\tau}$, which are the same as the zeros of the function $f(s) = 1 + L(s)$ where $L(s) = (a/s)e^{-s\tau}$. Use the Nyquist plot of $L(s)$ to determine how the zeros of $1 + L(s)$ and the poles of the original system depend on the parameters. Hint: Use τ in the range of 0 to 10.

[C,1es] loopanal:cruise-pictrl

RMM: Parameters don't match running

10.28 (Cruise control design) In this problem we will design a PI controller for a cruise control system, building on the example shown in class. Use the following transfer function to represent the vehicle and engine dynamics: †

$$P(s) = \frac{Tba/m}{(s+a)(s+c/m)}$$

where $b = 25$ is the transmission gain, $T = 200$ is the conversion factor between the throttle input and steady-state torque, $a = 0.2$ is the engine lag coefficient, $m = 1000$ kg is the mass of the car, and $c = 50$ N s/m is the viscous damping coefficient.

(a) Consider a proportional controller for the car, $u = k_\mathrm{p}(r - y)$. Assuming a unity gain feedback controller, this gives

$$C(s) = k_\mathrm{p}.$$

Set $k_\mathrm{p} = 0.1$ and compute the steady-state error, gain and phase margins, rise time, overshoot, and poles/zeros for the system. Remember that the gain and phase margins are computed based on the loop transfer function $L(s) = P(s)C(s)$; the remaining quantities should be computed for the closed loop system.

(b) Consider a proportional + integral controller for the car,

$$C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s}.$$

Fill in the following table (make sure to show your work), where g_m is the gain margin, φ_m the phase margin, SSerr the steady-state error, BW the bandwidth (you can use the bandwidth command in MATLAB, but you need to do so for the closed loop system), T_r the rise time, and M_p the overshoot (see Fig. 5.9 on p. 151 of the text, you do not need to be exact).

k_p	k_i	Stable?	g_m	φ_m	SSerr	BW	T_r	M_p
0.5	0.1							
0.05	1							
0.05	0.001							
0.005	0.001							

For each entry in the table, plot the pole zero diagram (pzmap) for the *closed loop* system and the step response. (Note that the steady-state error is zero in each stable case, due to the integral term in the control law.) (Suggestion: look for relationships between the various quantities you are computing and plotting. This problem should give you some insight into the relationship between some of the quantities.)

10.29 (Cruise control with time delay) Continuing the previous problem, we will now insert a small amount of time delay into the feedback path of the system. A pure time delay of τ seconds satisfies the equation [C,1es] loopanal:cruise-delay

$$y(t) = u(t - \tau)$$

This system is a linear input/output system and it can be shown that its transfer function is

$$G(s) = e^{-s\tau}.$$

Unfortunately, MATLAB is not able to perfectly represent a time delay in this form, and so we have to use a "Padé approximation", which gives a constant gain transfer function with phase that approximates a time delay. Using a 2nd order Padé approximation, we can approximate our time delay as

$$G(s) = \frac{1 - \tau s/2 + (\tau s)^2/12}{1 + \tau s/2 + (\tau s)^2/12}$$

This function can be computed using the `pade` function in MATLAB (although the numerator and denominator are scaled slightly differently).

Assume that there is a time delay of τ seconds, which we will insert between the output of the plant and the controller

(a) For the case $k_\mathrm{p} = 0.05$, $k_\mathrm{i} = 0.001$, insert time delays of $\tau = 0.25$ s and $\tau = 0.75$ s. Using a Padé approximation, compute the resulting gain and phase margin for each case and compute the overshoot and settling time (2%) for the step responses.

(b) *Optional:* Plot the Nyquist plot with the exact time delay and compare with the Pade approximation.

(c) Repeat part (a) using $k_\mathrm{p} = 0.02$, $k_\mathrm{i} = 0.0005$, and time delays of 0.75 s and 1.5 s.

(d) *Optional:* Plot the Nyquist plot for $k_\mathrm{p} = 0.02$, $k_\mathrm{i} = 0.0005$, $\tau = 0.75$ (with the exact time delay, not the Pade approximation).

10.30 For the unity feedback system with $P(s) = k/s$, does there exist a proper controller $C(s)$ such that the feedback system is internally stable for both $k = +1$ and $k = -1$? Explain your answer.

10.31 Consider an operational amplifier whose loop transfer function is [?,2e] loopanal:opamp-parasitic

$$G(s) = \frac{bcd}{(s + a)(s + b)(s + c)}$$

where the poles b and c represent undesirable but unavoidable features. The amplifier is coupled with unit gain feedback. Determine the conditions for stability of the closed loop.

[?,2e] loopanal:opamp-capload

10.32 Amplifier with capacitive load.

[?,2e] loopanal:opamp-inv-bode

RMM: Update grammar and style if kept

10.33 † The op amp way of using Bode plots. In control theory it is standard practice to explore stability using a Bode plot of the loop transfer function. A slightly different procedure is used by practitioners of operational amplifiers. They use Bode plots of the transfer function of the operational amplifier and of the inverse of the feedback loop. This is natural because the transfer function of the amplifier is typically fixed in a design and the only thing that changes is the feedback circuit. Show that a necessary condition for stability is that slope of the feedback circuit is less than 1.

[?,2e] loopanal:dcmotor-pdctrl

10.34

Comment [RMM, 28 Apr 2018]: This might appear as an example in the new version. There is also a variant in Ch 11 (PID)

Consider a simple direct current motor with inertia J and damping c. The transfer function is

$$P(s) = \frac{k_I}{Js^2 + cs},$$

where we take $k_I = 50$, $J = 2$, and $c = 1$. In this problem you will design some simple controllers to achieve a desired level of performance.

(a) Design a proportional control law, $C(s) = k_\text{p}$, that gives stable performance and has a gain crossover of at least 1 rad/s and a phase margin of at least 30 degrees. Plot the step response for the closed loop system using your controller.

(b) Consider a proportional + derivative controller (PD) of the form

$$C(s) = k_\text{p} + k_\text{d}\frac{s}{s + 100c/J}.$$

Note that the derivative term (k_d) is slightly modified so that we get a roll-off in controller response at high frequency. Design a controller (choose k_p and k_d) that gives closed loop bandwidth $\omega = 10$ rad/s and has phase margin of at least 30 degrees. Plot the step response for the closed loop system using your controller.

RMM: Update soluionts

*Instructor note:*The solutions below were generated for the case where $k_I = 1$, but in the problem we define $k_I = 50$.†

[?,2e] loopanal:stability-defns

10.35 Consider a linear input/output system Σ with a minimal realization given by (A, B, C, D) and let the associated transfer function be $H(s) = C(sI - A)^{-1}B + D$. For simplicity, you may also assume that the system is SISO.

(a) Show that if the linear system $\dot{x} = Ax$ is asymptotically stable then the induced input/output norm of the system Σ is bounded.

(b) Show that if a linear input/output system Σ has bounded induced input/output norm, then the linear system $\dot{x} = Ax$ is asymptotically stable.

(c) Show that if a linear system is input/output stable then $\|H\|_\infty$ is bounded.

(d) Show via example that $\|H\|_\infty$ being bounded is not a sufficient condition for stability of the underlying system.

Chapter 11 – PID Control

11.1 (Ideal PID controllers) Consider the systems represented by the block diagrams in Figure 11.1. Assume that the process has the transfer function $P(s) = b/(s+a)$ and show that the transfer functions from r to y are [B,1ep] pid:pid-ideal

(a) $G_{yr}(s) = \dfrac{bk_\mathrm{d}s^2 + bk_\mathrm{p}s + bk_\mathrm{i}}{(1+bk_\mathrm{d})s^2 + (a+bk_\mathrm{p})s + bk_\mathrm{i}}$,

(b) $G_{yr}(s) = \dfrac{bk_\mathrm{i}}{(1+bk_\mathrm{d})s^2 + (a+bk_\mathrm{p})s + bk_\mathrm{i}}$.

Pick some parameters and compare the step responses of the systems.

*Instructor note:*Add parameters to supplement version **KJA**

11.2 Consider a second-order process with the transfer function [B,1ep*] pid:secord-pid

$$P(s) = \frac{b}{s^2 + a_1 s + a_2}.$$

The closed loop system with a PI controller is a third-order system. Show that it is possible to position the closed loop poles as long as the sum of the poles is $-a_1$. Give equations for the parameters that give the closed loop characteristic polynomial

$$(s+\alpha_\mathrm{c})(s^2 + 2\zeta_\mathrm{c}\omega_\mathrm{c}s + \omega_\mathrm{c}^2).$$

11.3 Consider a system with the transfer function $P(s) = (s+1)^{-2}$. Find an integral controller that gives a closed loop pole at $s = -a$ and determine the value of a that maximizes the integral gain. Determine the other poles of the system and judge if the pole can be considered dominant. Compare with the value of the integral gain given by equation (11.6). [B,1ep*] pid:integral-twopole

11.4 (Tuning rules) Apply the Ziegler–Nichols and the modified tuning rules to design PI controllers for systems with the transfer functions [N,1ep] pid:zntuning-examples

$$P_1 = \frac{e^{-s}}{s}, \qquad P_2 = \frac{e^{-s}}{s+1}, \qquad P_3 = e^{-s}.$$

Compute the stability margins and explore any patterns.

Comment [RMM, 27 Dec 2019]: The problem below looks like a duplicate of the one above

11.5 (Ziegler–Nichols tuning) Consider a system with transfer function $P(s) = e^{-s}/s$. Determine the parameters of P, PI, and PID controllers using Ziegler–Nichols step and frequency response methods. Compare the parameter values obtained by the different rules and discuss the results. [C,1ep] pid:zntuning-delint

11.6 (Vehicle steering) Design a proportional-integral controller for the vehicle steering system that gives the closed loop characteristic polynomial [B,1ep] pid:steering-pi

$$s^3 + 2\omega_\mathrm{c}s^2 + 2\omega_\mathrm{c}^2 s + \omega_\mathrm{c}^3.$$

[B,2e] pid:pid-tar

11.7 (Average residence time with PID control) The average residence time is a measure of the response time of the system. For a stable system with impulse response $h(t)$ and transfer function $P(s)$ it can be defined as

$$T_{\mathrm{ar}} = \int_0^\infty t h(t)\, dt = -\frac{P'(0)}{P(0)}.$$

Consider a stable system with $P(0) \neq 0$ and a PID controller having integral gain $k_{\mathrm{i}} = k_{\mathrm{p}}/T_{\mathrm{i}}$. Show that the average residence time of the closed loop system is given by $T_{\mathrm{ar}} = T_{\mathrm{i}}/(P(0)k_{\mathrm{p}})$.

[B,2e] pid:webserver-dvfs

11.8 (Web server control) Web servers can be controlled using a method known as *dynamic voltage frequency scaling* in which the processor speed is regulated by changing its supply voltage. A typical control goal is to maintain a given service rate, which is approximately equal to maintaining a specified queue length. The queue length x can be modeled by equation (3.32),

$$\frac{dx}{dt} = \lambda - \mu,$$

where λ is the arrival rate and μ is the service rate, which is manipulated by changing the processor voltage. A PI controller for keeping queue length close to x_{r} is given by

$$\mu = k_{\mathrm{p}}(x - \beta x_{\mathrm{r}}) + k_{\mathrm{i}} \int_0^t (x - x_{\mathrm{r}})\, dt.$$

Choose the controller parameters k_{p} and k_{i} so that the closed loop system has the characteristic polynomial $s^2 + 1.6s + 1$, then adjust the setpoint weight β so that the response to a step in the reference signal has 2% overshoot.

[N,1ep] pid:motor-pdctrl
Exercise 3.7
Exercise 7.11

11.9 (Motor drive) Consider the model of the motor drive in Exercise 3.7 with the parameter values given in Exercise 7.11. Develop an approximate second-order model of the system and use it to design an ideal PD controller that gives a closed loop system with eigenvalues $-\zeta\omega_0 \pm i\omega_0\sqrt{1-\zeta^2}$. Add low-pass filtering as shown in equation (11.13) and explore how large ω_0 can be made while maintaining a good stability margin. Simulate the closed loop system with the chosen controller and compare the results with the controller based on state feedback in Exercise 7.11.

Exercise 7.11

[B,1ep] pid:antiwindup-intpi

11.10 (Windup and anti-windup) Consider a PI controller of the form $C(s) = 1 + 1/s$ for a process with input that saturates when $|u| > 1$, and whose linear dynamics are given by the transfer function $P(s) = 1/s$. Simulate the response of the system to step changes in the reference signal of magnitude 1, 2, and 10. Repeat the simulation when the windup protection scheme in Figure 11.11 is used.

[B,1ep] pid:antiwindup-condint

11.11 (Windup protection by conditional integration) Many methods have been proposed to avoid integrator windup. One method called *conditional integration* is to update the integral only when the error is sufficiently small. To illustrate this method we consider a system with PI control described by

$$\frac{dx_1}{dt} = u, \qquad u = \mathrm{sat}_{u_0}(k_{\mathrm{p}} e + k_{\mathrm{i}} x_2), \qquad \frac{dx_2}{dt} = \begin{cases} e & \text{if } |e| < e_0, \\ 0 & \text{if } |e| \geq e_0, \end{cases}$$

where $e = r - x$. Plot the phase portrait of the system for the parameter values $k_\mathrm{p} = 1$, $k_\mathrm{i} = 1$, $u_0 = 1$, and $e_0 = 1$ and discuss the properties of the system. The example illustrates the difficulties of introducing *ad hoc* nonlinearities without careful analysis.

11.12 (Windup stability) Consider a closed loop system with controller transfer function $C(s)$ and process transfer function $P(s)$. Let the controller have windup protection with the tracking constant k_aw. Assume that the actuator model in the anti-windup scheme is chosen so that the process never saturates. [C,2e] pid:antiwindup-stab

(a) Use block diagram transformations to show that the closed loop system with anti-windup can be represented as a connection of a linear block with transfer function (11.11) and a nonlinear block representing the actuator model.

(b) Show that the closed loop system is stable if the Nyquist plot of the transfer function (11.11) has the property $\mathrm{Re}\, H(i\omega) > -1$.

(c) Assume that $P(s) = k_\mathrm{v}/s$ and $C(s) = k_\mathrm{p} + k_\mathrm{i}/s$. Show that the system with windup protection is stable if $k_\mathrm{aw} > k_\mathrm{i}/k_\mathrm{p}$.

(d) Use describing function analysis to show that without the anti-windup protection, the system may not be stable and estimate the amplitude and frequency of the resulting oscillation.

(e) Build a simple simulation that verifies the results from part 0d.† **RMM**: Fix reference

11.13 Consider the system in Exercise 11.9 and investigate what happens if the second-order filtering of the derivative is replaced by a first-order filter. [C,1ep] Exercise 11.9 pid:demotor-derfilter

Supplemental Exercises Comment [RMM, 8 Sep 2018]: Karl: consider adding an example of PID control applied to queuing systems, motivated by cloud computing. This was something we discussed briefly in Lund. OK to omit.

Comment [RMM, 2017?]: Later: Adaptation in biology = integral feedback? Possibly build something around John D's PNAS paper.

11.14 Consider a second-order process of the form [C,1es] pid:pidmethods

$$P(s) = \frac{k}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \qquad k, \zeta, \omega_0 > 0.$$

In this problem we will explore various methods for designing a PID controller for the system.

(a) (Eigenvalue assignment) Suppose that we want the closed loop dynamics of the system to have a characteristic polynomial given by

$$p(s) = s^3 + a_1 s^2 + a_2 s + a_3.$$

Compute a formula for the controller parameters of a PID controller (k_p, k_i, and k_d) that gives the desired closed loop response.

(b) (Eigenvalue assignment) Let the process parameters be given by $k = 1$, $\zeta = 0.5$, and $\omega_0 = 2$. Using the formulas from part (a), compute a feedback control law that places the closed loop poles of the system at $\lambda = \{-1, -2 \pm i\}$. Plot the step response and frequency response for the closed loop system, and compute the gain and phase margins for your design.

(c) (Ziegler–Nichols step response) Using the same process parameters as above, plot the step response for the corresponding system and use one of the Ziegler–Nichols rules to design a PID controller. Plot the closed loop step response and frequency response for your design, and compute the gain and phase margins.

[C,1es] pid:pidexmps

11.15 For the control systems below, design a P, PI, PD, or PID control law that stabilizes the system, gives less than 1% error at zero frequency and gives at least 30° phase margin. You may use any method (loop shaping, Ziegler–Nichols, eigenvalue assignment, etc) and you only need to design one type of controller (as long as it meets the specification), but be sure to explain why you chose your controller, and include appropriate plots or calculations showing that all specifications are met. For the closed loop system, determine the steady-state error in response to a step input and the maximum frequency for which the closed loop system can track with less than 25% error.

(a) Drug administration/compartment model (Section 4.6):

$$P(s) = \frac{1.5s + 0.75}{s^2 + 0.7s + 0.05}$$

(b) Disk drive read head positioning system:

$$P(s) = \frac{1}{s^3 + 10s^2 + 3s + 10}$$

*Instructor note:*The given systems are stable with unity feedback - this could be rewritten with some performance specifications (perhaps a range of frequencies for tracking).

[D,1es] pid:compartment-monotone

11.16 (Compartment model) Compartment models and many systems encountered in industry have the property that their impulse responses are positive or, equivalently, that their step responses are monotone. Consider such a system with the transfer function $P(s)$. Show that the impulse response $h_n(t)$ of the normalized system $P(s)/P(0)$ has the properties $h_n(t) \geq 0$ and $\int_0^\infty h_n(t)dt = 1$. The function $h_n(t)$ can be interpreted as a probability density function—the probability that a particle entering the system at time 0 will exit at time t. Let

$$T_{\mathrm{ar}} = \int_0^\infty t h_n(t) dt$$

be the average residence time. Show that $T_{\mathrm{ar}} = -P'(0)/P(0)$ and that the tuning formula (11.6) can be written as $k_{\mathrm{i}} = 1/(T_{\mathrm{ar}} P(0))$.

11.17 A plasma oven can be used to retrieve rare metals from scrap. A key issue for this process is to maintain a stable plasma. This can be achieved by keeping the current through the plasma constant. The process can be described approximately by the model [N,1es] pid:plasma-oven

$$L\frac{dI}{dt} = V - (a + be^{-cI})$$

where the state I is the current through the plasma and the control variable is the voltage V of the amplifier that drives the system. Consider the plasma as a dynamical system with input V and output I. Linearize the equation around the operating condition I=20 A. Use the numerical values $L = 0.005$, a=2000 V, b=10,000 V, and c=0.05 A^{-1}.

(a) Write the linearized equations in standard form.

(b) Give the transfer function of the system.

(c) Is the system stable?

(d) Suggest a simple controller for keeping the current constant.

11.18 Consider a first-order system with a PI controller given by [D,1es] pid:pirlocus

$$P(s) = \frac{b}{s+a} \qquad C(s) = k_\mathrm{p}\Big(1 + \frac{1}{T_\mathrm{i} s}\Big).$$

In this problem we will explore how varying the gains k_p and T_i affect the closed loop dynamics.

(a) Suppose we want the closed loop system to have the characteristic polynomial

$$s^2 + 2\zeta\omega_0 s + \omega_0^2.$$

Derive a formula for k_p and T_i in terms of the parameters a, b, ζ, and ω_0.

(b) Suppose that we choose $a = 1$, $b = 1$ and choose ζ and ω_0 such that the closed loop poles of the system are at $\lambda = \{-20 \pm 10i\}$. Compute the resulting controller parameters k_p and T_i and plot the step and frequency responses for the system.

(c) Using the process parameters from part (b) and holding T_i fixed, let k_p vary from 0 to ∞ (or something very large). Plot the location of the closed loop poles of the system as the gain varies. You should plot your results in two different ways:

- A pair of plots showing the real and imaginary parts of the poles as a function of the gain k_p, similar to Figure 4.18a in the text.
- A parametric plot, showing the location of the eigenvalues on the complex plane, as k_p varies. Label the gains at which any interesting features in this plot occur. (This type of plot is called a *root locus* diagram.)

You may find it convenient to use the `subplot` command in MATLAB so that you can present all of your results in a single figure.

[D,1es] pid:pvtol-pid

11.19 In this problem we will design a PID compensator for a vectored thrust aircraft (see Example 2.9 in the text for a description). Use the following transfer function to represent the dynamics from the lateral input to the roll angle of the aircraft:

$$P(s) = \frac{r}{Js^2 + cs + mgl} \qquad \begin{array}{lll} g = 9.8 \text{ m/s}^2 & m = 1.5 \text{ kg} & c = 0.05 \text{ kg/s} \\ l = 0.05 \text{ m} & J = 0.0475 \text{ kg m}^2 & r = 0.25 \text{ m} \end{array}$$

(these parameters correspond to a laboratory-scale experiment that we have a Caltech). Design a feedback controller that tracks a given reference input with the following specifications:

- Steady-state error of less than 1%
- Tracking error of less than 5% from 0 to 1 Hz (remember to convert this to rad/s).
- Phase margin of at least 30°.

(a) Plot the open loop Bode plot for the system and mark on the plot the various frequency domain constraints in the above specification, as we did in class.

(b) Design a compensator for the system that satisfies the specification. You should include appropriate plots or calculations showing that all specifications are met.

(c) Plot the step and frequency response of the resulting closed loop control. For the step response, compute the steady-state error, rise time, overshoot, and settling time of your controller.

(Hint: you may not need all of the terms in a PID controller.)

[D,1es] pid:balance-pid

11.20 Consider the cart–pendulum system with the pendulum hanging *down* (you can think of this as the problem of moving the cart without exciting the pendulum too much; similar to walking without sloshing your coffee). The dynamics describing how the position of the cart depends on the applied force is given by the transfer function

RMM: Update parameters to match main text. $M = 10$, etc

$$P(s) = \frac{ls^2 + g}{Mls^4 + +bls^3 + (M+m)gs^2 + bgs} \qquad \begin{array}{ll} M = 0.5 \text{ kg} & m = 0.2 \text{ kg} \\ l = 0.3 \text{ m} & b = 0.1 \text{ N/m/s} \\ g = 9.8 \text{ kg m/sec}^2 & \end{array}$$

In this problem you will design a control law that satisfies the following specifications:

- 0.1% steady-state error
- Position (x) tracking within 10% up to 0.05 Hz
- Overshoot of less than 10% to step changes in x position
- Disturbance rejection of 10X for all disturbances above 10 Hz (for inputs with frequency above 10 Hz, the output of the system should be *smaller* than 0.1X the size of the input)

(a) Write the frequency domain portions of the specification as constraints on the loop transfer function in the appropriate frequency ranges. Estimate the phase margin requirement imposed by the step response using a second-order approximation. Show your results by sketching them on a Bode plot.

