Chapter 8

Transfer Functions

> *The typical regulator system can frequently be described, in essentials, by differential equations of no more than perhaps the second, third or fourth order. . . . In contrast, the order of the set of differential equations describing the typical negative feedback amplifier used in telephony is likely to be very much greater. As a matter of idle curiosity, I once counted to find out what the order of the set of equations in an amplifier I had just designed would have been, if I had worked with the differential equations directly. It turned out to be 55.*
>
> Henrik Bode, 1960 [Bod60].

This chapter introduces the concept of the *transfer function*, which is a compact description of the input/output relation for a linear system. Combining transfer functions with block diagrams gives a powerful method for dealing with complex linear systems. The relationship between transfer functions and other system descriptions of dynamics is also discussed.

8.1 Frequency Domain Analysis

Figure 8.1 shows a block diagram for a typical control system, consisting of a process to be controlled and a (dynamic) compensator, connected in a feedback loop. We saw in the previous two chapters how to analyze and design such systems using state space descriptions of the blocks. As was mentioned in Chapter 2, an alternative approach is to focus on the input/output characteristics of the system. Since it is the inputs and outputs that are used to connect the systems, one could expect that this point of view would allow an understanding of the overall behavior of the system. Transfer functions are the main tool in implementing this point of view for linear systems.

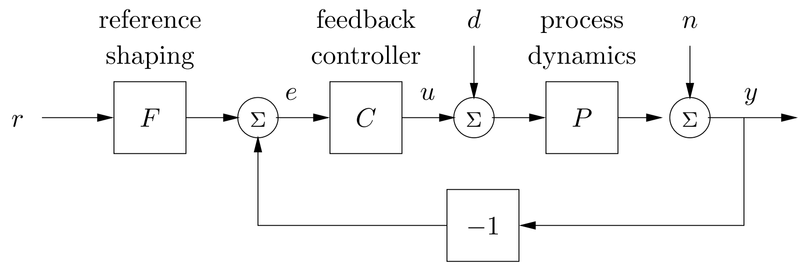


Figure 8.1: A block diagram for a feedback control system.

The basic idea of the transfer function comes from looking at the frequency response of a system. Suppose that we have an input signal that is periodic. Then we can always decompose this signal into the sum of a set of sines and cosines,

$$u(t) = \sum_{k=1}^{\infty} a_k \sin(k\omega t) + b_k \cos(k\omega t),$$

where ω is the fundamental frequency of the periodic input. Each of the terms in this input generates a corresponding sinusoidal output (in steady state), with possibly shifted magnitude and phase. The magnitude gain and phase at each frequency is determined by the frequency response, given in equation (5.21):

$$G(s) = C(sI - A)^{-1}B + D, \tag{8.1}$$

where we set $s = j(k\omega)$ for each $k = 1, \dots, \infty$. If we know the steady state frequency response $G(s)$, we can thus compute the response to any (periodic) signal using superposition.

The transfer function generalizes this notion to allow a broader class of input signals besides periodic ones. As we shall see in the next section, the transfer function represents the response of the system to an "exponential input," $u = e^{st}$. It turns out that the form of the transfer function is precisely the same as equation (8.1). This should not be surprising since we derived equation (8.1) by writing sinusoids as sums of complex exponentials. Formally, the transfer function corresponds to the Laplace transform of the steady state response of a system, although one does not have to understand the details of Laplace transforms in order to make use of transfer functions.

The power of transfer functions is that they allow a particularly convenient form for manipulating and analyzing complex feedback systems. As we shall see, there are many graphical representations of transfer functions that capture interesting properties of dynamics. Transfer functions also make it possible to express the changes in a system because of modeling error, which is essential when discussing sensitivity to process variations of the sort discussed in Chapter 12. In particular, using transfer functions it is possible to analyze what happens when dynamic models are approximated by static models or when high order models are approximated by low order models. One consequence is that we can introduce concepts that express the degree of stability of a system.

The main limitation of transfer functions is that they can only be used for linear systems. While many of the concepts for state space modeling and analysis extend to nonlinear systems, there is no such analog for transfer functions and there are only limited extensions of many of the ideas to nonlinear systems. Hence for the remainder of the text we shall limit ourselves to linear models. However, it should be pointed out that despite this limitation, transfer functions still remain a valuable tool for designing controllers for nonlinear systems, chiefly through constructing their linear approximations around an equilibrium point of interest.

8.2 Derivation of the Transfer Function

As we have seen in previous chapters, the input/output dynamics of a linear system has two components: the initial condition response and the forced response. In addition, we can speak of the transient properties of the system and its steady state response to an input. The transfer function focuses on the steady state response due to a given input, and provides a mapping between inputs and their corresponding outputs. In this section, we will derive the transfer function in terms of the "exponential response" of a linear system.