(b) Design a PID control law that satisfies the specification. Make sure to discuss how you determined the form of the control law and demonstrate that your control law satisfies all specifications.

(c) Determine whether your control law is robustly stable with respect to added sensor dynamics

$$G(s) = \frac{1}{s+1}.$$

These dynamics should be inserted in the feedback loop, as we did in Lecture 7.1. If your controller is not stable with for the perturbed system, redesign your control law to provide as much performance as you can for the nominal plant and still maintain stability for the perturbed plant.

(d) For the control law that provided robust stability in (a), check to see if the performance specifications are satisfied for the perturbed plant. If your controller is not robust, redesign your control law to provide robust stability and robust performance to the specified sensor dynamics. If necessary, relax the specifications according to the priorities.

(e) Consider now the case where we have a time delay of 1 second instead of $G(s)$. Determine whether your control law from part (b) provides robust stability and/or robust performance in the presence of the specified time delay. (You just need to check to see if it works; you don't need to redesign it.)

*Instructor note:*This problem is a pretty complicated transfer function and should only be used after students have had some experience with controller design. This could also be used as a (messy) exam problem, in which case it might be useful to give the locations of the poles and zeros (so that MATLAB is not necessarily required to complete the design).

11.21 (Congestion control) [D,1ep] pid:congctrl-pituning

Action: This exercise needs to be updated to make more clear whether we are designing the AQM controller of the TCP controller. Might be a good one to drop since it is a bit artificial. [RMM, 31 Dec 2018] **TBD**

A simplified flow model for TCP transmission is derived in [6, 7].† The linearized dynamics are modeled by the transfer function

RMM: Sort out citations for this work. We cite it in many places and don't do that with other applications. Ref Ch 5?

$$G_{bp}(s) = \frac{\gamma}{(s+a_1)(s+a_2)} e^{-s\tau_e},$$

which describes the dynamics relating the expected buffer length b to the expected packet drop p. The parameters are given by $a_1 = 2N^2/(c\tau_e^2)$, $a_2 = 1/\tau_e$, and $\gamma = c^2/(2N)$.† The parameter c is the bottleneck capacity, N is the number of sources feeding the link and τ_e is the round-trip delay time. Use the parameter values

RMM: check

RMM: update to match running

$N = 75$ sources, $c = 1250$ packets/s† and $\tau_e = 0.15$ sec and find the parameters of a PI controller using one of the Ziegler–Nichols rules and the corresponding improved rule. Simulate the responses of the closed loop systems obtained with the PI controllers.

[N,1es] pid:ballbeam-pid Supplement 3.26

11.22 (Ball and beam) Consider the ball and beam system described in Exercise **??**. Design a controller for the position of the ball by first designing a PD controller for the angle. If this regulator is well designed, the relation between the ball position and the reference angle will be approximately a double integrator. Next, design a PID controller for controlling the ball position using the commanded beam angle as an input. (This type of design is know as an inner loop/outer loop design and is described in more detail in Example 12.9 .)

Chapter 12 – Frequency Domain Design

12.1 Consider the system in Example 12.1, where the process and controller transfer functions are given by [B,1ep] loopsyn:rhppole-cancel

$$P(s) = 1/(s-a), \qquad C(s) = k(s-a)/s.$$

Choose the parameter $a = -1$ and compute the time (step) and frequency responses for all the transfer functions in the Gang of Four for controllers with $k = 0.2$ and $k = 5$.

12.2 (Equivalence of Figures 12.1 and 12.2) Consider the system in Figure 12.1 and let the outputs of interest be $\xi = (\mu, \eta)$ and the major disturbances be $\chi = (w, v)$. Show that the system can be represented by Figure 12.2 and give the matrix transfer functions $\mathcal{P}$ and $\mathcal{C}$. Verify that the elements of the closed loop transfer function $H_{\xi\chi}$ are the Gang of Four. [B,1ep*] loopsyn:gangoffour-hinfstruc

12.3 (Equivalence of controllers with two degrees of freedom) Show that the systems in Figures 12.1 and 12.13 give the same responses to command signals if $F_{\mathrm{m}}C + F_u = CF$. [A,1ep] loopsyn:twodof-equiv

12.4 (Web server control) Feedback and feedforward are increasingly used for complex computer systems such as web servers. Control of a single server is an example. A model for a virtual server is given by equation (3.32), [B,2e] loopsyn:webserver

$$\frac{dx}{dt} = \lambda - \mu,$$

where x is the queue length, λ is the arrival rate, and μ is the server rate. The objective of control is to maintain a given queue length. The service rate μ can be changed by dynamic voltage and frequency scaling (DVFS). Determine a PI controller that gives a closed loop system with the characteristic polynomial $s^2 + 4s + 4$. Use feedforward in the form of setpoint weighting to reduce the overshoot for step changes in reference signals; simulate the closed loop system to determine the setpoint weighting.

12.5 (Rise time-bandwidth product) Consider a stable system with the transfer function $G(s)$ whose frequency response is an ideal low-pass filter with $|G(i\omega)| = 1$ for $\omega \le \omega_{\mathrm{b}}$ and $|G(i\omega)| = 0$ for $\omega > \omega_{\mathrm{b}}$ and which has low-pass character. Define the rise time T_{r} as the inverse of the largest slope of the unit step response and the bandwidth as $\widetilde{\omega}_{\mathrm{b}} = \int_0^\infty |G(i\omega)|/G(0)\, d\omega$. Show that with this definition of the bandwidth the *rise time-bandwidth product* satisfies $T_{\mathrm{r}}\,\widetilde{\omega}_{\mathrm{b}} \ge \pi$. [B,1es] loopsyn:risetime-bandwidth

12.6 (Disturbance attenuation) Consider the feedback system shown in Figure 12.1. Assume that the reference signal is constant. Let y_{ol} be the measured output when there is no feedback and y_{cl} be the output with feedback. Show that $Y_{\mathrm{cl}}(s) = S(s)Y_{\mathrm{ol}}(s)$, where Y_{cl} and Y_{ol} are exponential signals and S is the sensitivity function. [A,1ep*] loopsyn:sensitivity-atten

12.7 (Approximate expression for noise sensitivity) Show that the effect of high-frequency measurement noise on the control signal for the system in Example 12.3 [B,1ep*] loopsyn:noiseapprox can be approximated by

$$CS \approx C = \frac{k_\mathrm{d} s}{(sT_\mathrm{f})^2/2 + sT_\mathrm{f} + 1},$$

and that the largest value of $|CS(i\omega)|$ is $k_\mathrm{d}/T_\mathrm{f}$ which occurs for $\omega = \sqrt{2}/T_\mathrm{f}$.

[B,2e*] loopsyn:tmwm-prod

12.8 (Peak frequency-peak time product) Consider the transfer function for a second order system

$$G(s) = \frac{\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2},$$

which has the unit step response

$$y(t) = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin \omega_0 t\sqrt{1-\zeta^2}.$$

Let $M_\mathrm{r} = \max_\omega |G(i\omega)|$ be the largest gain of $G(s)$, which is assumed to occur at ω_mr, and let $y_\mathrm{p} = \max_t y(t)$ be the largest value of $y(t)$, which is assumed to occur at t_p. Show that

$$t_p\omega_\mathrm{mr} = \frac{\arccos\zeta}{\sqrt{1-\zeta^2}}, \qquad \frac{y_\mathrm{p}}{M_\mathrm{r}} = 2\zeta e^{-\zeta\varphi},$$

and evaluate the right-hand sides of the above equations for $\zeta = 0.5$, 0.707, and 1.0.

[C,1ep] loopsyn:sensitivity-twopole

12.9 (Disturbance reduction through feedback) Consider a problem in which an output variable has been measured to estimate the potential for disturbance attenuation by feedback. Suppose an analysis shows that it is possible to design a closed loop system with the sensitivity function

$$S(s) = \frac{s}{s^2 + s + 1}.$$

Estimate the possible disturbance reduction when the measured disturbance response is

$$y(t) = 5\sin(0.1\,t) + 3\sin(0.17\,t) + 0.5\cos(0.9\,t) + 0.1\,t.$$

[C,2e] loopsyn:bode-formula

12.10 (Bode's formula) Consider the lead compensator

$$G(s) = 16\frac{s + 0.25}{s + 4}.$$

Verify Bode's phase area formula (12.8) and show that $G(\infty) = 16G(0)$ by numerical integration.

[B,2e] loopsyn:lead-lag

12.11 (Lead-lag compensation) Lead and lag compensators can be combined into a lead-lag compensator that has the transfer function

$$C(s) = k\frac{(s + a_1)(s + a_2)}{(s + b_1)(s + b_2)}.$$

Show that the controller reduces to a PID controller with special choice of parameters and give the relations between the parameters.

12.12 (Attenuation of low-frequency sinusoidal disturbances) Integral action eliminates constant disturbances and reduces low-frequency disturbances because the controller gain is infinite at zero frequency. A similar idea can be used to reduce the effects of sinusoidal disturbances of known frequency ω_0 by using the controller [A,1ep*] loopsyn:sinusoid-atten

$$C(s) = k_\mathrm{p} + \frac{k_s s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

This controller has the gain $C_s(i\omega_0) = k_\mathrm{p} + k_s/(2\zeta)$ for the frequency ω_0, which can be large by choosing a small value of ζ. Assume that the process has the transfer function $P(s) = 1/s$. Determine the Bode plot of the loop transfer function and simulate the system. Compare the results with PI control.† **RMM**: Be more specific

12.13 Consider the spring–mass system given by equation (3.16), which has the transfer function [B,1ep] loopsyn:springmass-feedforward

$$P(s) = \frac{1}{ms^2 + cs + k}.$$

Design a feedforward compensator that gives a response with critical damping ($\zeta = 1$).

Comment [RMM, 2018?]: Next three exercises are all about root locus. Pick 1–2? (All three are currently referenced in the text.)

Response [KJA, 16 Jul 2019]: There is actually a third root locus Exercise **??**. Suggest that we keep **??** and move the other to supplement, put them close together

Response [RMM, 17 Aug 2019]: Looks like at least one of the exercises went away? Sort out what to keep when deciding on final exercises.

12.14 (Asymptotes of root locus) Consider proportional control of a system with the transfer function [C,2e*] loopsyn:rootlocus-asymp

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots b_m}{s^n + a_1 s^{n-1} + \cdots a_n} = b_0 \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}.$$

Show that the root locus has asymptotes that are straight lines that emerge from the point

$$s_0 = \frac{1}{n_\mathrm{e}}\Bigl(\sum_{k=1}^n p_k - \sum_{k=1}^m z_k\Bigr),$$

where $n_\mathrm{e} = n - m$ is the pole excess of the transfer function.

12.15 (Real line segments of root locus) Consider proportional control of a process with a rational transfer function. Assuming that $b_0 k > 0$, show that the root locus has segments on the real line where there are an odd number of real poles and zeros to the right of the segment. [C,2e*] loopsyn:rootlocus-realline

12.16 Consider a lead compensator with the transfer function [B,1ep*] loopsyn:lead-phasecalc

$$C_n(s) = \Bigl(\frac{s\sqrt[n]{k} + a}{s + a}\Bigr)^n,$$

KJA: Add approximate value for large n

which has zero frequency gain $C(0) = 1$ and high-frequency gain $C(\infty) = k$. Show that the gain required to provide a given phase lead φ is

$$k = \Big(1 + 2\tan^2(\varphi/n) + 2\tan(\varphi/n)\sqrt{1+\tan^2(\varphi/n)}\Big)^n,$$

and that $\lim_{n\to\infty} k = e^{2\varphi}$. Discuss the practical consequences of the results.

[A,1ep*]
limits:sensitivity-pm

12.17 (Phase margin formulas) Show that the relationship between the phase margin and the values of the sensitivity functions at gain crossover is given by

$$|S(i\omega_{\text{gc}})| = |T(i\omega_{\text{gc}})| = \frac{1}{2\sin(\varphi_{\text{m}}/2)}.$$

RMM: Check for extra vertical space in pdf version

[B,2e*]
loopsyn:rootlocus-initial

RMM: Referenced in the text

12.18 (Initial direction of root locus)† Consider proportional control of a system with the transfer function

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots b_m}{s^n + a_1 s^{n-1} + \cdots a_n} = b_0 \frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}.$$

Let p_j be an isolated pole and assume that $kb_0 > 0$. Show that the root locus starting at p_j has the initial direction.

$$\angle(s-p_j) = \pi + \Sigma_{k=1}^m \angle(p_j - s_k) - \Sigma_{k\neq j}\angle(p_j - p_k).$$

Give a geometric interpretation of the result.

[D,2e*]
loopsyn:crossover-frequencies

RMM: Low grade, but referenced in the text

12.19 (Gain crossover frequency properties)† Consider a system where the loop transfer function has monotone gain and phase.

(a) Show that the gain crossover frequency ω_{gc}, the sensitivity crossover frequency ω_{sc}, and the crossover frequency ω_{tc} for the complementary sensitivity function $T(s)$ are equal if the phase margin $\varphi_{\text{m}} = 60°$.

(b) Show that $\omega_{\text{sc}} < \omega_{\text{gc}} < \omega_{\text{tc}}$ if $\varphi, m < 60°$ and that $\omega_{\text{sc}} > \omega_{\text{gc}}$ if $\varphi_{\text{m}} > 60°$.

(c) Show that $\omega_{\text{sc}} \approx P(0)k_{\text{i}}$ for a process with integral action.

(d) Find an approximate relation between the bandwidth ω_{bw} and ω_{tc}.

Supplemental Exercises

[C,2e]
loopsyn:phasearea

12.20 (Bode's phase area formula) Consider a transfer function $G(s)$ with no poles and zeros in the right half-plane and assume that $G(0)$ is not zero. Prove Bode's phase area formula, equation (12.8), by contour integration of the function $F(s) = (1/s)\log\big(G(s)/G(0)\big)$.

12.21 Let $F(s)$ be the Laplace transform of a signal $f(t)$. Show that the Taylor series expansion for $f(t)$ for small t and its Laplace transform are related by [C,1es] loopsyn:laplace-smalltime

$$F(s) = \frac{1}{s} f(0) + \frac{1}{s^2} f'(0) + \frac{1}{2s^3} f''(0) + \cdots$$

Use this result to show that the behavior of a time function for small t is related to the Laplace transform for large s. Derive a similar result for large values of t. Hint: Use the definition of Laplace transform, introduce $v = st$ and use

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \frac{1}{s} \int_0^\infty e^{-v} f\Big(\frac{v}{s}\Big)\, dv.$$

12.22 (Robustness of closed loop systems obtained with PI tuning rules) Consider a system with the transfer function [N,1es] pid:znlimits

$$P(s) = \frac{K}{1 + sT} e^{-s\tau}$$

Compute the the phase lag at the gain crossover frequency for the systems obtained with the Ziegler–Nichols tuning rule based on the step response and the modified rule given by equation (**??**) for different ratios of $\tau/(T + \tau)$.

12.23 Consider a process with the transfer function [D,2e] loopsyn:rhp-pzpair2B

$$P(s) = \frac{s - 1}{(s - 5)(s + 0.05)}.$$

Find the lowest order controller that gives the closed loop poles -2, -2, and -8. Determine the poles and zeros of the controller and plot the root locus of the system. Discuss the root locus.† **RMM**: Vague; what are we looking for?

Comment [KJA, 16 Jul 2019]: This exercise should be combined with 12.36, because they are really similar, the only difference is that 12.22 has a fast rhp zero and a slow rhp pole, while 12.26 has a fast rhp pole and a slow rhp zero. The problem is best solved by root locus. 12.22 requires a controller with a right half plane pole because the only way to move the root locus into the left hp is to go around the slow rhp zero, while 12.36 can be solved with a stable controller. The problem can be solved straightforwardly by pole placement but the interesting discussion is really around the root locus.

12.24 (Vehicle steering in reverse) Consider the system for vehicle steering in Example 12.6. Assume that the vehicle is driven in reverse gear so that $\gamma < 0$. Compute the Gang of Four and discuss the behavior of the closed loop system when there are load disturbances and measurement noise. Simulate the system to verify your conclusions. [B,1es] loopsyn:steering-reverse

12.25 Regenerate the controller for the system in Example 12.5 and use the frequency responses for the Gang of Four to show that the performance specification is met. [D,2e] loopsyn:pvtol-gangoffour

12.26 Consider the system in Figure 12.13. Derive the transfer functions that correspond to the Gang of Six. [D,1es] loopsyn:pvtol-gangofXfeedforward

[N,1es]
loopsyn:secord-perf
RMM: Reference Exercise 9.9

12.27 (Performance characteristics of second-order systems) Consider a control system with with process and controller dynamics given by

$$P(s) = \frac{1}{s(s+1)} \qquad C(s) = k.$$

where $k > 0$.

(a) Compute the closed loop transfer functions for the "Gang of Four" and give an analytical formula for the poles and zeros for each in terms of k.

(b) Plot the following performance characteristics as a function of k, with k varying between 0.1 and 10 (using logarithmic spacing):

- The gain crossover frequency, ω_{gc}
- The phase margin, φ_{m}, for the system (using $L = PC$)
- Magnitude of resonant peak, M_{r}, for $T = H_{yr}$
- Percentage overshoot, M_{p}, for the step response of r to y

Include enough resolution in k to see all interesting features. (Hint: Use `logspace` to set the values of k and use the outputs from the `margin`, `bode`, and `step` commands in MATLAB, along with the `max` function.)

(c) Using your data from part (b), plot the overshoot M_{p} and the gain crossover ω_{gc} as a function of the phase margin φ_{m}.

(d) Generate step responses for the gains corresponding to a phase margin of $\varphi_{\mathrm{m}} = 30^\circ$, 45°, and 60°. Explain the main features of the plot in terms of your results from part (c).

[N,1es]
loopsyn:secord-perf-alt
RMM: Reference Exercise 9.9

12.28 (Performance characteristics of second-order systems) Consider a control system with process and controller dynamics given by

$$P(s) = \frac{1}{s(s+c)} \qquad C(s) = k.$$

where $k > 0$.

(a) Show that the closed loop (tracking) response of the system can be written as

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

and give formulas for ζ and ω_0 in terms of c and k.

(b) Show that the phase margin for the system is given by

$$\varphi_{\mathrm{m}} = \tan^{-1}\Bigl(\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\Bigr)$$

(Hint: compute the frequency at which $|L(i\omega)| = 1$ and then find the phase at that frequency.)

12.29 Consider a second-order system with transfer function [N,1es] loopsyn:secord-rhpz

$$P(s) = \frac{-s+1}{(s+10)^2}.$$

(a) Plot the Bode plot for the system. Find another transfer function with the same magnitude but whose phase is less negative than the phase of $P(s)$.

(b) Plot the step response of the open loop system and show that the response initially moves in the opposite direction of the step.

(c) Consider a proportional controller $C(s) = k_\mathrm{p}$. Compute the range of gains for which the controller stabilizes the system and show that as $k_\mathrm{p} \to \infty$, one of the poles of the closed loop transfer function approaches the zero at $s = 1$.

12.30 (Lead compensator for an insect flight control system) Consider the problem of stabilizing the orientation of a flying insect, modeled as a rigid body with moment of inertia $J = 0.41$ and damping constant $D = 1$.[1] We assume there is a small delay $\tau = 0.01$ sgiven by the neural circuitry that implements the control system. The resulting transfer function for the system is taken to be [N,1es] loopsyn:flylead

$$P(s) = \frac{1}{Js^2 + Ds} e^{-\tau s}.$$

(a) Suppose that we can measure the orientation of the insect relative to its environment and we wish to design a control law that gives zero steady-state error, less than 10% tracking error from 0 to 0.5 Hz and has an overshoot of no more than 10%. Convert these specifications to appropriate bounds on the loop transfer function and sketch the resulting constraints on a Bode plot. (Hint: Try using problem 12.28 to convert the overshoot requirement to a phase margin requirement.)

(b) Using a lead compensator, design a controller that meets the specifications in part (a). Provide whatever plots are required to verify that the specification is met. You may use a Padé approximation for the time delay, but make sure that it is a good approximation over a frequency range that includes your gain crossover frequency.

(c) Plot or sketch the Nyquist plot corresponding to your controller and the process. You can again use a Padé approximation for the time delay. Show the gain and phase margin on your plot.

(d) Plot the "Gang of 4" for the system. If any of the magnitudes of the closed loop transfer functions are substantially greater than one in some frequency range, explain the consequences of this in terms of one of the input/output responses of your system. (You are not required to fix these problems.)

(e) Extra credit: genetically modify a fly to implement your controller, using the fly visual system as your input.

[1] Based loosely on "Biologically Inspired Feedback Design for Drosophila Flight", M. Epstein, S. Waydo, S. B. Fuller, W. Dickson, A. Straw, M. H. Dickinson, and R. M. Murray, 2007 American Control Conference.

[N,1es] loopsyn:maglev

12.31 (Control of a magnetic levitation system) Consider the dynamics of the magnetic levitation system from lecture. The transfer function from the electromagnet input voltage to the IR sensor output voltage is given by

$$P(s) = \frac{k}{s^2 - r^2}$$

with $k = 4000$ and $r = 25$ (these parameters are slightly different than those used in the MATLAB files distributed with the lecture).

(a) Design a proper stabilizing compensator for the process, assuming unity feedback. Compute the poles and zeros for the loop transfer function and for the closed loop transfer function between the reference input and measured output.

(b) Plot the Nyquist plot corresponding to your compensator and the process dynamics, and verify that the Nyquist criterion is satisfied.

(c) Plot the log of the magnitude of the sensitivity function, $\log|S(i\omega)|$, versus ω on a *linear* scale and numerically verify that the Bode integral formula is (approximately) satisfied. (Hint: you can do the integration numerically in MATLAB, using the `trapz` function. Make sure to choose your frequency range sufficiently large.)

[N,1es] loopsyn:pvtol-gangof4

12.32

Comment [RMM, 11 Nov 2018]: Compare to exercise **??**

RMM: Exercise filename misleading; rename to 'redesign?'

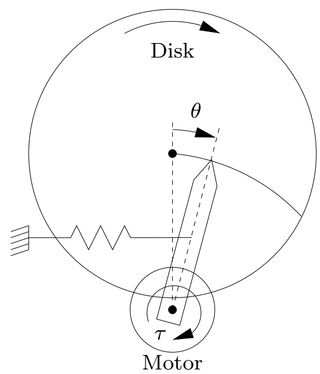
The controller for the vectored thrust aircraft designed Example 12.9 achieves the performance specifications but the Gang of Four in Figure 12.22 reveals that the noise and load sensitivity functions have high gain in some frequency ranges. Redesign the controller to meet the original performance specification but improving the high-frequency response to noise and low-frequency response to disturbances.

[N,1es] loopsyn:invpend-accsens
KJA: New exercise

12.33 (Stabilization of inverted pendulum with acceleration sensor) It has been proposed to stabilize an inverted pendulum where the pendulum angle is measured using an accelerometer (by computing the angle of the gravitational force). Explore if this is feasible. The mass of the pendulum is m, its moment of inertia with respect to the pivot is J. The distance between the center of mass and the pivot is l and the accelerometer is placed at the distance h from the pivot. Consider the cases when a) the pendulum is a point mass and $h = l$ and b) when the pendulum is a homogeneous rod and the mass of the accelerometer is negligible.

[N,1es] loopsyn:diskdrive-design

12.34 The figure below shows a simple mechanism for positioning a disk drive read head and the associated equations of motion:



$$J\ddot\theta = -b\dot\theta - kr\sin\theta + \tau_m$$
$$\dot\tau_m = -a(\tau_m - u)$$

RMM: Check font sizes

The system consists of a spring-loaded arm that is driven by a small motor. The motor applies a force against the spring and pulls the head across the platter. The input to the system is the desired motor torque, u. In the diagram above, the force exerted by the spring is a nonlinear function of the head position due to the way it is attached. All constants are positive. We wish to design a controller that holds the drive head at a given location θ_d.

(a) Show that the transfer function of the process can be written as

$$P(s) = \frac{a}{a+s}\cdot\frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

(b) Assume that the system parameters are such that $K = 0.001$, $\zeta = 0.5$, $\omega_n = 0.1$, and $a = 1$. Design a compensator that provides tracking with less than 10% error up to 1 rad/s and has a phase margin of 60°.

(c) Plot the Nyquist plot for the (open loop) system corresponding to your control design and compute the gain margin, phase margin, and stability margin.

(d) Compute and plot the Gang of Four for your system. Comment on any of the transfer functions that might lead to large errors or control signals and indicate the conditions under which this might occur.

Comment [RMM, 12 May 2018]: The following exercises were commented out. Make sure that we don't want to keep them (or that they appear elsewhere. If not used here, delete source files.

12.35 (Pole in the right half-plane and time delay) The non-minimum phase part of the transfer function for a system with one pole in the right half-plane and a time delay τ is [B,1es] loopsyn:rhppole-delay

$$P_\mathrm{ap}(s) = \frac{s+p}{s-p}e^{-s\tau}.$$

Using the gain crossover frequency inequality, compute the limits on the achievable bandwidth of the system.

12.36 Consider a process with the transfer function [D,2e] loopsyn:rhp-pzpair1B

$$P(s) = \frac{s-5}{(s-1)(s+0.05)}.$$

Find the lowest order controller that gives the closed loop poles -2, -2, and -8. Determine the poles and zeros of the controller and plot the root locus of the system. Discuss the root locus plot for the system.†

RMM: Vague; what are we looking for? Also, wording is slightly different for rhp-pzpair2B.tex. OK?

Comment [RMM, 2017?]: Possible additions (3e?):

- X-29: closeness of poles and zeros
- Internet design example
- Design example: bicycle balancing

Chapter 13 – Robust Performance

13.1 Consider systems with the transfer functions $P_1 = 1/(s+1)$ and $P_2 = 1/(s+a)$. Show that P_1 can be changed continuously to P_2 with bounded additive and multiplicative uncertainty if $a > 0$ but not if $a < 0$. Also show that no restriction on a is required for feedback uncertainty.

[B,1ep] robperf:uncertainty-onepole

13.2 Consider systems with the transfer functions $P_1 = (s+1)/(s+1)^2$ and $P_2 = (s+a)/(s+1)^2$. Show that P_1 can be changed continuously to P_2 with bounded feedback uncertainty if $a > 0$ but not if $a < 0$. Also show that no restriction on a is required for additive and multiplicative uncertainties.

[B,1ep] robperf:uncertainty-twopole

13.3 (Difference in sensitivity functions) Let $T(P, C)$ be the complementary sensitivity function for a system with process P and controller C. Show that

$$T(P_1, C) - T(P_2, C) = \frac{(P_1 - P_2)C}{(1 + P_1C)(1 + P_2C)},$$

and compare with equation (13.6). Derive a similar formula for the sensitivity function.

[B,1ep] robperf:sensitivity-difference

13.4 (Vinnicombe metrics) Consider the transfer functions

$$P_1(s) = \frac{k}{4s+1}, \qquad P_2(s) = \frac{k}{(2s+1)^2}, \qquad P_3(s) = \frac{k}{(s+1)^4}$$

Compute the Vinnicombe metric for all combinations of the transfer functions when $k = 1$ and $k = 2$. Discuss the results.

[C,2e] robperf:sensitivity-vinmetrics

13.5 (Sensitivity of feedback and feedforward) Consider the system in Figure 13.11 and let G_{yr} be the transfer function relating the measured signal y to the reference r. Show that the sensitivities of G_{yr} with respect to the feedforward and feedback transfer functions F and C are given by $dG_{yr}/dF = CP/(1+PC)$ and $dG_{yr}/dC = FP/(1+PC)^2 = G_{yr}S/C$.

[B,1ep*] loopsyn:sensitivity-gyr

13.6 (Guaranteed stability margin) The inequality given by equation (13.10) guarantees that the closed loop system is stable for process uncertainties. Let $s_\mathrm{m}^0 = 1/M_\mathrm{s}^0$ be a specified stability margin. Show that the inequality

$$|\delta(i\omega)| < \frac{1 - s_\mathrm{m}^0|S(i\omega)|}{|T(i\omega)|} = \frac{1 - |S(i\omega)|/M_\mathrm{s}^0}{|T(i\omega)|}, \qquad \text{for all } \omega \geq 0,$$

guarantees that the closed loop system has a stability margin greater than s_m^0 for all perturbations (compare with equation (13.10)).

[B,2e*] robperf:sensitivity-maxguaran

13.7 (Stability margins) Consider a feedback loop with a process and a controller having transfer functions P and C. Assume that the maximum sensitivity is $M_\mathrm{s} = 2$. Show that the phase margin is at least 30° and that the closed loop system will be stable if the gain is changed by 50%.

[B,1ep*] robperf:sensitivity-maxpm

13.8 (Bode's ideal loop transfer function) Bode's ideal loop transfer function is given in Example 13.9. Show that the phase margin is φ_m =180°–90°n and that the stability margin is $s_\mathrm{m} = \sin\pi(1 - n/2)$. Make Bode and Nyquist plots of the transfer function for n=5/3.

[B,1ep] robperf:idealbode-margins

[B,1ep] robperf:delay-compensation

13.9 (Ideal delay compensator) Consider a process whose dynamics are a pure time delay with transfer function $P(s) = e^{-s}$. The ideal delay compensator is a controller with the transfer function $C(s) = 1/(1 - e^{-s})$. Show that the sensitivity functions are $T(s) = e^{-s}$ and $S(s) = 1 - e^{-s}$ and that the closed loop system will be unstable for arbitrarily small changes in the delay.

[C,2e*] robperf:sensitivity-feedforward

13.10 (Sensitivity of two degree-of-freedom controllers to process variations) Consider the two degree-of-freedom controller shown in Figure 12.13, which uses feedforward compensation to provide improved response to reference signals and measured disturbances. Show that the input/output transfer functions and the corresponding sensitivities to process variations for the feedforward, feedback, and combined controllers are given by

Controller	G_{yr}	$\dfrac{dG_{yr}}{G_{yr}}$	G_{yv}	$\dfrac{dG_{yv}}{dP_1}$
Feedforward ($C = 0$)	F_m	$\dfrac{dP}{P}$	0	$-\dfrac{P_2}{P_1}$
Feedback (F_r, $F_v = 0$)	TF_m	$S\,\dfrac{dP}{P}$	$S\,P_2$	$-S\,\dfrac{P_2}{P_1}$
Feedforward and Feedback	F_m	$S\,\dfrac{dP}{P}$	0	$S\,\dfrac{P_2}{P_1}$

[X,1ep] robperf:hinf-singval

13.11 (H_∞ control) Consider the matrix $H(P, C)$ in equation (13.32). Show that it has the singular values

$$\sigma_1 = 0, \qquad \sigma_2 = \bar\sigma = \sup_\omega \frac{\sqrt{(1 + |P(i\omega)|^2)(1 + |C(i\omega)|^2)}}{|1 + P(i\omega)C(i\omega)|} = \|H(P, C))\|_\infty.$$

Also show that $\bar\sigma = 1/\delta_\nu(P, -1/C)$, which implies that $1/\bar\sigma$ is a generalization of the closest distance of the Nyquist plot to the critical point and hence also serves as a measure of the stability margin.

[C,2e] robperf:robust-distweighting

13.12 (Disturbance weighting) Consider an H_∞ control problem with the disturbance weight W ($\overline{P} = PW$ and $\overline{C} = W^{-1}C$). Show that

$$\|\mathcal{G}(\overline{P}, \overline{C})\|_\infty \geq \sup_\omega \big(|S(i\omega)| + |T(i\omega)|\big).$$

[C,1ep*] robperf:idealbode-fracsys

13.13 Consider a process with the transfer function $P(s) = k/(s(s+1))$, where the gain can vary between 0.1 and 10. A controller that has a phase margin close to $\varphi_\mathrm{m} = 45°$ for the gain variations can be obtained by finding a controller that gives the loop transfer function $L(s) = 1/(s\sqrt{s})$. Suggest how the transfer function can be implemented by approximating it by a rational function.

Supplemental Exercises

[X,1es] robperf:robustness-conditions

13.14 (Robustness inequalities) Derive the inequalities given in Table 13.1.

13.15 (Robustness using the Nyquist criterion) Another view of robust performance can be obtained through appeal to the Nyquist criterion. Let $S_{\max}(i\omega)$ represent a desired upper bound on our sensitivity function. Show that the system provides this level of performance subject to additive uncertainty Δ if the following inequality is satisfied: [D,1ep] robperf:robust-nyquist

$$|1+\tilde{L}| = |1+L+C\Delta| > \frac{1}{|S_{\max}(i\omega)|} \qquad \text{for all } \omega \geq 0. \tag{S13.1}$$

Describe how to check this condition using a Nyquist plot.

*Instructor note:*The solution is very long. Instructors should consider splitting it up into two exercises: make one exercise on state feedback in the State Feedback chapter, make another exercise on the Kalman filter and the output feedback in the Output Feedback chapter, and keep the rest of the exercise here.

13.16 Let P and C be matrices whose entries are complex numbers. Show that the singular-values of the matrix [X,1es] robperf:hinf-gangof4sv

$$H(P,C) = \begin{pmatrix} \dfrac{1}{1+PC} & \dfrac{P}{1+PC} \\ \dfrac{C}{1+PC} & \dfrac{PC}{1+PC} \end{pmatrix}$$

are

$$\underline{\sigma} = 0 \qquad \bar{\sigma} = \sup_{\omega} \frac{\sqrt{(1+|P(i\omega)|^2)(1+|C(i\omega)|^2)}}{|1+P(i\omega)C(i\omega)|}.$$

13.17 (Pole/zero cancellation) Consider the system in Figure 13.11, where the process and the controller have the transfer functions [?,2e] robperf:design-pzcanc

$$P(s) = \frac{1}{s+1}, \qquad C(s) = \frac{5(s+1)}{s}.$$

Notice that the process pole $s = -1$ is canceled by the zero of the controller. Derive the transfer functions G_{yv}, G_{yw}, G_{uv}, and G_{uw}. Compute the Kalman decomposition of a realization of the closed loop system with one state chosen as the process state and the other as the controller (integrator) state.

(a) Discuss reachability with respect to the inputs v and w and observability from the signals y and u.

(b) Discuss relations between pole/zero cancellations and the Kalman decomposition.

13.18 (Diametrically opposite points) Consider the points $z = a$ and $z = -1/a$ in the complex plane. Show that their projections on the Riemann sphere are diametrically opposite and hence their chordal distance is 1. [C,2e*] robperf:sensitivity-diaopp

13.19 Show that [N,1es] robperf:vinnicombe-sm

$$\sup_{w} \frac{|1+P(i\omega)C(i\omega)|}{\sqrt{(1+|P(i\omega)|^2)(1+|C(i\omega)|^2)}} = d(P, -1/C).$$

[B,1es]
robperf:uncertainty-rhp

13.20 Show that a stable additive perturbation Δ can create right half-plane zeros but not right half-plane poles, and that a stable feedback perturbation Δ_{fb} can create right half-plane poles but not right half-plane zeros. Give constructive examples of each.

[B,1es]
robperf:vinnicombe-nonunity

13.21 The distance measure $\delta_\nu(P_1, P_2)$ is closely related to closed loop systems with unit feedback. Show how the measure can be modified to measure the distance between closed loop systems with a fixed controller $C(s)$ and process dynamics P_1 and P_2.

[N,1es]
robperf:vinnicombe-onepole

13.22 Compute the Vinnicombe metric between the systems

$$P_1(s) = \frac{k}{s+1} \quad \text{and} \quad P_2(s) = \frac{k}{s-1}$$

for $k = 1$, 2, and 5.

[C,1ep]
robperf:smith-predictor

13.23 (Smith predictor) The *Smith predictor*, a controller for systems with time delays, can be obtained using the Youla parameterization by taking $P(s) = e^{-s\tau} P_0(s)$ and $Q(s) = C_0(s)/(1 + C_0(s)P_0(s))$. The controller $C_0(s)$ is designed to give good performance for the process $P_0(s)$.

Comment [RMM, 5 Dec 2019]: This problem could use a bit of cleaning up. See the paper "Future of the Smith Predictor Based Regulators Comparing to Youla Parametrization" by L. Keviczky and Cs. Banyasz for a good description.

(a) Compute the transfer function for the compensator and show it consists of a controller that uses a delayed version of the process output compared with the current value of the process output.

(b) Prove that the resulting control system is stable as long as the pair (P_0, C_0) is stable.

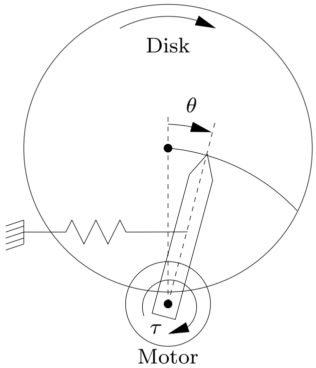
(c) Show that the sensitivity and complementary senstivity functions for the control system are given by

$$S(s) = \frac{1 + (1 - e^{-s\tau})P_0(s)C_0(s)}{1 + P_0(s)C_0(s)}, \qquad T(s) = \frac{P_0(s)C_0(s)}{1 + P_0(s)C_0(s)} e^{-s\tau}.$$

(It follows that a system with a Smith predictor gives an reference tracking response that is a delayed version of the tracking response for a system with no delay.)

[N,1es]
robperf:diskdrive-tracking

13.24 (Disk drive tracking) The figure below shows a simple mechanism for positioning a disk drive read head and the associated equations of motion:



$$J\ddot{\theta} = -b\dot{\theta} - kr\sin\theta + \tau_m$$

$$\dot{\tau}_m = -a(\tau_m - u)$$

RMM: Check font sizes

The system consists of a spring-loaded arm that is driven by a small motor. The motor applies a force against the spring and pulls the head across the platter. The input to the system is the desired motor torque, u. In the diagram above, the force exerted by the spring is a nonlinear function of the head position due to the way it is attached. All constants are positive. We wish to design a controller that holds the drive head at a given location θ_{d}. Assume that the system parameters are such that $K = 0.001$, $\zeta = 0.5$, $\omega = 0.1$, and $a = 1$. Design a compensator that provides tracking with less than 10% error up to 1 rad/s and has a phase margin of 60°. Describe the sets of additive and multiplicative uncertainty that can be accommodated by your design.

13.25 (AFM nanopositioning system) Consider the design in Example 13.11 and explore the effects of changing parameters α_0 and ζ_0. [C,1ep] robperf:afm-robustpid

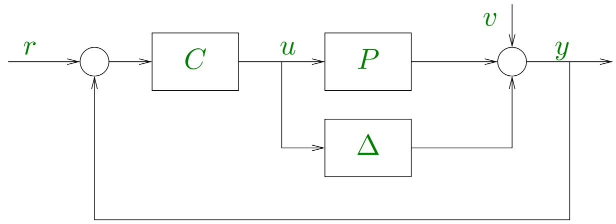
13.26 (Necessity of checking robustness) Consider the system [N,1ep] robperf:robust-statespace

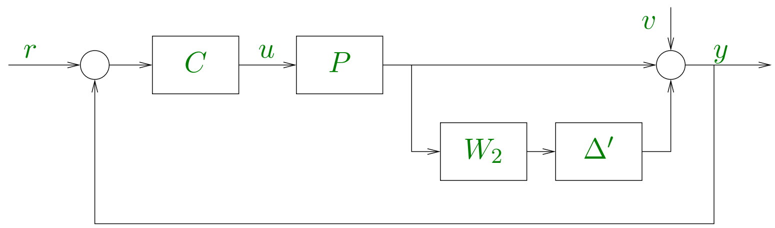
$$\frac{dx}{dt} = Ax + Bu = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} a-1 \\ 1 \end{pmatrix} u, \qquad y = Cx = \begin{pmatrix} 0 & 1 \end{pmatrix} x,$$

with $a = 1.25$. In Exercise 7.9 we designed a state feedback that gave the characteristic polynomial $\det(sI - BK) = s^2 + 2\zeta_{\mathrm{c}}\omega_{\mathrm{c}} s + \omega_{\mathrm{c}}^2$ and in Exercise 8.6 we designed an observer with the characteristic polynomial $\det(sI - LC) = s^2 + 2\zeta_{\mathrm{o}}\omega_{\mathrm{o}} s + \omega_{\mathrm{o}}^2$. The numerical values used were $a = 1.25$, $\omega_{\mathrm{c}} = 5$, $\zeta_{\mathrm{c}} = 0.6$, $\omega_{\mathrm{o}} = 10$, and $\zeta_{\mathrm{o}} = 0.6$. Compute the eigenvalues of the nominal system and the perturbed system where the process gain is increased by 2%. Also compute the loop transfer function and the sensitivity functions. Is there a way to know beforehand that the system will be highly sensitive?

Exercise 7.9
Exercise 8.6

13.27 In this problem we will transform an additive uncertainty problem to a multiplicative uncertainty problem in order to get a closed form solution for the robust performance problem. Consider the two unity feedback loops shown below with the uncertainty in the first system given by Δ stable, $\|\Delta\|_\infty \leq 1$ and the P stable, minimum phase, and biproper. [N,1es] robperf:uncertainty-add2mult





(a) Find a stable W_2 and stable Δ' with $\|\Delta'\|_\infty \leq 1$ such that the second system is the same as the first.

(b) Consider the performance specification given by $\|H_{uv}\|_\infty < 1$ for all Δ, where H_{uv} is the transfer function from v to u. Derive a necessary and sufficient condition for robust performance in terms of the complementary sensitivity function for the nominal plant and the weight W_2.

(c) Which of the following conditions is necessary in order for the above procedure to work:

(i) P stable

(ii) P minimum phase

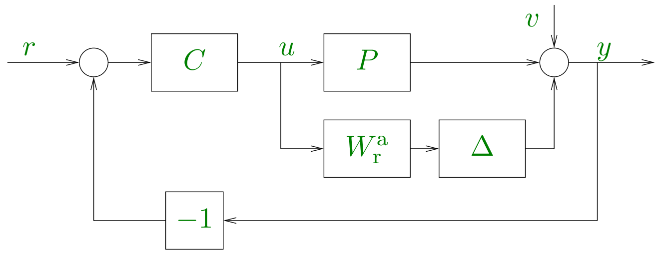
(iii) P biproper

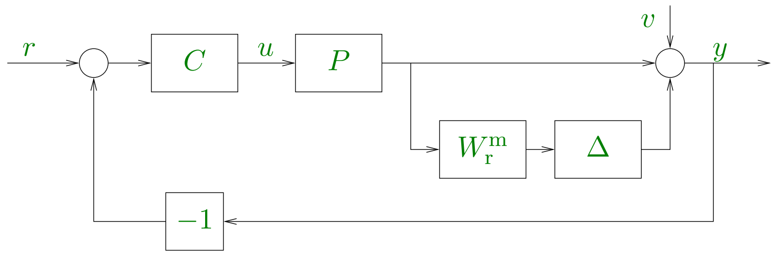
Explain your answer.

*Instructor note:*This problem is not particularly well formulated. Although a naive conversion of one problem to the other would seem to require that P not have any zeros in the right half-plane, it fact it is possible to derive robust stability and performance conditions for both cases (standard ones in DFT) without such assumptions $\implies$ the conditions are a bit artificial.

[N,1es] robperf:uncertainty-mult2add

13.28 Consider the two unity feedback loops shown below with the uncertainty in each system given by Δ stable, $\|\Delta\|_\infty \leq 1$.





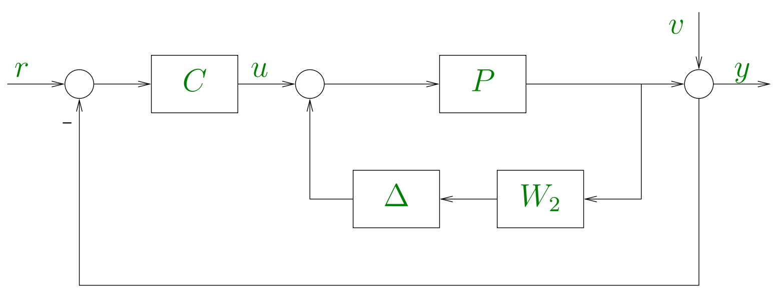
(a) Derive conditions under which the mutliplicative uncertainty problem is equivalent to the additive uncertainty problem by appropriate choice of the weight W_r^a. (Make sure to consider the case where P may have right half-plane poles or zeros.)

(b) Under what conditions is an additive uncertainty problem equivalent to a multiplicative uncertainty problem?

(c) Consider the performance specification given by $\|W_p H_{uv}\|_\infty < 1$ for all Δ, where H_{uv} is the transfer function from v to u and W_p is a weighting function. Derive a necessary and sufficient condition for robust performance in the presense of multiplicative uncertainty with weight W_r.

*Instructor note:*2019 TA notes: P6(a, b): 40 min, still a little bit confused about the notation in the diagram. For (b), I think an additional requirement for P is it has no RHP zeros? (c): 18 min. Proof using similar block diagram argument and small gain theorem.

13.29 Consider the system shown below. The performance objective is $\|W_1 H_{uv}\|_\infty < 1$ for all $\|\Delta\|_\infty \leq 1$, where H_{uv} is the transfer function from v to u. [N,1es] robperf:robperf-conditions



(a) Derive a set of necessary and sufficient conditions for robust stability of the system.

(b) Derive one or more sufficient conditions for robust performance. These conditions may be written in terms of W_1, W_2, L and P, but should not contain C or Δ.

(c) Design a lead compensator that provides robust stability and nominal performace for the following case:

$$P(s) = \frac{1}{s}, \qquad W_1(s) = \frac{1}{20}, \qquad W_2(s) = 5.$$

Check to see if your controller provides robust performance.

*Instructor note:*For CDS 131: the firstt part of this problem is the same as DFT 4.9. Make sure not to include both of them in the same problem set. The second part of the problem should perhaps be rewritten to given the bound and ask students to derive it?

Chapter 14 – Limits

The exercises below were commented out in source, but they are referenceed in the text. Need to decide whether to keep or move to supplemental? If not used, remove source files from repository. **RMM**

14.1 (Right half-plane pole/zero pair PI control) Consider a process with the transfer function [?,2e*] limits:rhp-pzpair1

$$P(s) = \frac{s-z}{s-p}.$$

(a) Show that the system can be controlled by a PI controller and design a PI controller that gives a closed loop system with poles at $s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$.

(b) Calculate the maximum sensitivity of the closed loop system as a function of ω_0 and compare with the bound imposed by the the right half-plane poles and zeros of the system. Discuss the differences between the cases $z > p$ and $z < p$.

(c) Plot the root locus of the process with the PI controller and qualitatively describe how it changes with the process pole and the process zero. Use the numerical values $\omega_0 = 1, \zeta = 1$, $p = 1$, $z = 5$ and $p = 5$, $z = 1$.

14.2 (Right half-plane pole/zero pair lag control) A process with the transfer function [?,2e*] limits:rhp-pzpair3

$$P(s) = \frac{s-0.1}{s-1}$$

cannot be controlled by a stable controller. Show that the controller

$$C = \frac{b}{s-2}$$

with $b = 6$ stabilizes the system. Plot the Nyquist curve of the closed loop system and determine the maximum sensitivities. Also plot the root locus with respect to the parameter b.

14.3 (Right half-plane pole/zero pair lag control) Consider a process with the transfer functions [C,2e*] limits:rhp-pzpair2

$$P(s) = \frac{s-z}{s-p}.$$

(a) Show that the system can be controlled by a controller with the transfer function $C(s) = b/(s-a)$. Design such a controller that gives the closed loop poles $s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$.

(b) Calculate the maximum sensitivities of the closed loop system as a function of ω_b and compare the bound imposed by the right half-plane poles and zeros of the system. Discuss the differences between the cases $z > p$ and $z < p$.

(c) Plot the root locus of the process with the PI controller and describe qualitative the differences between the cases $z > p$ and $z < p$. Use the numerical values $\omega_0 = 2, \zeta = 1, p = 1, z = 5$ and $p = 5, z = 1$.

[B,2e*] limits:bode-rolloff RMM: Update [see externote]

14.4 (Effect of roll-off) † Consider a closed loop system consisting of a first-order process and a proportional controller. Let the loop transfer function be

$$L(s) = P(s)C(s) = \frac{k}{s+1},$$

where parameter $k > 0$ is the controller gain. Show that the sensitivity function can be made arbitrarily small.

*Instructor note:*Ayush: For HW 9 problem 3, I think the wording of the problem can be a made a bit more specific since $\|S\|_\infty = 1$ for a fixed k, although $|S(j\omega)|$ can be made arbitrarily small with k. So, we could probably change it to say something like the magnitude of the sensitivity function can be made arbitrarily small?

[B,2e*] limits:bodeint

14.5 (Bode's integral formula) In Theorem 14.1 it was assumed that $sL(s)$ goes to zero as $s \to \infty$. Assume instead that $\lim sL(s) = a$ and show that

$$\int_0^\infty \log |S(i\omega)|\, d\omega = \int_0^\infty \log \frac{1}{|1 + L(i\omega)|}\, d\omega = \pi \sum p_k - a\,\frac{\pi}{2},$$

where p_k are the poles of the loop transfer function $L(s)$ in the right half-plane.

[C,1ep*] limits:bode-compsens

14.6 (Integral formula for complementary sensitivity)Prove the formula (14.7) for the complementary sensitivity.

[B,2e*] limits:hydroelectric

14.7 (Water turbine dynamics) Consider the problem of power generation in an hydroelectric power station. Let the control signal be the opening area a at the turbine entrance and ℓ be the length of the tube, which has area A. Formulate a mathematical model for the system, then linearize the model around a nominal valve opening $u_0 = a/A$ and a nominal power P_0. Show that the linearization is non-minimum phase, with transfer function

$$G(s) = \frac{P_0}{a_0}\,\frac{1 - 2u_0 s\tau}{1 + u_0 s\tau},$$

where $\tau = \ell/\sqrt{2gh}$ and g is the acceleration due to gravity.

[A,1ep*] limits:rhppzpair

14.8 (The pole/zero ratio) Consider a process with the loop transfer function

$$L(s) = k\frac{z - s}{s - p},$$

with positive z and p. Show that the system is stable if $p/z < k < 1$ or $1 < k < p/z$ and that the largest stability margin is $s_m = |p - z|/(p + z)$, which is obtained for $k = 2p/(p + z)$. Determine the pole/zero ratios that give the stability margin $s_m = 2/3$.

[C,2e*] limits:rhpzpair-rhppoledelay

14.9 (Phase lag of systems with right half-plane pole/zero pair and delay and right half-plane pole) Consider the transfer functions for a process with a right half-plane pole and right half-plane zero in Example 14.7 and a right half-plane pole and a time delay in Example 14.8. The phase lags of their all-pass factors are given in equations (14.15) and (14.16). Show that the largest phase lags are

$$\begin{aligned}\varphi_{\mathrm{ap1}} &= -\arg P_{pz}(i\omega) \le 2\arctan\big(2\sqrt{pz}/|z-p|\big),\\ \varphi_{\mathrm{ap2}} &= -\arg P_{p\tau}(i\omega) \le \sqrt{p\tau(2-p\tau)} + 2\arctan\sqrt{p\tau/(2p-p\tau)}\end{aligned}$$

and that they occur for $\omega_1 = \sqrt{pz}$ and $\omega_2 = \sqrt{2p/\tau - p^2}$ respectively.

14.10 (X-29) A simplified model of the X-29 aircraft in a certain flight condition has a right-hand pole/zero pair with $p = 6$ rad/s and $z = 26$ rad/s. Estimate the achievable stability margins and compare with the results in Example 14.4. [?,2e*] limits:rhp-pzpair-X29

14.11 (Sensitivity inequalities)Prove the inequalities given by equation (14.22). (Hint: Use the maximum modulus theorem.) [A,1ep*] limits:rhp-complexzero-cond

14.12 (Sensitivity limits due to poles in the right half-plane) Let $T_{\mathrm{r}} = M_{\mathrm{t}}\, b/(s + b)$ represent an upper bound on the desired sensitivity and let ω_{tc} represent the complementary sensitivity crossover frequency. Show that for a process $P(s)$ with a right half-plane pole $s = p$ but no other singularities in the right half-plane, the following inequalities hold: [C,2e*] limits:mmp-rhp-poles

$$b \ge \frac{p_{\mathrm{re}} + \sqrt{M_{\mathrm{t}}^2 p_{\mathrm{re}}^2 + (M_{\mathrm{t}}^2 - 1)p_{\mathrm{im}}^2}}{M_{\mathrm{t}}^2 - 1}, \qquad \omega_{\mathrm{tc}} \le \frac{p_{\mathrm{re}} + \sqrt{M_{\mathrm{t}}^2 p_{\mathrm{re}}^2 + (M_{\mathrm{t}}^2 - 1)p_{\mathrm{im}}^2}}{\sqrt{M_{\mathrm{t}}^2 - 1}}. \tag{S14.1}$$

14.13 (Maximum complementary sensitivity for multiple right half-plane poles and zeros) Consider a process $P(s)$ with the right half-plane zeros z_k and right half-plane poles p_k. Introduce the polynomial $n(s)$ with zeros $s = z_k$ and the polynomial $d(s)$ with zeros $s = p_k$. Show that the complementary sensitivity function has the property [C,2e*] limits:rhpz-Mt

$$M_{\mathrm{t}} \ge \max_k \left|\frac{n(-p_k)}{n(p_k)}\right|.$$

Also show that the equations (14.29) hold.

14.14 (Vehicle steering) Consider the Nyquist curve in Figure 14.12. Explain why part of the curve is approximately a circle. Derive a formula for the center and the radius and compare with the actual Nyquist curve. [C,1ep] limits:steering-nyquist

14.15 Consider a process with the transfer function [B,1ep] limits:design-fourthord

$$P(s) = \frac{(s+3)(s+200)}{(s+1)(s^2+10s+40)(s+40)}.$$

Discuss suitable choices of closed loop poles for a design that gives dominant poles with undamped natural frequency 1 and 10.

14.16 (Large signals) Verify Figure 14.1 by hand calculation. [A,1ep*] limits:largesignals

[A,1ep] limits:noisebw

14.17 (Noise limits bandwidth) Consider PI control of an integrator, where the transfer functions of the process and the controller are

$$P(s) = \frac{1}{s}, \qquad C(s) = k_\text{p} + \frac{k_\text{i}}{s},$$

and $k_\text{p} = 2\zeta\omega_0$ and $k_\text{i} = \omega_0^2$, with $\zeta = 0.707$. Assume that the inputs and outputs range from 0 to 10 V, that there is measurement noise with a standard deviation of 10 mV, and that the largest permissible variation in the control signal due to noise is 2 V. Show that the bandwidth, defined as $\omega_\text{bw} = 2\omega_0$, cannot be larger than 283.

Supplemental Exercises

[?,2e] limits:stabilizability-rank

14.18 (Stabilizability) Consider a linear system A, B with n state variables. Show that a system is stabilizable if

$$\operatorname{rank}\begin{pmatrix} A - \lambda I & B \end{pmatrix} = n$$

holds for $\lambda \in \text{RHP} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$ and that it is reachable if the condition holds for all $\lambda \in \mathbb{C}$. (The reachability condition is known as the Popov-Belevitch-Hautus (PBH) test..)

Comment [RMM, 26 Dec 2019]: Not clear that this results belongs in this chapter. Move to state feedback?

Comment [RMM, 26 Dec 2019]: The next two exercises are variants of ones included above. Create exernotes to this effect.

[B,1es] limits:bode-compsens-norolloff

14.19 (Bode's integral formula without rolloff) Bode's integral formula in Theorem 14.1 and the corresponding formula for the complementary sensitivity function assume that the loop transfer function has the property $\lim_{s\to\infty} sL(s) = 0$. Assume instead that the limit is not zero and prove that

$$\int_0^\infty \log|S(i\omega)|\,d\omega = \int_0^\infty \log\frac{1}{|1+L(i\omega)|}\,d\omega = \pi\sum p_k - \frac{\pi}{2}\lim_{s\to\infty} sL(s)$$
$$\int_0^\infty \log\left|T\Big(\frac{1}{i\omega}\Big)\right|d\omega = \pi\sum\frac{1}{z_k} - \frac{\pi}{2\lim\limits_{s\to 0} sL(s)}$$

[B,1es] limits:bode-compsens-alt

14.20 (Bode's integral formula for complementary sensitivity) Let z_k be the right half-plane zeros of the loop transfer function. Prove the following integral formula for the complementary sensitivity

$$\int_0^\infty \log\left|T\Big(\frac{1}{i\omega}\Big)\right|d\omega = \pi\sum\frac{1}{z_k}.$$

Comment [RMM, 8 Jul 2019]: Looks like there is something wrong in the solution. Double check before including in solutions manual.

14.21 (Limits on achievable phase lag) Derive the analytical formulas corresponding to the plots in Figure 14.8. [N,1es] limits:rhppole-limitations

14.22 (Time delay and a pole in the right half-plane) Consider a process with the transfer function [C,2e*] limits:mmp-rhp-pole-delay

$$P(s) = \frac{e^{-s\tau}}{s-p}\bar{P}(s),$$

where $\bar{P}(s)$ has no poles and zeros in the right half-plane. Show that the sensitivity functions have the properties listed in Table 14.1:

$$M_{\mathrm{t}} \geq e^{p\tau}, \qquad M_{\mathrm{s}} \geq e^{p\tau} - 1.$$

14.23 (Stabilization of an inverted pendulum with visual feedback) Consider stabilization of an inverted pendulum based on visual feedback using a video camera with a 50-Hz frame rate. Let the effective pendulum length be l. Assume that we want the loop transfer function to have a slope of $n_{\mathrm{gc}} = -1/2$ at the crossover frequency. Use the gain crossover frequency inequality to determine the minimum length of the pendulum that can be stabilized if we desire a phase margin of 45°. [C,1ep*] limits:balance-visual

14.24 (Design with poor robustness) Consider a process with the transfer function [C,2e] limits:poor-robustness

$$P(s) = \frac{1}{s+1}.$$

Design a PI controller that gives a closed loop system with the characteristic polynomial $s^2 + 0.2s + 0.01$. Plot the Nyquist curve of the loop transfer function and determine the gain, phase, and stability margins.

14.25 (Rear-steered bicycle) Consider the simple model of a bicycle in equation (4.5), which has one pole in the right half-plane. The model is also valid for a bicycle with rear wheel steering, but the sign of the velocity is then reversed and the system also has a zero in the right half-plane. Use the results of Exercise 14.8 to give a condition on the physical parameters that admits a controller with the stability margin s_{m}. [C,1ep] limits:bicycle-rearsteer

Exercise 14.8

Chapter 15 – Architecture

Appendix A – CDS 101/110a Exam Problems

The problems in this chapter come from midterm and final exams for courses using Åström and Murray [2] as a textbook and from qualifying examinations based on this material.

A.1

A.2 Choose any two of the feedback systems listed below. For each system you choose, answer the following questions:

i. Draw a block diagram for the system consisting of the process dynamics, the sensing and actuation subsystems, and the control law (similar to Figure 3.1 in the course text, shown below). You should label your diagram in a descriptive fashion and include any external inputs to the blocks.

ii. For each block in your diagram, describe a plausible model of the subsystem and give the state, inputs and outputs, and the dynamics of the model. You may give your answer in words, but please be as precise as possible.

iii. Describe the effect of the feedback controller, either in terms of robustness with respect to uncertainty or the modifications of the (open loop) dynamics. In other words, describe *why* feedback might be useful for this example.

Systems to choose from:

1.1 Temperature control of a greenhouse.

1.2 Aircraft autopilot system.

1.3 Walmart supply chain system for regulating inventory levels.

1.4 Human balancing on a bicycle.

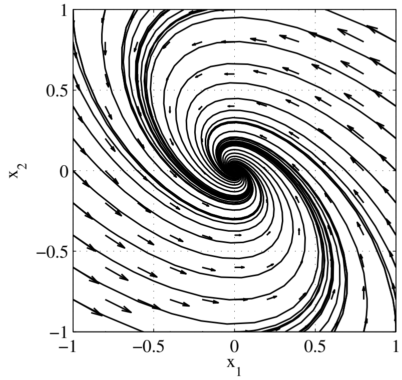
1.5 Population control for a lynx/hare ecosystem.

1.6 Regulation of oxygen concentration in the bloodstream.

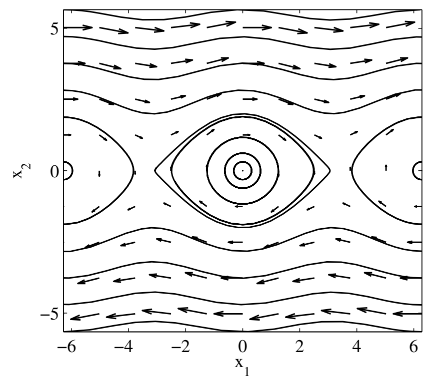
Please make sure to identify the number of the system when responding to the three questions above. Your answers will be graded based on your grasp of control concepts and not how well you know the details of the particular system (so it is OK if you make up some plausible description of the system).

A.3

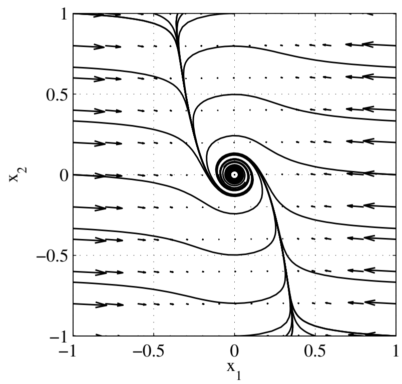
A.4 In this problem, you will answer a series of questions about each of the systems illustrated in the phase curves below.



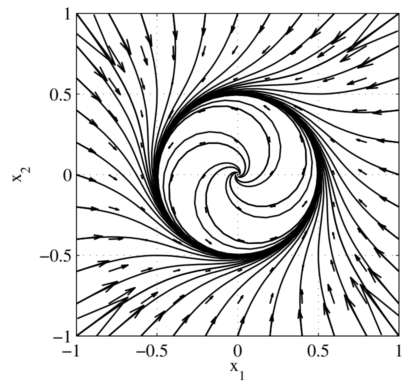
A.6-1



A.6-2



A.6-3



A.6-4

For each of the phase curves above, answer the following:

i. [5 points] Is the equilibrium point at the origin stable, asymptotically stable, or unstable?

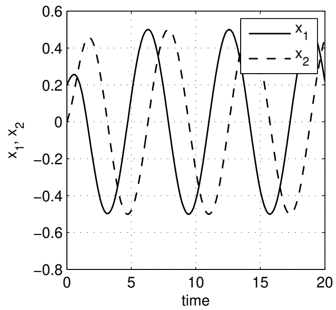
ii. [5 points] Which of the following sets of equations can be used to describe the phase portrait? Assume all constants are strictly positive (> 0) and that they may be chosen as needed to match the phase portrait. More than one equation may be applicable to a given phase portrait, so list *all* equations that are consistent.

$$\begin{bmatrix} \dot{x}_1 = -ax_1 - bx_2 \\ \dot{x}_2 = ax_1 \end{bmatrix} \quad (1) \qquad \begin{bmatrix} \dot{x}_1 = -ax_1^3 - bx_2 \\ \dot{x}_2 = ax_1 \end{bmatrix} \quad (2)$$

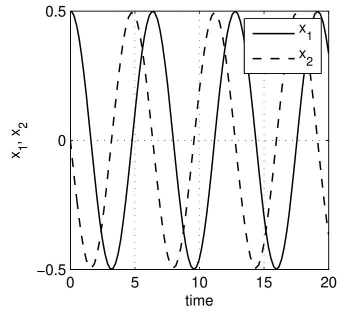
$$\begin{bmatrix} \dot{x}_1 = ax_1 - bx_2 - c(x_1x_2^2 + x_1^3) \\ \dot{x}_2 = ax_1 + bx_2 - c(x_1^2x_2 + x_2^3) \end{bmatrix} \quad (3) \qquad \begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g\sin(x_1) \end{bmatrix} \quad (4)$$

Hint: you should not need to solve these equations in detail to determine your answer.

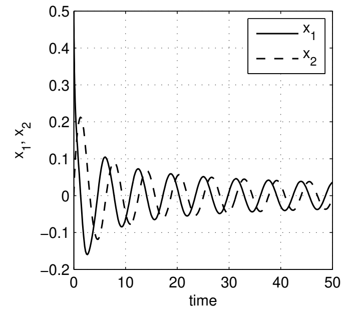
iii. [5 points] Which of the following sets of initial condition responses are possible for the given phase portrait? (Note that more than one response may be applicable to a given phase portrait, depending on the initial condition, so list all responses that are consistent.)



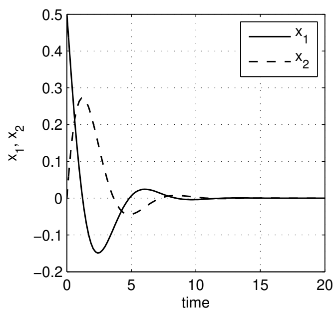
R1



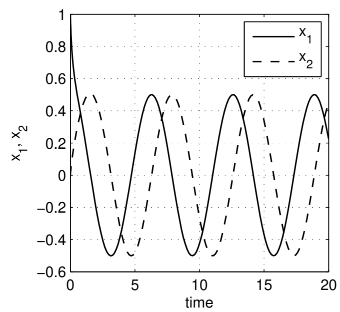
R2



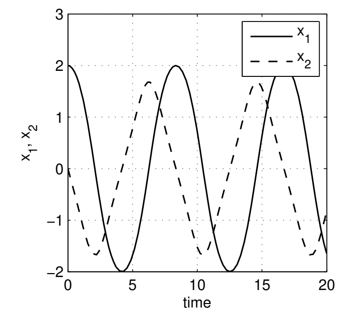
R3



R4



R5



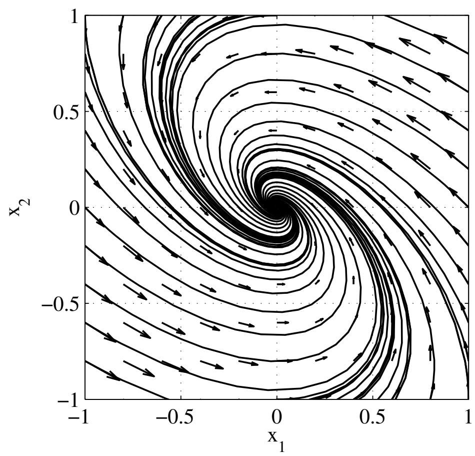
R6

Please transcribe the following table into your notes to summarize your answers. Note that you should *not* just fill out this table. You must *explain* your answers and provide justification to receive full credit for the problem.

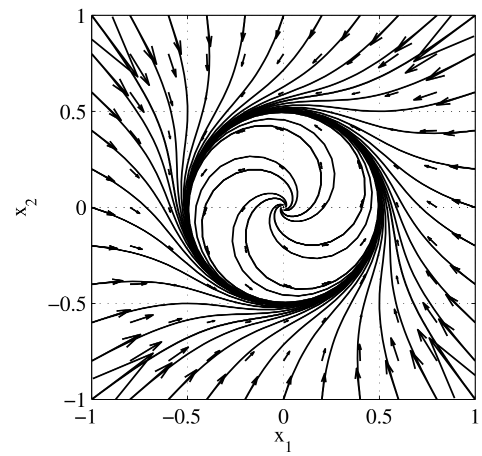
Part	Sample	System 2.1	System 2.2	System 2.3	System 2.4
(a) stable, asy stable, unstable	stable				
(b) Equation (1)–(4)	(2), (3)				
(c) Responses R1–R6	R1, R4				
(d) Globally asy stable?	Y				
(e) Exponentially stable?	N				

A.5

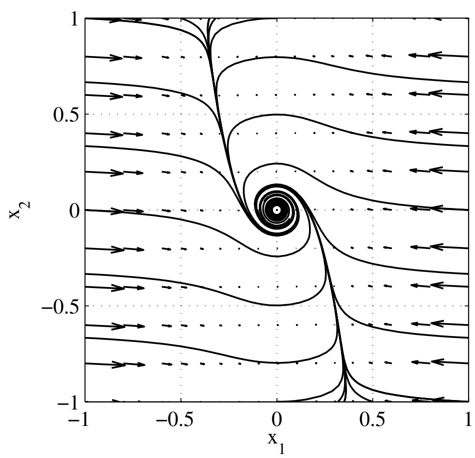
A.6 In this problem, you will answer a series of questions about each of the systems illustrated in the phase curves below.



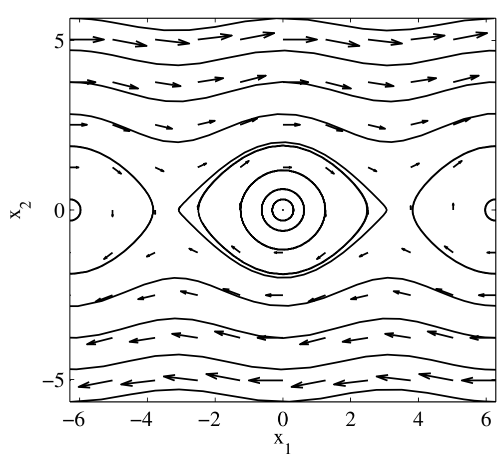
A.6-1



A.6-2



A.6-3



A.6-4

For each of the phase curves above, answer the following questions:

i. [5 points] Is the equilibrium point at the origin stable, asymptotically stable, or unstable?

ii. [5 points] Which of the following sets of equations can be used to describe the phase portrait? Assume all constants are strictly positive (> 0) and that they may be chosen as needed to match the phase portrait. More than one equation may be applicable to a given phase portrait, so list *all* equations that are consistent.

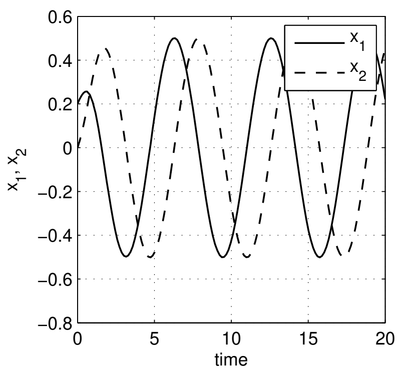
$$\begin{bmatrix} \dot{x}_1 = -ax_1 - bx_2 \\ \dot{x}_2 = ax_1 \end{bmatrix} \quad (1) \qquad \begin{bmatrix} \dot{x}_1 = -ax_1^3 - bx_2 \\ \dot{x}_2 = ax_1 \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} \dot{x}_1 = ax_1 - bx_2 - c(x_1x_2^2 + x_1^3) \\ \dot{x}_2 = ax_1 + bx_2 - c(x_1^2x_2 + x_2^3) \end{bmatrix} \quad (3) \qquad \begin{bmatrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g\sin(x_1) \end{bmatrix} \quad (4)$$

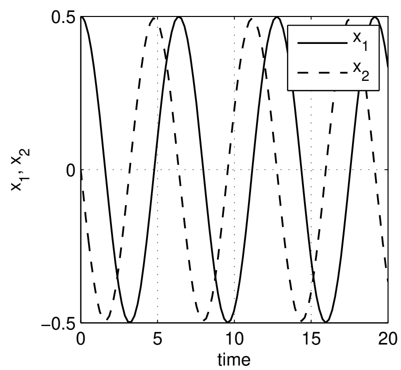
Hint: you should not need to solve these equations in detail to determine your answer.

iii. [5 points] Sketch a respresentative initial condition response showing the initial value of the state, any important features in the response, and the steady-state behavior of the system. If there are qualitatively different plots for different initial conditions, include a plot for each different behavior.

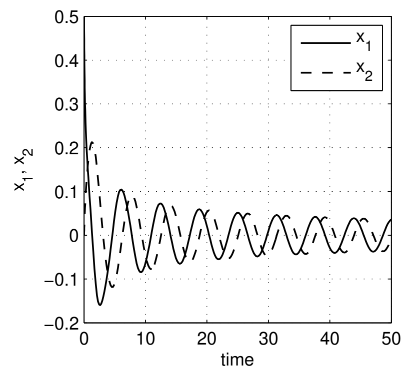
Two sample responses are show below:



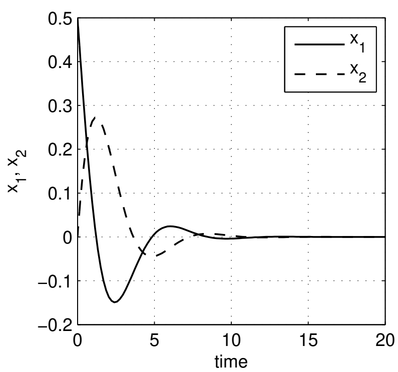
R1



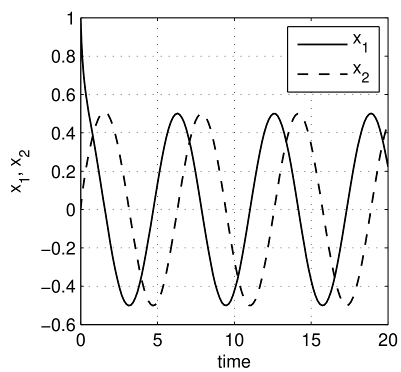
R2



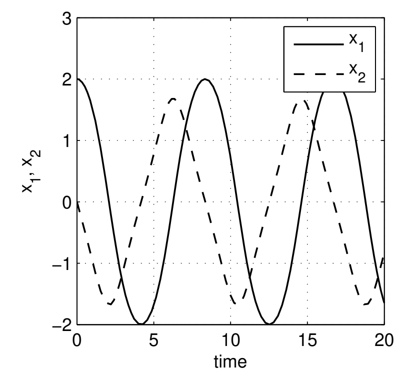
R3



R4



R5



R6

A.7

A.8 Write the form of the solution for a linear system

$$\frac{dx}{dt} = Ax + Bu$$

to an arbitrary input u. Use the form of the solution to show that the system is reachable if and only if the reachability matrix

$$W_\mathrm{r} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}.$$

is full rank.

A.9

A.10 In this problem you will analyze the performance of a congestion control method known as FAST TCP. It is not necessary to understand the derivation of these equations in order to complete the problem.

A single queue with a constant rate c fed by a FAST TCP sender $x(t)$ and a malicious sender $u(t)$ can be modeled by the following equations:

$$\dot{x} = \frac{1}{1+p}\left(\alpha - x\,(p-1) - \frac{x^2}{c} - \frac{xu}{c}\right)$$
$$\dot{p} = \frac{1}{c}\,(x - c + u)$$

where $x(t)$ is the FAST TCP sender rate (in bits/sec), $u(t)$ is the malicious sender rate (in bits/sec), and $p(t)$ is the queuing delay (in sec). Here $c > 0$ and $\alpha > 0$ are given constants, representing link capacity and target queue length, respectively.

i. Suppose the equilibrium malicious sender rate is $u_\mathrm{e} = 0$. What is the equilibrium state $(x_\mathrm{e}, p_\mathrm{e})$?

ii. Keeping $u_\mathrm{e} = 0$, let $z = (x - x_\mathrm{e}, p - p_\mathrm{e})$ be the state of the linearized system around the equilibrium point $(x_\mathrm{e}, p_\mathrm{e}, u_\mathrm{e})$, with $v = u - u_\mathrm{e}$ the input, and $y = p - p_\mathrm{e}$ the output. Show that the linearized system is described by:

$$\dot{z} = Az + Bv$$
$$y = Cz$$

where

$$A = \begin{pmatrix} -1 & -\frac{c^2}{c+\alpha} \\ \frac{1}{c} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -\frac{c}{c+\alpha} \\ \frac{1}{c} \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

iii. Suppose $v(t) = 0$ for all t. Show that the linearized system is asymptotically stable for any positive finite c and α.

iv. Take $c = \alpha = 1$. Suppose the malicious sender $v(t)$ can observe $p(t)$ but not $x(t)$ and tries to distabilize the system $z(t)$ using linear state feedback of the form

$$v = Kz = \begin{pmatrix} 0 & k \end{pmatrix} z$$

for some $k \in \mathbb{R}$. Show that $z(t)$ remains asymptotically stable if and only if $k < 1$.

A.11

A.12 In this problem, you will answer a series of questions about each of the following *linear* control systems:

$$\left[\begin{array}{l} \dot{x}_1 = -bx_1 + u \\ \dot{x}_2 = -ax_1 + bx_2 \end{array}\right. \qquad (3.1)$$

Assume all parameters are strictly positive (> 0).

i. [5 points] Let $u = 0$. Compute the eigenvalues for the equilibrium point at the origin and determine the stability of the origin.

ii. [5 points] Determine whether the system is reachable. If the answer depends on the specific values of parameters, indicate any conditions under which the system is *not* reachable.

A.13

A.14 Consider a linear process with transfer function

$$P(s) = \frac{s + a}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

i. Find a state space linear system whose steady-state input/output response has transfer function $H_{yu}(s) = P(s)$.

ii. Show that the process dynamics can be written in state space using input/output dynamics of the form

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 & -a_2 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} b_1 & b_2 \end{pmatrix} x$$

and find expressions for a_1, a_2, b_1, and b_2.

iii. Assume that the states of your model for part 1 are available. Design a state space controller that places the closed loop eigenvalues of the system at $\lambda_1 = -1$ and $\lambda_2 = -2$. Is it possible to do this for all values of a, ζ and ω_0?

iv. Suppose $a > \omega_0 > 0$ and $0 < \zeta < 0.5$, and consider a PI compensator of the form

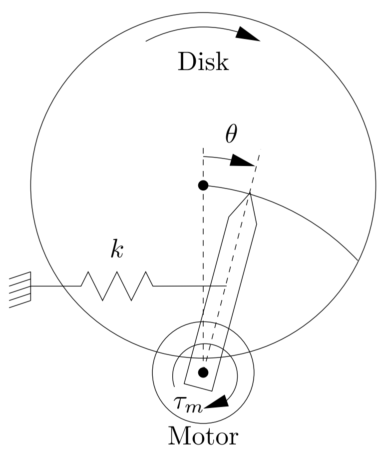
$$C(s) = \frac{k_\mathrm{p} s + k_\mathrm{i}}{s}.$$

Sketch the Bode and Nyquist plots for the loop transfer function and use these to determine if the bandwidth of the closed loop system can be set arbitrarily high using this compensator. What is the maximum phase margin your controller can achieve?

v. Suppose now that $-\omega_0 < a < 0$. Design a controller that gives step response with less than 1% steady-state error and that provides disturbance attentuation of a factor of 10 up to frequency ω_0.

A.15

A.16 The figure below shows a simple mechanism for positioning a disk drive read head and the associated equations.

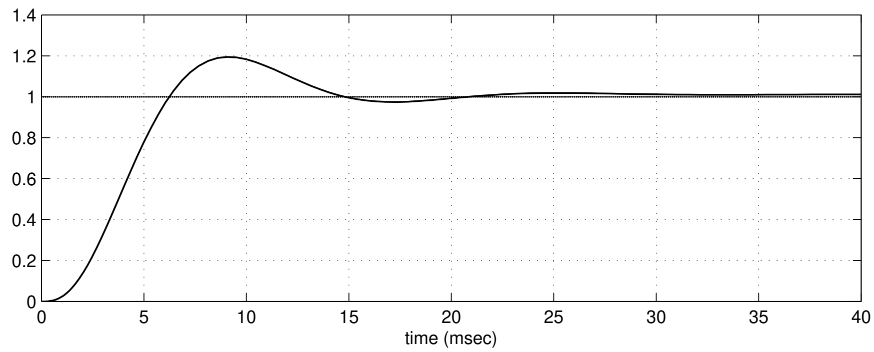


$$J\ddot{\theta} = -b\dot{\theta} - kr\sin\theta + \tau_m$$
$$\dot{\tau}_m = -a(\tau_m - u)$$

The system consists of a spring loaded arm that is driven by a small motor. The motor applies a force against the spring and pulls the head across the platter. The input to the system is the desired motor torque, u. In the diagram above, the force exerted by the spring is a nonlinear function of the head position due to the way it is attached. All constants are positive.

i. Write the equations of motion for the system in state space form ($\dot{x} = f(x, u)$). You should order your states so that the states corresponding to the arm dynamics come first and the motor dynamics second.

ii. Plotted below is the step response of the nominal closed loop system.



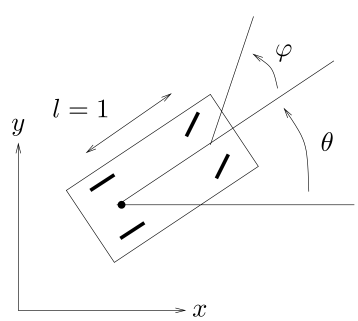
Estimate the rise time, overshoot, settling time (2%), and steady-state error. Indicate your answers on a (hand drawn) sketch of response, with appropriate notation to understand how you did your calculation.

iii. Suppose we wish to design a state space control law around a specified read area on the disk, given by $\theta_{\rm d} \gg 0$. Compute the linearization of the system at the desired equilibrium point and verify reachability of the system. Show by direct calculation that a state feedback control law of the form $u = u_{\rm e} - K(x - x_{\rm e})$ can be used to arbitrarily place the closed loop eigenvalues of the system. You may find the following calculation for the determinant of a 3×3 matrix helpful:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

A.17

A.18 Consider the problem of lateral steering for the Alice, Caltech's autonomous vehicle that competed in the 2007 Urban Challenge. The goal is to design a control law that adjusts the turning velocity of the steering wheel so as to cause the vehicle to follow a horizontal line. The system and its equations of motion are:



$$\dot y = v_0 \sin\theta$$
$$\dot\theta = v_0 \tan\varphi$$
$$\dot\varphi = u$$

Here y is the distance of a point at the rear of the car from center of the lane, θ is the angle that the car makes with respect to the horizontal, φ is the angle of the steering wheel, v_0 is the forward velocity of the car (assumed here to be constant), and u is the rate at which the steering wheel is turned. The horizontal position, x, is ignored. (Don't worry if you can't derive these equations, just assume they are right.)

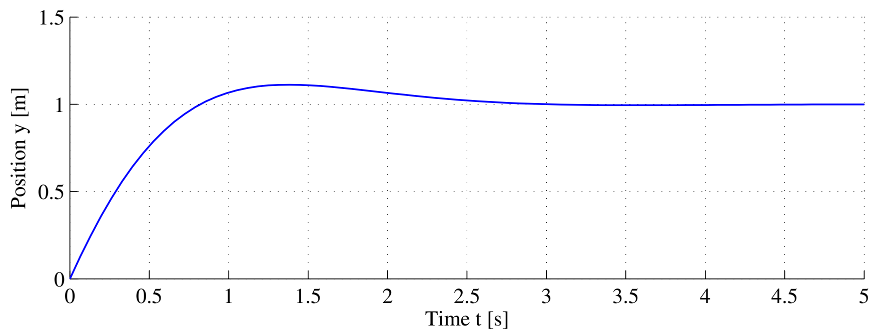
i. [5 points] Compute the equilibrium points for the system, assuming $\varphi \in [0, \pi/4]$ and $v_0 > 0$. (Hint: there is more than one equilibrium point.)

ii. [5 points] Show that the linearized dynamics of the system are given by

$$\dot{z} = \begin{pmatrix} 0 & v_0 & 0 \\ 0 & 0 & v_0 \\ 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w$$

where $z = x - x_\text{e}$ is the state and $w = u - u_\text{e}$ the input. Write down the dynamics of the closed loop system under a state feedback of the form $w = -Kz$ with $K = \begin{pmatrix} v_0 & 3v_0 & 3v_0 \end{pmatrix}$ and determine if the closed loop system is stable, asymptotically stable, or unstable.

iii. [5 points] Plotted below is the step response of the system under state feedback (not necessarily the same one as the previous part), as it moves from one y position to another:



Estimate the rise time, overshoot, settling time (2%), and steady-state error. Indicate your answers on a (hand drawn) sketch of response, with appropriate notation to understand how you did your calculation.

iv. [10 points] Suppose we wish to design a state space control law $u = -K(x - x_\text{e})$ for the original nonlinear system. Compute the linearization of the *closed loop* system $\dot{x} = f(x, -K(x - x_\text{e}))$ around a relevant equilibrium point and calculate the gains necessary to place the eigenvalues of that linearized model at the locations $\lambda_1, \lambda_2, \lambda_3$.

You may find the following calculations helpful:

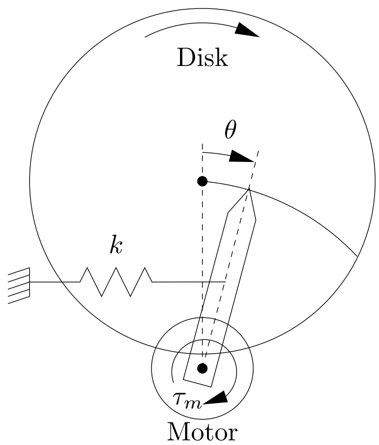
$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$\frac{\partial}{\partial \theta}(\sin\theta) = \cos\theta$$

$$\frac{\partial}{\partial \varphi}(\tan\varphi) = \frac{1}{\cos^2\varphi}$$

A.19

A.20 The figure below shows a simple mechanism for positioning a disk drive read head and the associated equations.



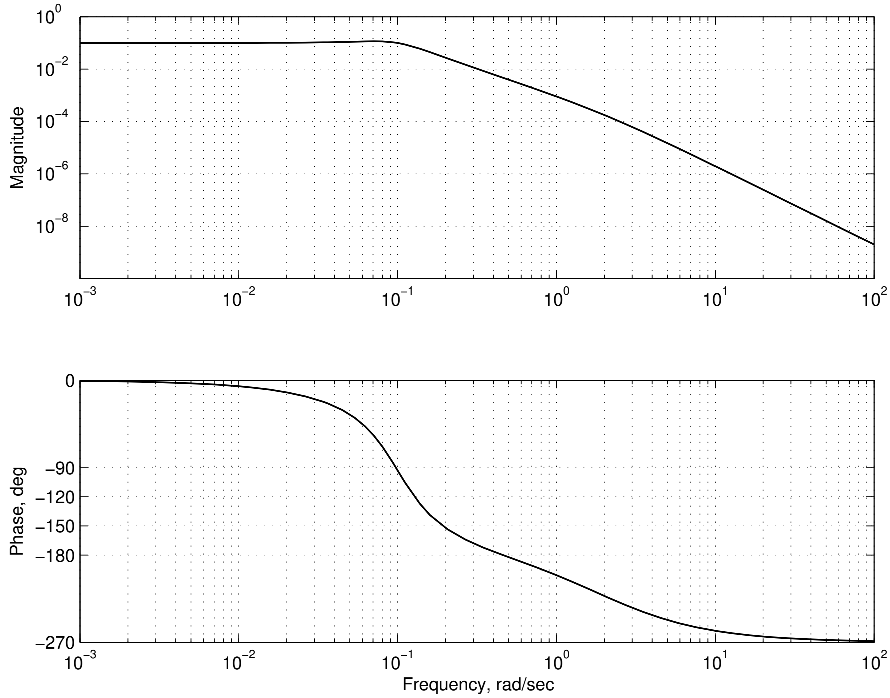
$$J\ddot{\theta} = -b\dot{\theta} - kr\sin\theta + \tau_m$$
$$\dot{\tau}_m = -a(\tau_m - u)$$

The system consists of a spring loaded arm that is driven by a small motor. The motor applies a force against the spring and pulls the head across the platter. The input to the system is the desired motor torque, u. In the diagram above, the force exerted by the spring is a nonlinear function of the head position due to the way it is attached. All constants are positive. We wish to design a controller that holds the drive head at a given location $\theta_{\rm d}$.

i. (5 points) Suppose we wish to design a linear control law around the desired drive head position $\theta = \theta_{\rm d} \gg 0$. Compute linearization of the system (in state space form) at the desired equilibrium point and verify reachability of the system.

ii. (5 points) Show that the transfer function from the input u to the drive angle θ has the form
$$H_{yu} = \frac{K}{(s^2 + 2\zeta\omega_0 s + \omega_0^2)(s+\alpha)}.$$
Find explicit expressions for K, ζ, ω_0, and α in terms of the parameters of the physical system and the desired drive head position. (Hint: you can do this without inverting a 3×3 matrix by thinking of the system as the series connection between two simpler systems.)

iii. (5 points) Assume that the system parameters are such that
$$K = 0.001 \qquad \zeta = 0.5 \qquad \omega_0 = 0.1 \qquad \alpha = 1.$$
The Bode plot for the transfer function H_{yu} is shown below:



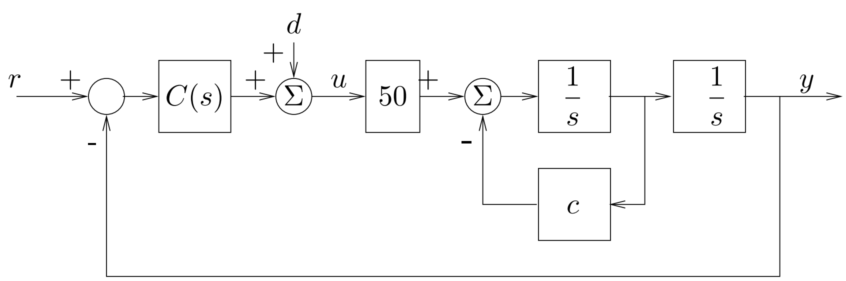
Design a proportional controller to achieve a steady-state error of 10% and determine the phase margin for the resulting controller. All answers should be accurate to within 10%.

iv. (CDS 110 only; 10 points) Design a compensator that provides tracking with less than 10% error up to 1 rad/sec and has a phase margin of at least 60°. Make sure to explain your design and verify that the specifications are satisfied, including sketching a Nyquist plot to demonstrate stability.

A.21

A.22 Consider the block diagram given below:

RMM: Update figure to match AM08 standard



i. Using block diagram algebra, redraw the block diagram so that it consists of a controller $C(s)$ and process $P(s)$ in a unity feedback system. Give an expression for $P(s)$ in terms of s, c and any other numerical factors that are required.

ii. Sketch the frequency response of the loop transfer function with $C(s) = 1$. Label the magnitudes, breakpoints, and slopes of the key features in your sketch.

iii. Design a controller for your system that has a gain crossover frequency of at least $20c$ rad/sec and gives a phase margin of at least $45°$. Demonstrate that your controller satisfies the specification by plotting the loop transfer function and labeling it appropriately.

iv. Setting $c = 1$, compute the gain, phase, and stability margins for your controller and label these on a Nyquist plot.

A.23

A.24 Consider a predator-prey system with forcing whose dynamics are given by a modified version of the Lotka-Volterra equations:

$$\begin{aligned}\frac{dx_1}{dt} &= (b+u)x_1 - ax_1x_2,\\ \frac{dx_2}{dt} &= ax_1x_2 - dx_2.\end{aligned}$$

In this equation, x_1 represents the prey population, x_2 represents the predator population and u is an input. The parameters b, a and d are all positive and we take our time unit to be in years (rather than seconds).

i. Compute the equilibrium point(s) for the unforced ($u = 0$) system and determine whether the equilibrium point with $x_1, x_2 > 0$ is stable, asymptotically stable, or unstable. (You only need to do your calculation for the linearized system.)

ii. Returning to the unforced equilibrium point for the system, show that the transfer function from the input u to the prey population $y = x_1$ has the form

$$P(s) = \frac{k(s-\alpha)}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

and give expressions for k, α, ζ, and ω_0.

iii. Design a controller that tracks a reference value r and provides less than 10% error at up to 1 rad/yr. If you need to use parameters for your design, you can take $b = 1$, $d = 2$, and $a = 0.01$.

*Instructor note:*This is a bit of a misleading question, but one that has appeared on previous exams (including ones posted on the web).

*Instructor note:*A variant on this problem is to use the predator population as the controlled variable. This is probably a better choice since the input can set the level of the prey population.

iv. The Bode integral formula

$$\int_0^\infty \log|S(i\omega)|\,d\omega = \pi \sum_{\lambda_k \in \mathrm{RHP}} \operatorname{Re}\lambda_k$$

limits the performance of a closed loop system. Describe how these limits affect your control design.

A.25

A.26 Consider a linear system with transfer function

$$P(s) = \frac{20}{(s+1)^3(s+20)}$$

i. Sketch the frequency response to a sinusoidal input $u = A\sin\omega t$. Make sure to label all important features and their appropriate values.

ii. Suppose that we want to track a reference signal r. Design a feedback compensator for the system that gives a stable closed loop system and steady-state tracking error $(y - r)$ less than 5% from 0 to 1 Hz.

iii. For the compensator you designed in question 0()ii, show how to compute the gain and phase margin using a Nyquist plot.

A.27

A.28 Consider a mechanical system with dynamics

$$m\ddot{q} + b\dot{q} + k\sin(q) = u$$

where $q \in \mathbb{R}$ is the configuration variable for the system, u is the input, and m, b, and k are all positive constants.

i. Write the state space equations for the system with output $y = \dot{q}$. Consider the linearization of the system around the point $q = \dot{q} = 0$. Is the linearized system reachable? Observable?

ii. Write the transfer function for the linearized system and sketch the frequency response to a sinusoidal input $u = A\sin\omega t$. Make sure to label all important features and their appropriate values (in terms of m, b, and k).

iii. Suppose that we want $y = \dot{q}$ to track a reference signal r. Design a feedback compensator for the system that gives steady state tracking error $(y - r)$ less than 5% from 0 to 10 Hz. Is the resulting closed loop system internally stable?[1] If not, how might you modify the performance criterion to allow internal stability?†

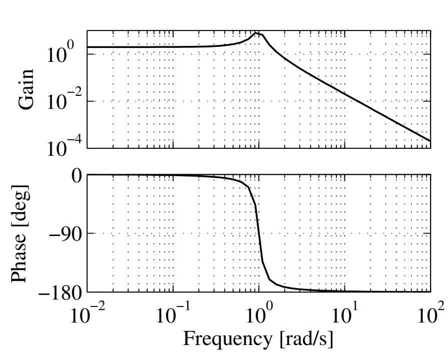
RMM: Make sure this is defined

[1]A controller is said to be internally stable if all possible transfer functions between any two points on the block diagram are stable.

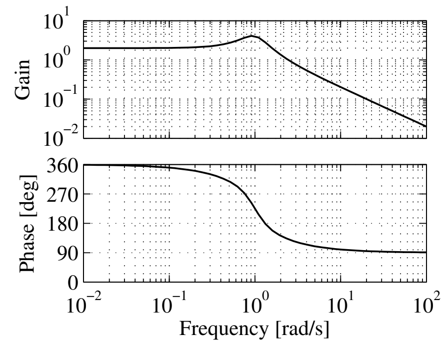
iv. For the compensator you designed in question 0()ii, show how to compute the gain and phase margin using a Nyquist plot.

A.29

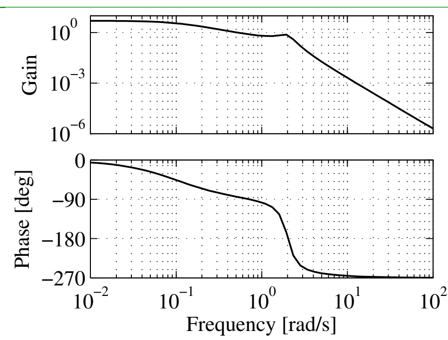
A.30 In this problem, you will answer a series questions about each of the systems illustrated in the Bode plots below. You should *summarize* your answer to this question by creating a table with the rows corresponding to each system and the columns corresponding the the answers for questions a–c. You should also include the analysis you used to obtain your answers (separate from the summary chart). For example, relate the phase and/or frequency of the gain crossover in the Bode plot to the equivalent features in the Nyquist plot and/or step response. If you use MATLAB, you should still give a description of the key features that justify your answers.



2.1



2.2



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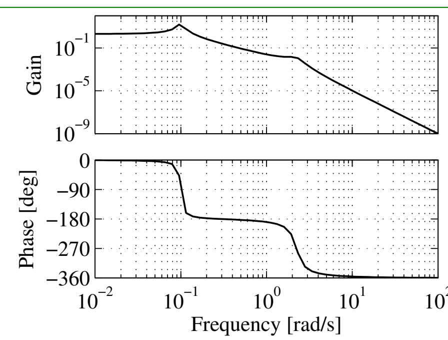
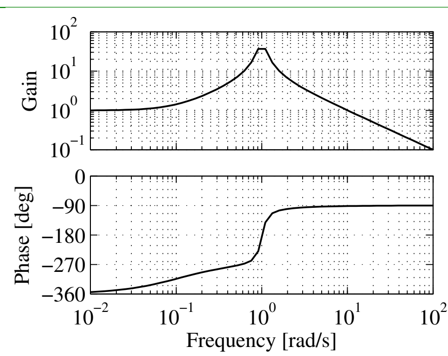
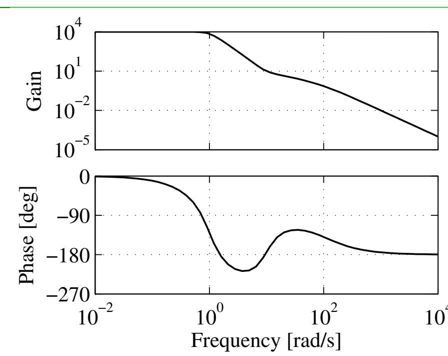


2.4



2.5

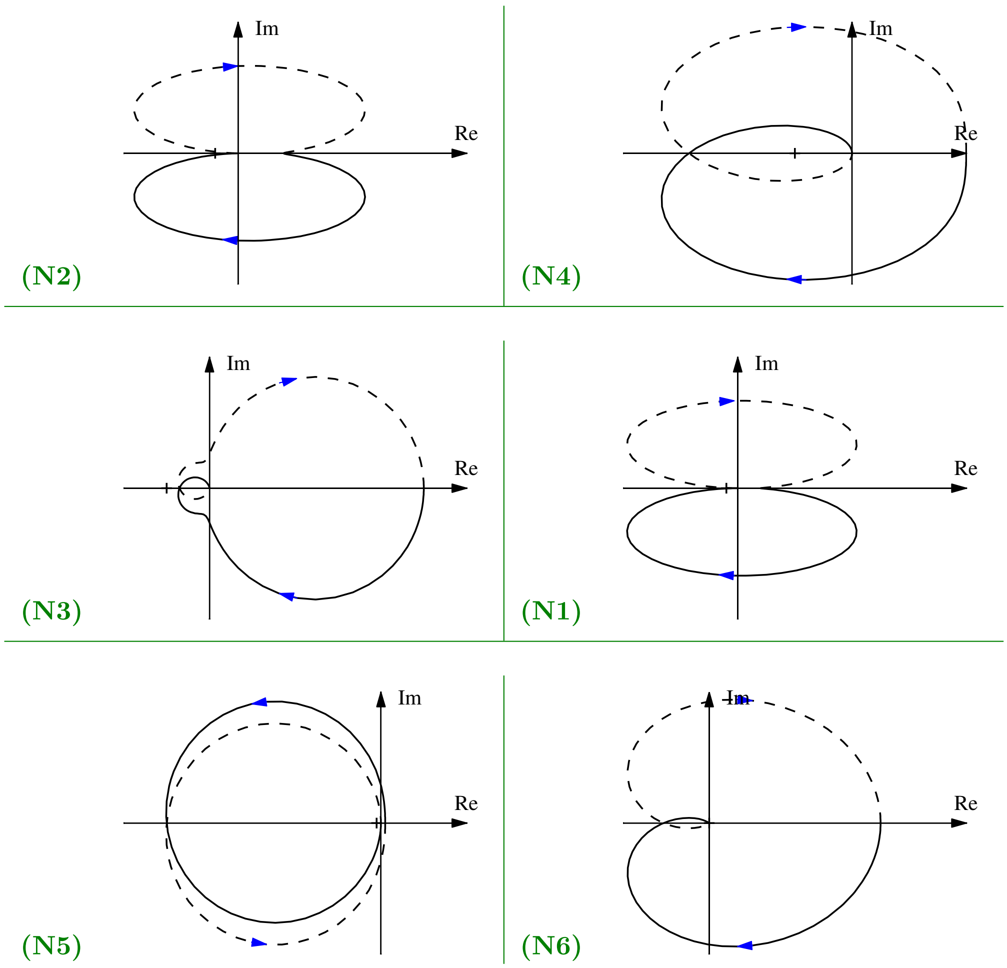


2.6

For each of the *open loop* Bode plots above, answer the following:

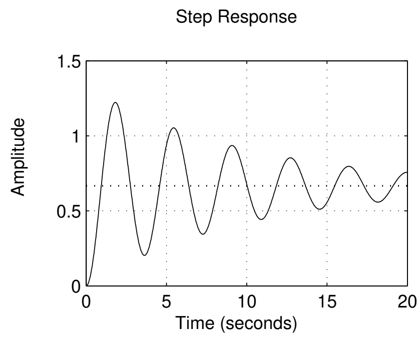
i. Determine the zero frequency gain and gain crossover frequency $\omega_{\rm gc}$ for each of the systems above. Assuming the closed loop systems are stable, compute the steady-state error to a step input for a unity gain feedback system.

ii. Which of the following Nyquist plots corresponds to the Bode plots? Assume that all open loop poles have non-positive real part (they might have zero real part).

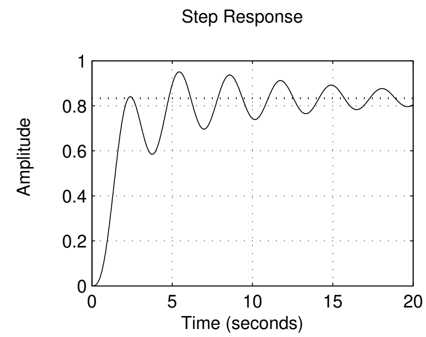


Please note that not all features of the plots above are completely discernible (a limitation of Nyquist plots), so you may need to use multiple features to sort things out. Don't forget to describe how you obtained your answer.

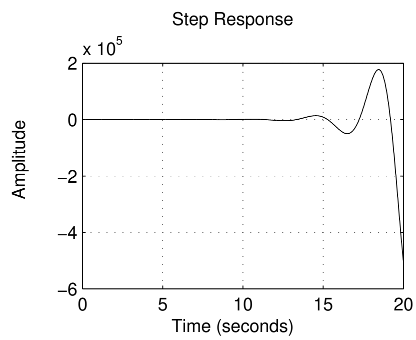
iii. Suppose that we create a closed loop system by setting $u = -y$ (the standard unity gain, negative feedback loop). Which of the following sets of closed loop, unit step responses are possible for the given open loop frequency response? Note that the scales are different on each plot.



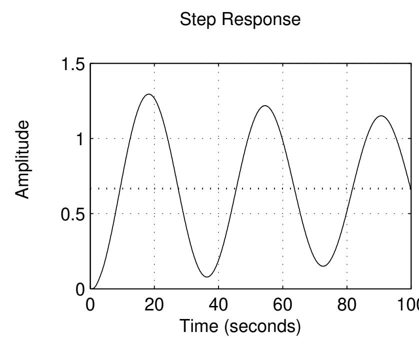
(S2)



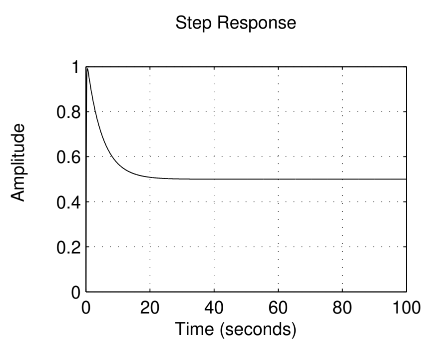
(S1)



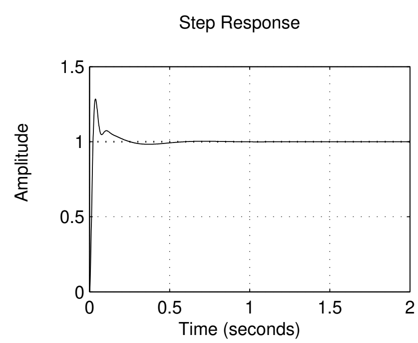
(S3)



(S4)



(S3)



(S4)

What is the maximum amount of additional phase lag (within 10%) for which each system is stable?

A.31

A.32 Consider a system with transfer function

$$P(s) = \frac{1}{(s+1)^2(a-s)} \qquad 0.02 < a < 0.2.$$

Consider the possibility of using a proportional controller of the form $C(s) = k_{\mathrm{p}}$ where $k_{\mathrm{p}} \in \mathbb{R}$.

A. Sketch the Bode plot for the loop transfer function, labeling all key features (breakpoints in gain and phase, zero frequency gain, phase at zero and infinity, etc).

B. Sketch the Nyquist plot for the system and determine whether the system is stable for $k_\mathrm{p} = 2a$ and $k_\mathrm{p} = -2a$.

C. Using your Nyquist plot or other means, determine the range of values of k_p for which the controller stabilizes the closed loop system. (Keep in mind that k_p can be positive or negative.)

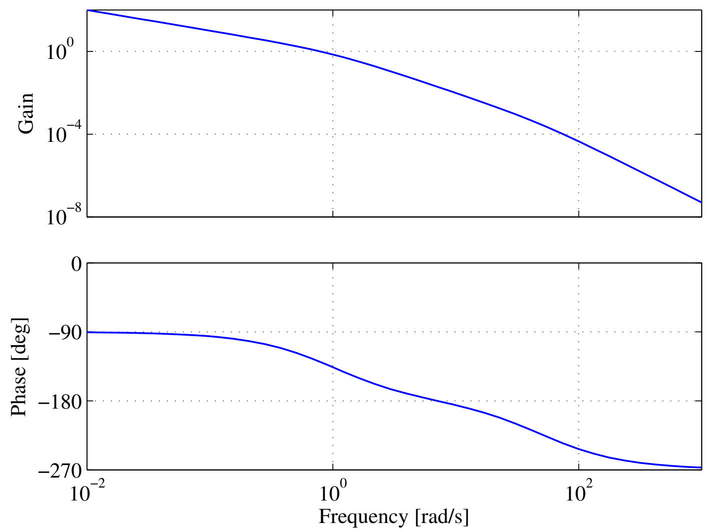
A.33

A.34 Consider a unity feedback control system with plant and controller dynamics given by

$$P(s) = \frac{1}{(s+1)(s+50)}$$

$$C(s) = \frac{k_\mathrm{i}}{s}.$$

The Bode plot for the process transfer function, P, and the loop transfer function, $L = PC$, with $k_\mathrm{i} = 50$ are shown on the right.



A. Sketch the Nyquist plot for the system and use it to estimate the largest gain, k_max, for which the closed loop system is stable. Your answer should be within 10% of the exact value.

B. For $k_\mathrm{i} = 50$, plot the Gang of Four transfer functions and identify any input/output pairs and corresponding frequency ranges for which there exists an input that can produce an output with gain larger than 10.

C. Consider the following performance specification:

- Steady-state error to a unit step input is less than 1%.
- Tracking error of less than 10% up to 1 rad/sec.
- The unit step response has an overshoot of less than 20%.

For the step response, you may assume that a phase margin of 60 degrees is sufficient to achieve the specification.

Does the controller $C(s)$ with $k_i = 50$ satisfy the performance specification? If so, compute the steady-state error, bandwidth, and phase margin for the system. If not, design a new compensator for the system, $C'(s)$, that satisfies the given specifications.

D. (CDS 110 only) For the original integral compensation $C(s)$, suppose we choose $k_i = 100$ and add a sensor in the feedback path with dynamics

$$G(s) = \frac{1}{\tau s + 1}$$

with $\tau = 0.5$. Draw the block diagram for the system with the sensor dynamics added, compute the new loop transfer function, and sketch the revised open loop Bode plot, indicating where the sensor dynamics begin to affect the frequency response (gain and phase). From the Bode plot, determine if the closed loop dynamics are still stable.

integrate these two problems **RMM**

A.35

A.36 Consider a unity feedback control system with plant and controller dynamics given by

$$P(s) = \frac{1}{(s+1)(s+25)} \qquad C(s) = \frac{k_i}{s}$$

A. Sketch the Bode plot for the loop transfer function and use it to estimate the largest gain, $k_{\max}$, for which the closed loop system is stable. Your answer should be within 10% of the exact value.

B. Consider the following performance specification:

- Steady-state error to a unit step input is less than 1%.
- Tracking error of less than 10% up to 1 rad/sec.
- Phase margin of 60 degrees.

Does the controller $C(s)$ with $k_i = 10$ satisfy the performance specification? If so, compute the steady-state error, bandwidth, and phase margin for the system. If not, design a new compensator for the system, $C'(s)$, that satisfies the given specifications.

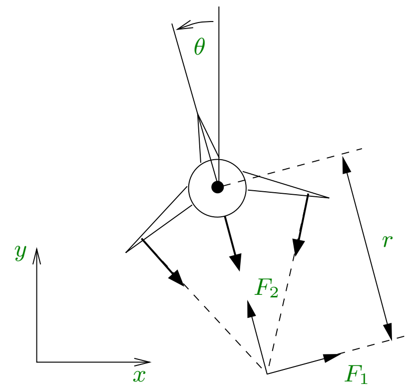
C. **(CDS 110 only)** For the original integral compensation $C(s)$, suppose we choose $k_i = 100$ and add a sensor in the feedback path with dynamics

$$G(s) = \frac{1}{\tau s + 1}$$

with $\tau = 0.5$. Draw the block diagram for the system with the sensor dynamics added, compute the new loop transfer function, and sketch the revised open loop Bode plot, indicating where the sensor dynamics begin to affect the frequency response (gain and phase). From the Bode plot, determine if the closed loop dynamics are still stable.

A.37

A.38 Consider the motion of vectored thrust aircraft, such as the Harrier "jump jet" shown below:



The Harrier is capable of vertical takeoff by redirecting its thrust downward and through the use of smaller maneuvering thrusters located on its wings. A simplified model of the Harrier is shown in on the right, where we focus on the motion of the vehicle in a vertical plane through the wings of the aircraft. We resolve the forces generated by the main downward thruster and the maneuvering thrusters as a pair of forces f_1 and f_2 acting at a distance r below the aircraft (determined by the geometry of the engines).

Let (x, y, θ) denote the position and orientation of the center of mass of aircraft. Let m be the mass of the vehicle, J the moment of inertia, g the gravitational constant, and c the damping coefficient. Then the equations of motion for the fan are given by:

RMM: $f_i \to F_i$?

$$
\begin{aligned}
m\ddot{x} &= f_1 \cos\theta - f_2 \sin\theta - c\dot{x} \\
m\ddot{y} &= f_1 \sin\theta + f_2 \cos\theta - mg - c\dot{y} \\
J\ddot{\theta} &= r f_1.
\end{aligned}
$$

A. Compute the equilibrium point for the aircraft corresponding to hovering and find the linearization of the dynamics about that equilibrium point.

B. Assume that the vertical force f_2 is fixed at its equilibrium value. Compute the transfer function $H(s)$ from f_1 to the lateral position x.

C. Find the poles and zeros of the transfer function H and sketch the Nyquist plot for the open loop system. Describe any limits on achievable performance that arise from the structure of the system (i.e., right half-plane poles or zeros, shape of the Nyquist plot) and discuss what type of controller might be needed to provide stability and good performance.

D. Suppose that we can measure the position and orientation of the center of mass of the aircraft, (x, y, θ), but we cannot directly measure any of the velocities corresponding to these quantities. Describe how to estimate the complete state from these measurements. What other information, if any, do you need in order to design your estimator?

E. Assume that the orientation of the aircraft is stabilized at $\theta = 0$ and suppose we wish to find the optimal trajectory to lower the aircraft to the ground from a height h in a fixed time T. Set this up as an optimal control problem and show how to solve it (providing whatever equations are needed to define your solution).

*Instructor note:*This exercise is mostly worked out in the text, so should only be used for a closed book exam.

A.39

A.40 Consider a nonlinear system with dynamics

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + x_1^2 \\ \dot{x}_2 &= -x_2 + u\end{aligned}$$

$$y = x_1$$

In this problem you will design a local linear controller for the system about one of its equilibrium points.

A. Assuming no input ($u = 0$), compute all of the equilibrium points for the system above and use the linearization about each equilibrium point to determine if that point is locally stable, asymptotically stable, or unstable.

B. For the equilibrium point at the origin, verify that the linearization is reachable and compute a state space feedback $u = Kx$ that places the closed loop poles at -1 and -2.

For the next part, you should use the linearization of the system dynamics around the origin.

A. Compute the transfer function from u to y for the open loop system (no state feedback) and sketch the frequency response using a Bode plot. Label all significant features in your plot with the appropriate numerical values.

B. Assuming the model above describes the process dynamics in a unity feedback system, design a frequency domain controller that satisfies the following performance specifications:

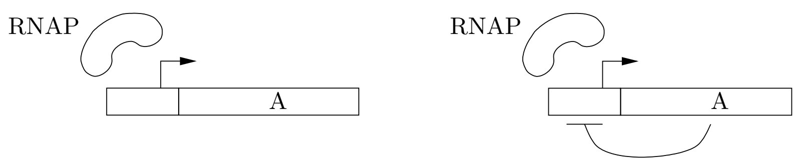
- Steady-state error to a unit step input in r is less than 1/100.
- Tracking error of less than 10% up to 10 rad/sec.
- The unit step response from r to y has an overshoot of less than 20%.

Your computations do not have to be exact, but you should include enough detail to indicate how and why the specifications are satisfied.

(CDS 210 only) Describe the maximum amount of multiplicative process uncertainty that your system can tolerate and still maintain internal stability. Your description should be in terms of a frequency bound on the magnitude of the uncertainty.

A.41

A.42 Consider a genetic circuit consisting of a single gene. We wish to study the response of the protein concentration to fluctuations in the mRNA dynamics. We consider two cases: a *constitutive promoter* (no regulation) and self-repression (negative feedback), illustrated below



(a) Open loop (b) Negative feedback

We describe the dynamics of the system using the model

$$\frac{dp}{dt} = \beta m - \gamma p, \qquad \frac{dm}{dt} = \alpha(p) - \delta m + d,$$

where d is a disturbance term that affects mRNA transcription. The function $\alpha(p)$ has the form

$$\alpha(p) = \frac{\alpha_1}{1 + kp^n} + \alpha_0$$

where $\alpha_1 = 0$ corresponds to the open loop system.

A. Assuming $d = 0$, determine the equilibrium points for each system and determine whether the system is stable, asymptotically stable or unstable at each equilibrium point. You do not need to solve for the equilibrium points analytically, but should give appropriate expressions in terms of $\alpha(p)$ and $\alpha'(p)$.

B. Consider the linear model that approximates the system around its stable equilibrium point. Is the system reachable? observable?

C. Compute the transfer function from d to p for each circuit around its stable equilibrium point and sketch the frequency response from u to p for each system, labelling all relevant features (zero frequency gain, bandwidth, etc).

D. Suppose that we connect two repressors in series, so that the output of the first circuit is the input to the next circuit. Write down the dynamics for the system and compute the transfer function from the input of the first circuit to the output of the second.

E. (CDS 110) Suppose that we wish to replace the static function $\alpha(p)$ with a controller that maintains the stable equilibrium point in the presence of a disturbance input d. Design a frequency domain controller that provides zero steady-state error and can track a reference concentration with less than 10% error up to the natural bandwidth of the system.

F. (CDS 210) Suppose that the parameter δ (which correlates with the growth rate of the cells) is uncertain and time-varying. Give the conditions on δ for robust stability of the closed loop system.

G. Suppose we want to estimate the states of the single repressor system given (noisy) measurements of its inputs and outputs. Describe how to design an estimator that provides an estimate of the mean and covariance of the states as a function of time.

A.43

A.44 Consider a nonlinear system with dynamics

$$m\ddot{q} + c\dot{q} + k\sin(q) = u$$

where $q \in \mathbb{R}$, u is the input, and m, c, and k are all positive constants. We will consider a relatively well damped system, with $c^2 - 4km > 0$.

A. Compute the equilibrium point that corresponds to $q(t) = \pi/4$ and find the linearization of the system around that equilibrium point. Using $x = (q, \dot{q})$, show that the linearized system can be written as

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -\alpha & -\beta \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

and give expressions for z, v, α, and β.

B. Determine whether the system is reachable and, if so, design a state feedback controller that $u = -Kz + k_\text{f} r$ so that the response of the state z_1 to a step input with magnitude r has overshoot less than 20% and zero steady-state error. You should express the form of your controller in terms of α and β.

C. Compute the transfer function $P(s)$ for the linearized system in (a), determine its poles and zeros, and sketch a Bode plot for the frequency response from the input v to the output $y = q - q_\text{e}$. You should do all computations in terms of α and β.

A.45

A.46 Suppose that a process that we wish to control with transfer function

$$P(s) = \frac{s-a}{s(s+b)}$$

where $a, b > 0$. Your friend from MIT suggests the following control law, based on process inversion:

$$C(s) = \frac{\omega_0}{s}\frac{1}{P(s)} = \frac{\omega_0(s+b)}{(s-a)},$$

which has a loop transfer function given by $L(s) = \omega_0/s$.

In answering the questions below, you can assume that $a < b < \omega_0$.

A. Sketch the Bode plot, Nyquist for the loop transfer function designed by your friend, labeling your plot with key frequencies, magnitudes and slopes. Using your plot, compute the gain and phase margins for the system, marking the phase margin on your Nyquist plot.

B. Analytically compute the Gang of 4 for this system, sketch their Bode plots (magnitude only), and describe any transfer functions for which the system may not provide good input/output response. (Hint: the sensitivity function for the system is

$$S = \frac{1}{1+L} = \frac{s}{s+\omega_0}.$$

If you compute the gang of 4 using this, you should get pretty simple formulas. At least one of the gang of 4 has one or more problems with it.)

C. (CDS 110 only) Sketch a loop shape for the system that satisfies the following specifications:

- Step response from the reference input to the process output has zero steady-state error and overshoot less than 20%;
- Bandwidth of H_{yr} is ω_0 or greater;
- Steady-state response to a step disturbance has zero error;
- Sensor noise at the process output is attenuated by the controller so that $|H_{un}|$ decreases at slope -1 or steeper above $10\omega_0$.

You should annotate your sketch showing the different specifications that must be satisfied in different regions of the plot. You do not have to find a specific controller that satisfies this loop shape specification.

RMM: Set up problem using xkeyval

†

Appendix B – CDS 131 (Theory) Problems

The problems in this chapter come from CDS 131, a course on linear systems theory taught at Caltech that builds on the material in Åström and Murray [2]. These problems are more theory-oriented than the problems in the main text.

B.1 Consider a linear system $\dot{x} = Ax$ with the matrix A given by

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

i. Find the stable, unstable, and center subspaces E^s, E^u, and E^c for $\lambda_1 > 0$ and $\lambda_2 < 0$.

ii. Qualitatively sketch the phase portrait of the system:

   A. For $\lambda_1, \lambda_2 > 0$

   B. For $\lambda_1, \lambda_2 < 0$

   C. For $\lambda_1 > 0$ and $\lambda_2 < 0$

iii. Compute the matrix exponential, e^{At} for the system for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

iv. From part (a), verify that $\mathbb{R}^2 = E^s \oplus E^u \oplus E^c$, where $\oplus$ represents the direct-sum of the vector spaces. Also verify that these subspaces are invariant under e^{At}.

v. Give an example of a *non-hyperbolic* (Definition 2.2 FBS2s) linear system ($\dot{x} = Ax + Bu$, $y = Cx$). For all bounded inputs to your system, is the output bounded? Prove or give a counter example.

B.2 Consider a n-dimensional linear discrete time system with p inputs and q outputs

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k],$$

where $x[0] = x_0$ is the unknown initial condition, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{q\times n}$. Prove that the system is observable if and only if the $n \times n$ observability Gramian for the discrete-time system has full rank:

$$W_\mathrm{o}[n] = \sum_{i=0}^{n-1} (A^T)^i C^T C A^i.$$

Describe how the observability condition is different for a discrete-time system from the continuous-time dynamics studied in class (and the notes).

B.3 It can be shown that a linear differential equation of the form

$$\frac{dx}{dt} = Ax$$

is asymptotically stable if and only if there exists a positive definite, symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that $PA + A^T P$ is a *negative* definite, symmetric matrix. (One way think about this is to note that for a diagonalizable matrix A the matrix P can be taken as the identity matrix in transformed coordinates.)

Use this fact to show the following results:

i. Suppose that we solve a linear quadratic regulator problem of that minimizes the cost function

$$J = \int_0^\infty \left(x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)\right)\, d\tau.$$

where $Q > 0$ and $R > 0$. Use the stability condition above to show that the resulting control law $u = -R^{-1}B^TP$ is asymptotically stable if $P > 0$ is the solution to the algebraic Riccati equation.

ii. Suppose that we only care about the output of the system $y = Cx$, so that we attempt to minimize

$$J = \int_0^\infty \left(y^T(\tau)y(\tau) + u^T(\tau)Ru(\tau)\right)\, d\tau. \tag{SB.1}$$

Show that if the system is not observable then we are not guaranteed that the solution to the Riccati equation is positive definite and hence the optimal controller may not be stabilizing.

iii. Give an example of a controllable linear system for which an optimal controller using the cost in the form given in equation (SB.1) is not stabilizing. Does there exist a stabilizing optimal compensator? Is it unique?

Instructor note:

- P3 (a) - 5 mins - There is a typo in the statement of part (a). Also, does this problem need conditions on Q and R that Q is p.s.d and R is p.d.? Another possible typo in this problem is in the equation for u, I think it should be $-R^-1B^TP$. The problem itself, if I understood correctly, is asking to prove the standard LQR proof. If that's the case, then OBC Section 2.4 gives the proof sketch.
- P3 (b) - 5 mins
- P3 (c) - 10 mins - In this problem it says "only output cost". Does this mean the cost is only on the outputs and not on inputs as well (i.e. R = 0?), it would probably be helpful clarify this.

B.4kalman,xferfcn Consider a linear input/output system with dynamics

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} x.$$

Unless otherwise specified, your answers below should include all possible values of γ.

i. What are the stable and unstable subspaces for the open loop ($u = 0$) system?

ii. For what values of γ is the system reachable? stabilizable?

iii. For appropriate values of γ, find a state feedback such that the eigenvalues of the system can be placed at $\{-1, -2, -3\}$.

iv. For what values of γ is the system observable? detectable?

v. Find an input $u(t)$ that steers the system from $(0, 0, 1)$ to $(0, 1, 0)$ in time $T = 1$. You may leave your answer in terms of matrices, integrals and inverses, but you must show that the equations are well-formed (e.g,, if you invert a matrix, you must provide that it is invertible).

*Instructor note:*Fall 2019 TA comments:

- P2 (a) - 5 mins. This part is pretty straightforward and similar to HW 2 P6.
- P2 (b) and (c) - From the controllability matrix, I am getting that the system is not reachable but it is stabilizable. I wanted to point this out just to make sure I am doing it correctly. This makes the next part (c) a bit tricky because it asks to place eigenvalues for the system (but the system is not controllable). But since the uncontrollable mode is not being placed (it is already stable, lamda = -1) , we can still find a state-feedback controller. Timing wise, I had to spend a little above 20 mins in total for both parts to make sure I was calculating everything correctly.
- P2 (d) and (e) - I am getting that the system is observable for all values of gamma =/= 0. It took me around 10 mins to complete both parts. Also for these parts, I just wanted to make sure if the stabilizable and detectable concepts were covered in class. If not, we might need to give definitions with the problem.
- P2 (f) - 10 mins in total but I didn't compute everything to its final form. The problem itself should be straightforward as it is similar to HW 3 problem 1.
- P2 (g) - 10 mins - It took me a while to figure out what "Kalman decomposition subspaces" mean.

B.5 Consider a discrete time system having dynamics

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k],$$

where $x[k] \in \mathbb{R}^n$ is the state of the system at time $k \in \mathbb{Z}$, $u[k] \in \mathbb{R}$ is the (scalar) input for the system, $y[k] \in \mathbb{R}$ is the (scalar) output for the system and A, B, and C are constant matrices of the appropriate size. We use the notation $x[k] = x(kh)$ to represent the state of the system at discrete time k where $h \in \mathbb{R}$ is the sampling time (and similarly for $u[k]$ and $y[k]$).

Let $\mathcal{T} = [0, h, \ldots, Nh]$ represent a discrete time range, with $N \in \mathbb{Z}$.

i. Considered as a dynamical system over $\mathcal{T}$, what is the input space $\mathcal{U}$, output space $\mathcal{Y}$, and state space Σ corresponding to the dynamics above? Show that each of these spaces is a linear space by verifying the required properties (you may assume that $\mathbb{R}^p$ is a linear space for appropriate p).

ii. What is the state transition function $s(t_1, t_0, x_0, u(\,\cdot\,))$? Show that this function satisfies the state transition axiom and the semi-group axiom.

iii. What is the readout function $r(t, x, u)$? Show that the input/output system is a linear input/output dynamical system over $\mathcal{T}$.

iv. What is the zero-input response for the system? What is the zero-state response for the system?

B.6 Consider a second order mechanical system with transfer function

$$\widehat{G}(s) = \frac{1}{s^2 + 2\omega_n \zeta s + \omega_n^2}$$

(ω_n is the natural frequency of the system and ζ is the damping ratio). Setting $\omega_n = 1$, plot the ∞-norm as a function of the damping ratio $\zeta > 0$. (You may use a computer to to this, but if you do then make sure to turn in a copy of your code with your solutions.)

B.7 Consider the double integrator system $\ddot{y} = u$. Use the controllability Gramian to compute an input that steers the system for the origin to a state x_{f} in time T. What happens as $T \to 0$ and as $T \to \infty$?

B.8 Consider a linear system that is not controllable but whose dynamics in the unreachable subspace are asymptotically stable. Show that it is possible to steer the system to any point in the reachable subspace of the system dynamics.

B.9 Show that for a linear time-invariant system, the following notions of controllability are equivalent:

i. Reachability to the origin ($x_0 \rightsquigarrow 0$).

ii. Reachability from the origin ($0 \rightsquigarrow x_{\mathrm{f}}$).

iii. Small-time local controllability ($x_0 \rightsquigarrow B(x_0, \epsilon)$).

B.10 Consider a system with the state x and z described by the equations

$$\frac{dx}{dt} = Ax + Bu, \qquad \frac{dz}{dt} = Az + Bu.$$

If $x(0) = z(0)$ it follows that $x(t) = z(t)$ for all t regardless of the input that is applied. Assuming that the pair (A, B) is controllable, compute the rank of the reachability Grammian W_{c} and use this to determine the reachable space of the system starting from the origin and its dimension.

B.11 Show that the set of unobservable states for a linear system with dynamics matrix A and output matrix C is an A-invariant subspace and that it is equal to the largest A-invariant subspace annihilated by C.

B.12 Consider the following LTI system with state $x \in \mathbb{R}^3$:

$$\frac{dx}{dt} = \begin{pmatrix} -1 & \gamma & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u \qquad y = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} x$$

In answering the questions below, make sure to keep track of how the answer depends on $\gamma \in \mathbb{R}$.

i. Suppose that we set $u = 0$. Qualitatively describe the set of all possible outputs for the system for different initial conditions (and values of γ).

ii. Describe the set of states that are reachable from the origin in time T.

iii. Consider a state $x_\mathrm{f} = (0, 1, 1)$. Describe an input that will steer the system from the origin to this state in time T. (Note: you don't have to compute the input explicitly, but you should provide a description that is a computable formula.)

iv. Suppose $\gamma = 0$. Design a state feedback compensator that stabilizes the origin of the system.

B.13 This problem concerns robust stability of the unity-feedback system. Suppose that P and C are nominal transfer functions for which the feedback system is internally stable. Instead of allowing perturbations in just P, this problem allows perturbations in C too. Suppose that P may be perturbed to

$$P(1 + \Delta_1 W_1)$$

and C may be perturbed to

$$C/(1 + \Delta_2 W_2)$$

The transfer functions W_1 and W_2 are fixed, while Δ_1 and Δ_2 are variable transfer functions having ∞-norms no greater than 1. Making appropriate additional assumptions, find a sufficient condition, depending only on the four functions P, C, W_1, W_2, for robust stability. Prove sufficiency. (A weak sufficient condition is the goal; for example, the condition $W_1 = W_2 = 0$ would be too strong.)

Appendix C – CDS Qualifying Exam Problems

The problems in this chapter come from qualifying exams given in the Control and Dynamical Systems PhD program at Caltech. Students taking this class have typically taken the course that uses Åström and Murray [2] as a textbook. Most of the questions try to cut across different aspects of the material covered across the text.

ME Controls Candidacy Exam, April 2002

C.1 Consider the following transfer function

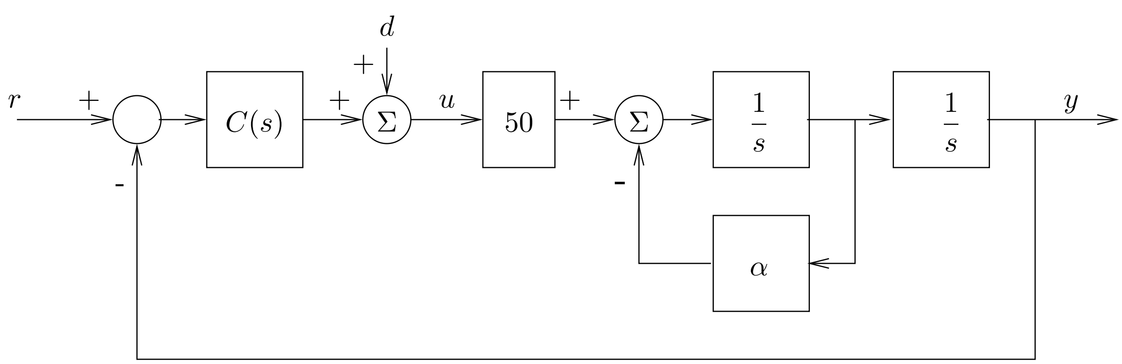
$$G(s) = \frac{s - z}{s(s + p)} \qquad \text{where } p = 10z > 0.$$

i. Sketch the Bode plot for the system, labelling all breakpoints and slopes.

ii. Sketch the Nyquist plot and label the $\omega = 0$ point as well as the $\omega > 0$ and $\omega < 0$ branches and other important features.

iii. Sketch the pole/zero locations for a unity gain, closed loop plant having $kG(s)$ as the loop transfer function, as a function of $k > 0$.

C.2 Consider the block diagram given below:



i. Compute the plant transfer function $G(s)$ that takes u to y.

ii. Plot the frequency response of the open loop system with $D(s) = 1$

iii. Design a compensator that satisfies the following specifications:

- Steady-state error to a unit ramp input is less than 1/150.
- The unit step response has an overshoot of less than 25%.
- The bandwidth for the compensated system is no less than that of the uncompensated system.

Your computations do not have to be exact, but you should include enough detail to indicate how and why the specifications are satisfied.

C.3 Describe the difference between robust stability, nominal performance, and robust performance of a linear control system. Give conditions for robust stability in the presence of (weighted) additive uncertainty.

ME Controls Candidacy Exam, April 2003

C.4 Consider a mechanical system with dynamics

$$m\ddot{q} + b\dot{q} + kq = u$$

where $q \in \mathbb{R}$ is the configuration variable for the system, u is the input, and m, b, and k are all positive constants.

i. Write the state space equations for the system with output $y = \dot{q}$. Is the system controllable? Observable?

ii. Design a full-order observer for the system and discuss how the observer gains should be chosen.

iii. Write the transfer function for the system and sketch the frequency response to a sinusoidal input $u = A \sin \omega t$. Make sure to label all important features and their appropriate values (in terms of m, b, and k).

iv. Suppose that we want $y = \dot{q}$ to track a reference signal r. Design a feedback compensator for the system that gives steady state tracking error $(y - r)$ less than 5% from 0 to 10 Hz. Is your controller internally stable? If not, how might you modify the performance criterion to allow internal stability?

v. For the compensator you designed in question 0()ii, show how to compute the gain and phase margin using a Nyquist plot.

vi. Show how to check whether a stabilizing compensator for this system provides robust stability with respect to a 10% variation in b.

ME Controls Candidacy Exam, October 2003

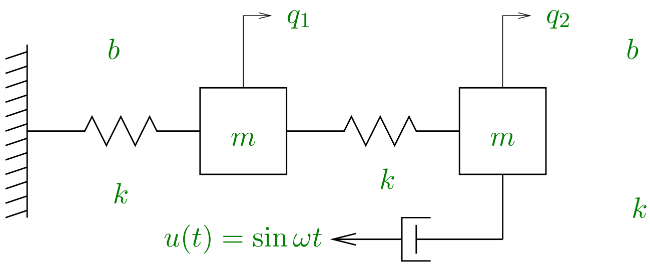
C.5 Controllability:

i. For the system $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$, define controllability on the interval $t \in [0, T]$.

ii. Give 3 tests for controllability for the linear system $\dot{x} = Ax + Bu$.

iii. Prove that any two of your tests are equivalent. (Hint: if you get stuck, you might want to leave this question until the end of the exam.)

C.6 Consider the spring-mass system shown below, where the control acts on the second mass through a dashpot. For simplicity, assume both masses are unity.



i. Is this system controllable?

ii. A control designer tries to simplify the system by choosing the input u as

$$\dot{u} = \frac{k}{b}(q_2 - q_1 - w) + \dot{q}_2$$

(Hint: this should give $\ddot{q}_2 = -kw$.)
Derive the transfer function from the new input w to q_1.

iii. Can one stabilize the system with feedback $w = \alpha q_1 + \beta \dot{q}_1$? Why or why not?

CDS 110, January 2005

C.7 Consider a mechanical system with dynamics

$$m\ddot{q} + b\dot{q} + k\sin(q) = u$$

where $q \in \mathbb{R}$ is the configuration variable for the system, u is the input, and m, b, and k are all positive constants.

i. Write the state space equations for the system with output $y = \dot{q}$. Consider the linearization of the system around the point $q = \dot{q} = 0$. Is the linearized system reachable? Observable?

ii. Write the transfer function for the linearized system and sketch the frequency response to a sinusoidal input $u = A \sin \omega t$. Make sure to label all important features and their appropriate values (in terms of m, b, and k).

iii. Suppose that we want $y = \dot{q}$ to track a reference signal r. Design a feedback compensator for the system that gives steady state tracking error $(y - r)$ less than 5% from 0 to 10 Hz. Is the resulting closed loop system internally stable?[1] If not, how might you modify the performance criterion to allow internal stability?

iv. For the compensator you designed in question 0()ii, show how to compute the gain and phase margin using a Nyquist plot.

[1] A controller is said to be internally stable if all possible transfer functions between any two points on the block diagram are stable.

CDS 110, January 2006

C.8 Consider a linear system with transfer function

$$P(s) = \frac{1}{(s+1)^4}$$

i. Sketch the frequency response to a sinusoidal input $u = A \sin \omega t$. Make sure to label all important features and their appropriate values.

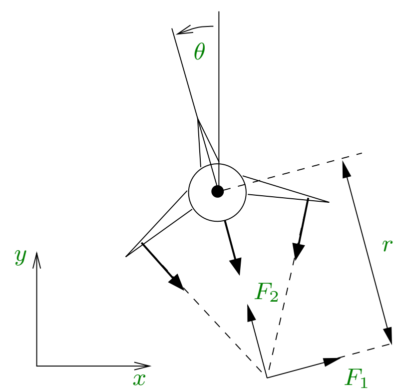
ii. Suppose that we want to track a reference signal r. Design a feedback compensator for the system that gives a stable closed loop system and steady-state tracking error $(y - r)$ less than 5% from 0 to 1 Hz.

iii. For the compensator you designed in question 0()ii, show how to compute the gain and phase margin using a Nyquist plot.

iv. Find a state space system whose transfer function is equal to $P(s)$ and design a state feedback compensator such that the eigenvalues of the closed loop system are all at -10. Does this compensator satisfy the performance condition in part 0()ii?

ME Controls Candidacy Exam, October 2006

C.9 Consider the motion of vectored thrust aircraft, such as the Harrier "jump jet" shown below:



The Harrier is capable of vertical takeoff by redirecting its thrust downward and through the use of smaller maneuvering thrusters located on its wings. A simplified model of the Harrier is shown in on the right, where we focus on the motion of the vehicle in a vertical plane through the wings of the aircraft. We resolve the forces generated by the main downward thruster and the maneuvering thrusters as a pair of forces f_1 and f_2 acting at a distance r below the aircraft (determined by the geometry of the engines). Let (x, y, θ) denote the position and orientation of the center of mass of aircraft. Let m be the mass of the vehicle, J the moment of inertia,

g the gravitational constant, and c the damping coefficient. Then the equations of motion for the fan are given by:

$$m\ddot{x} = f_1 \cos\theta - f_2 \sin\theta - c\dot{x}$$
$$m\ddot{y} = f_1 \sin\theta + f_2 \cos\theta - mg - c\dot{y}$$
$$J\ddot{\theta} = r f_1.$$

i. Compute the equilibrium point for the aircraft corresponding to hovering and find the linearization of the dynamics about that equilibrium point.

ii. Assume that the vertical force f_2 is fixed at its equilibrium value. Compute the transfer function $H(s)$ from f_1 to the lateral position x.

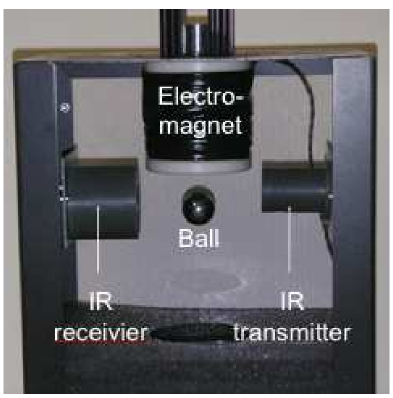
iii. Find the poles and zeros of the transfer function H and sketch the Nyquist plot for the open loop system. Describe any limits on achievable performance that arise from the structure of the system (i.e., right half-plane poles or zeros, shape of the Nyquist plot) and discuss what type of controller might be needed to provide stability and good performance.

iv. Suppose that we can measure the position and orientation of the center of mass of the aircraft, (x, y, θ), but we cannot directly measure any of the velocities corresponding to these quantities. Describe how to estimate the complete state from these measurements. What other information, if any, do you need in order to design your estimator?

v. Assume that the orientation of the aircraft is stabilized at $\theta = 0$ and suppose we wish to find the optimal trajectory to lower the aircraft to the ground from a height h in a fixed time T. Set this up as an optimal control problem and show how to solve it (providing whatever equations are needed to define your solution).

CDS 110a, March 2007

C.10 Consider a magnetic suspension system



$$
\begin{aligned}
m\ddot{z} &= \frac{k_m i_m^2}{z^2} - mg \\
v_{\mathrm{ir}} &= k_T z + v_o
\end{aligned}
\tag{SC.1}
$$

where $z < 0$ is the displacement of the ball (with $z = 0$ corresponding to the location of the electromagnet), i_m is the current applied to the electromagnet, and v_{ir} is the voltage from an infrared sensor that measure the gap between the ball and the magnet. The parameters k_m, k_T, m, and g are all positive constants.

i. Compute the equilibrium point(s) for the system and show that the linearized dynamics about an appropriate equilibrium point can be described by the transfer function

$$P_{yu}(s) = \frac{k}{s^2 - r^2} \qquad k, r > 0.$$

ii. Suppose that we wish to hold the ball at a reference position r. Show that the system cannot be made asymptotically stable using proportional gain on the error $r - y$.

iii. Show that the system can be stabilized using a compensator with transfer function

$$C(s) = k\frac{s+a}{s+b}$$

and sketch the Nyquist plot that demonstrates the stability of the system.

iv. The Bode integral formula

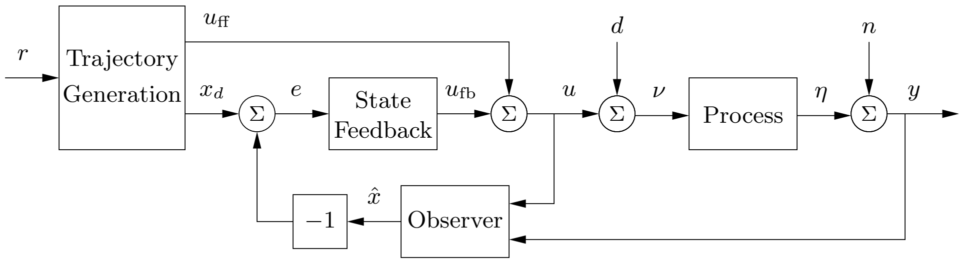
$$\int_0^\infty \log|S(i\omega)|\,d\omega = \pi \sum_{\lambda_k \in \mathrm{RHP}} \operatorname{Re}\lambda_k$$

limits the performance of the closed loop system. Describe how these limits affect your control design.

v. Discuss the robustness of your controller to unmodeled dynamics in the amplifier that commands the current to the electromagnet.

ME Controls Candidacy Exam, October 2007

C.11 Consider the trajectory generation and tracking problem show in the figure below:



Suppose that the process is a mechanical system with transfer function

$$P(s) = \frac{s-a}{s(s+b)} \qquad 0 < a < b.$$

i. (Process dynamics) Compute a state space realization (A, B, C, D) for the process such that $P(s) = C(sI - A)^{-1}B + D$. Is your realization unique?

ii. (Trajectory generation) Compute an open loop input u_{d} that steers the system between output $y = 0$ at time $t = 0$ and output $y = y_{\mathrm{f}}$ at $t = T$. Your input should be chosen such that the system remains at $y = y_{\mathrm{f}}$ for $t > T$.

iii. (State feedback) Assume the full state is available for measurement. Design a state space control law $u_{\mathrm{fb}} = -Ke$ that places the closed loop eigenvalues at the roots of $\lambda_{\mathrm{c}}(s) = s^2 + 2\zeta_{\mathrm{c}}\omega_{\mathrm{c}}s + \omega_{\mathrm{c}}^2 = 0$.

iv. (Observer) Suppose now that only the output y is available for measurement. Design an observer that minimizes the estimation error for d a zero mean, Gaussian random process with covariance R_{d} and n a zero mean, Gaussian random process with covariance R_n (d and n independent).

v. (Robustness) Suppose that your observer has eigenvalues given by the roots of $\lambda_{\mathrm{o}}(s) = s^2+2\zeta_{\mathrm{o}}\omega_{\mathrm{o}}s+\omega_{\mathrm{o}}^2 = 0$ with $\omega_{\mathrm{o}} \gg b$. Compute the loop transfer function for the complete system and find the gain, phase, and stability margins for the system.

CDS 110a, January 2008

C.12 Consider a predator-prey system with forcing whose dynamics are given by a modified version of the Lotka-Volterra equations:

$$\begin{aligned}\frac{dx_1}{dt} &= (b+u)x_1 - ax_1x_2,\\ \frac{dx_2}{dt} &= ax_1x_2 - dx_2.\end{aligned}$$

In this equation, x_1 represents the prey population, x_2 represents the predator population and u is an input. The parameters b, a, and d are all positive and we take our time unit to be in years (rather than seconds).

i. Compute the equilibrium point(s) for the unforced ($u = 0$) system and determine whether the equilibrium point with $x_1, x_2 > 0$ is stable, asymptotically stable, or unstable. (You only need to do your calculation for the linearized system.)

ii. Give conditions under which we can find an equilibrium point $(x_\mathrm{e}, u_\mathrm{e})$ that corresponds to a prey population of $r > 0$ and determine whether or not we can asymptotically stabilize the resulting equilibrium point using state feedback.

iii. Returning to the unforced equilibrium point for the system, show that the transfer function from the input u to the prey population $y = x_1$ has the form

$$P(s) = \frac{k(s - \alpha)}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

and give expressions for k, α, ζ, and ω_0.

iv. Design a controller that tracks a reference value r and provides less than 10% error at up to 1 rad/yr. If you need to use parameters for your design, you can take $b = 1$, $d = 2$, and $a = 0.01$.

v. The Bode integral formula

$$\int_0^\infty \log |S(i\omega)|\, d\omega = \pi \sum_{\lambda_k \in \mathrm{RHP}} \operatorname{Re} \lambda_k$$

limits the performance of a closed loop system. Describe how these limits affect your control design.

CDS 110a, April 2008

C.13 Consider a linear process with transfer function

$$P(s) = \frac{s + a}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

i. Find a state space linear system whose steady-state input/output response has transfer function $H_{yu}(s) = P(s)$.

ii. Assume that the states of your model for part 1 are available. Design a state space controller that places the closed loop eigenvalues of the system at $\lambda_1 = -1$ and $\lambda_2 = -2$. Is it possible to do this for all values of a, ζ, and ω_0?

iii. Suppose $a > \omega_0 > 0$ and $0 < \zeta < 0.5$, and consider a PI compensator of the form

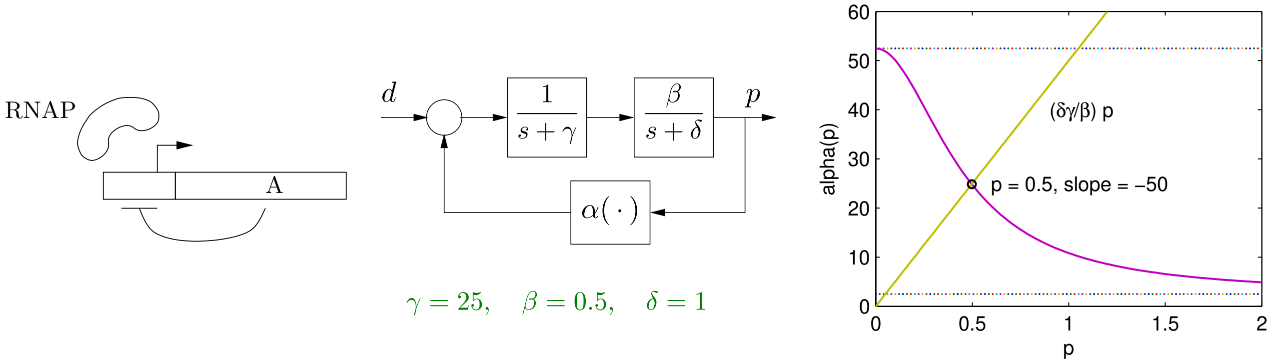
$$C(s) = \frac{k_\mathrm{p} s + k_\mathrm{i}}{s}.$$

Sketch the Bode and Nyquist plots for the loop transfer function and use these to determine if the bandwidth of the closed loop system can be set arbitrarily high using this compensator. What is the maximum phase margin your controller can achieve?

iv. Suppose now that $a < 0$. Sketch the design of a controller that gives step response with less than 1% steady-state error and that provides disturbance attentuation of a factor of 10 up to frequency ω_0.

CDS 110a/210, January 2009

C.14 Consider a genetic circuit consisting of a single gene under negative feedback (self-repression), illustrated below



$\gamma = 25, \quad \beta = 0.5, \quad \delta = 1$

(a) Self-repression (b) Block diagram (c) Feedback nonlinearity

We describe the dynamics of the system using the model

$$\frac{dm}{dt} = \alpha(p) - \gamma m + d, \qquad \frac{dp}{dt} = \beta m - \delta p, \qquad \alpha(p) = \frac{\alpha_1}{1 + kp^2} + \alpha_0,$$

where d is a disturbance term that affects mRNA transcription and $\alpha(p)$ is plotted in (c).

i. Assuming $d = 0$, determine the equilibrium points for each system and determine whether the system is stable, asymptotically stable or unstable at each equilibrium point. You do not need to solve for the equilibrium points analytically, but should give appropriate expressions in terms of $\alpha(p)$ and $\alpha'(p)$.

ii. Consider the linearized state space model that approximates the system around its stable equilibrium point. Is the approximate model reachable? observable? What can you say about the original system based on these two properties?

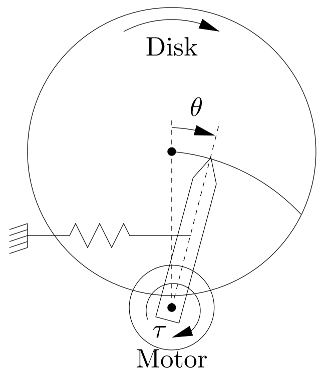
iii. (CDS 110) Compute the transfer function from d to p for the closed loop circuit around its stable equilibrium point and sketch the frequency response from d to p, labelling all relevant features (zero frequency gain, bandwidth, etc).

iv. Suppose that we wish to replace the static function $\alpha(p)$ with a controller that perfectly maintains the stable equilibrium point in the presence of a disturbance input d. Design a frequency domain controller that provides zero steady-state error. CDS 110: Use a Nyquist plot to compute the gain and phase margin for your design. CDS 210: Does your controller give internal stability?

v. (CDS 210) Suppose that the parameter δ is uncertain and time-varying. Give the conditions on δ for robust stability of the closed loop system using the controller you designed in problem 0()iv. Describe the amount of uncertainty that is allowable at low frequency, in the neighborhood of the crossover region and at high frequency.

CDS 110a/210, April 2009

C.15 The figure below shows a simple mechanism for positioning a disk drive read head and the associated equations.



$$J\ddot{\theta} = -b\dot{\theta} - kr\sin\theta + T_{\mathrm{m}}$$
$$\dot{T}_{\mathrm{m}} = -a(T_{\mathrm{m}} - u)$$

The system consists of a spring loaded arm that is driven by a small motor. The motor applies a force against the spring and pulls the head across the platter. The input to the system is the desired motor torque, u. In the diagram above, the force exerted by the spring is a nonlinear function of the head position due to the way it is attached. The system parameters are given by

$$k = \sqrt{2}, \qquad J = 100, \qquad b = 10, \qquad r = 1, \qquad a = 1.$$

The following formulas may also be useful:

$$\sin\tfrac{\pi}{4} = \cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}, \qquad (s - \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i)(s - \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i) = s^2 + s + 1.$$

i. Compute the equilibrium points for the system and determine the linearization about an arbitrary equilibrium point $(x_{\mathrm{e}}, u_{\mathrm{e}})$. Is the system stable for all equilibrium points?

ii. Design a state space controller for the system that stabilizes the system about $\theta_{\mathrm{e}} = 45°$ and sets the closed loop eigenvalues to $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and $\lambda_3 = -a$. (You don't need to work through the detailed algebra; just show how the calculation can be done and explain why these pole locations will be achievable.)

iii. Sketch the step response of the system for a change in commanded position near this equilibrium position.

iv. Compute the transfer function H_{yu} for the system around the equilibrium point and sketch the frequency response of the open loop system.

v. Design a compensator that provides tracking with less than 10% error up to 1 rad/sec and has a phase margin of at least 60°.

vi. Suppose that the controller has a time-delay of $0 < \tau_\mathrm{d} \leq 0.01$ sec. Determine whether your frequency domain controller is robust with respect to this amount of time delay. If not, describe how to improve the controller design to be robust with respect to this delay.

CDS 110a Qualifying Exam, January 2010

C.16 The following "delayed-oscillator" dynamics description is common in systems where the restoring effect has a time-delay, such as in El Niño, cavity flow oscillations, chatter in machining processes, etc.; here we also include a forcing term u:

$$\dot{T} = aT - bT(t-\tau) + u \tag{SC.2}$$

Choose the time delay $\tau = 2$ (we can do this without loss of generality by scaling the time units). A finite-dimensional system can be obtained by substituting a Padé approximation for the time delay, so writing

$$z = T(t-\tau) \qquad \text{where} \qquad \frac{z(s)}{T(s)} = \frac{1 - s\tau/2}{1 + s\tau/2}$$

Use the Padé approximation for the following:

i. Write the system (SC.2) in state space form where the Padé approximation has been used for the time-delay. What are the equilibrium point(s) of the system? Choose $b = 1$ for simplicity and $0 < a \ll 1$; is the system stable?

ii. Is the system observable if we measure T? If we measure $z = T(t-\tau)$? Is the system controllable from input u?

iii. With the same numerical values as before, compute the transfer function and sketch the Bode plot from input u to output T. How does this change if we measure z?

iv. Design a controller such that the closed loop system is stable. Can you do this using only a measurement of z and not T?

C.17 Systems such as the delayed oscillator motivate discrete-time state space representations. Define reachability and observability for the discrete-time system

$$x[k+1] = Ax[k] + Bu[k] \qquad y[k] = Cx[k]$$

and derive necessary and sufficient conditions for observability.

CDS 110a Qualifying Exam, April 2010

C.18 Aeroelastic flutter can result if the structural deflection (e.g. of an airplane wing, a stop sign, the Tacoma Narrows bridge, etc.) causes a change in aerodynamic forces that in turn influences the structure. In the case of an airplane wing, the aileron can be used to control the motion, and the simplest representative equation for this type of system is of the form:

$$m\ddot{q} + kq - c_1\dot{q} + c_2\dot{q}^3 = u \tag{SC.3}$$

where q is the deflection, u the control, and the parameters m, k, $c_1 < \sqrt{mk}$, and c_2 are all positive.

i. What are the equilibrium point(s) of the system? Are they stable? Sketch the phase portrait for this system.

ii. Linearize the system about the origin, and derive the transfer function between the input u and the deflection q.

iii. For constant gain feedback $u = \alpha q$, sketch the Nyquist and Bode plots for this system. Can constant gain feedback give a stable closed loop system?

iv. Design a controller such that the linearized system is stable, and can track a reference command with less than 10% steady-state error. (You may choose $m = k = 1$, $c_1 = c_2 = 0.1$ if you prefer to work with numerical values.) Sketch the Nyquist and Bode plots, and indicate the gain and phase margin on the Nyquist plot. Can you make a guess as to whether the controlled nonlinear system is globally asymptotically stable?

CDS 110a Qualifying Exam, Jan 2011

C.19 Consider the following nonlinear system:

$$\dot{x} = x - y - (xy^2 + x^3) + u\dot{y} \quad = x + y - (x^2y + y^3)$$

i. For the unforced system ($u = 0$), compute the dynamics and equilibrium points for the variable $z = x^2 + y^2$, and use this to sketch the phase portrait for the system above. (Can you use this to guess whether there are any equilibrium points of the unforced system other than the origin?)

ii. Linearize the system about the origin, and derive the transfer function between the input u and output x.

iii. For constant gain feedback $u = \alpha x$, sketch the Nyquist and Bode plots for the open loop linearized system. Can constant gain feedback give a stable closed loop system?

iv. Indicate how you would design a controller such that the linearized system is stable. Is this easier if instead you have output of the second state y? If you have output of both x and y available?

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