Transmission of Exponential Signals

To formally compute the transfer function of a system, we will make use of a special type of signal, called an *exponential signal,* of the form e^{st} where $s = \sigma + j\omega$ is a complex number. Exponential signals play an important role in linear systems. They appear in the solution of differential equations and in the impulse response of linear systems, and many signals can be represented as exponentials or sums of exponentials. For example, a constant signal is simply $e^{\alpha t}$ with $\alpha = 0$. Damped sine and cosine signals can be represented by

$$e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos\omega t + i\sin\omega t),$$

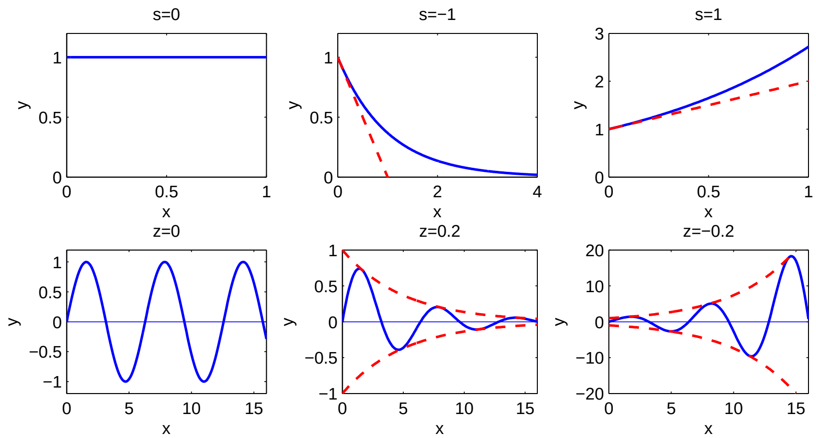


Figure 8.2: Examples of exponential signals.

where $\sigma < 0$ determines the decay rate. Many other signals can be represented by linear combinations of exponentials. Figure 8.2 give examples of signals that can be represented by complex exponentials. As in the case of sinusoidal signals, we will allow complex valued signals in the derivation that follows, although in practice we always add together combinations of signals that result in real-valued functions.

To investigate how a linear system responds to the exponential input $u(t) = e^{st}$ we consider the state space system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned} \tag{8.2}$$

Let the input signal be $u(t) = e^{st}$ and assume that $s \neq \lambda_i(A)$, $i = 1, \dots, n$, where $\lambda_i(A)$ is the ith eigenvalue of A. The state is then given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau}\, d\tau = e^{At}x(0) + e^{At}\int_0^t e^{(sI-A)\tau} B\, d\tau.$$

If $s \neq \lambda(A)$ the integral can be evaluated and we get

$$\begin{aligned} x(t) &= e^{At}x(0) + e^{At}(sI-A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\ &= e^{At}x(0) + e^{At}(sI-A)^{-1}\Big(e^{(sI-A)t} - I\Big)B \\ &= e^{At}\Big(x(0) - (sI-A)^{-1}B\Big) + (sI-A)^{-1}Be^{st}. \end{aligned}$$

The output of equation (8.2) is thus

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + \Big(C(sI - A)^{-1}B + D\Big)e^{st}, \end{aligned} \tag{8.3}$$

a linear combination of the exponential functions e^{st} and e^{At}. The first term in equation (8.3) is the transient response of the system. Recall that e^{AT} can be written in terms of the eigenvalues of A (using the Jordan form) and hence the transient response is a linear combinations of terms of the form $e^{\lambda_i t}$, where λ_i are eigenvalues of A. If the system is stable then $e^{AT} \to 0$ as $t \to \infty$ and this term dies away.

The second term of the output (8.3) is proportional to the input $u(t) = e^{st}$. This term is called the *pure exponential response*. If the initial state is chosen as

$$x(0) = (sI - A)^{-1}B,$$

then the output only consists of the pure exponential response and both the state and the output are proportional to the input:

$$\begin{aligned} x(t) &= (sI - A)^{-1}Be^{st} = (sI - A)^{-1}Bu(t) \\ y(t) &= \big(C(sI - A)^{-1}B + D\big)e^{st} = \big(C(sI - A)^{-1}B + D\big)u(t). \end{aligned}$$

The map from the input to output,

$$G_{yu}(s) = C(sI - A)^{-1}B + D, \tag{8.4}$$

is the *transfer function* of the system (8.2); the function

$$G_{xu}(s) = (sI - A)^{-1}B$$

is the transfer function from input to state. Note that this latter transfer function is actually a *vector* of n transfer functions (one for each state). Using transfer functions the response of the system (8.2) to an exponential input is thus

$$y(t) = Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + G_{yu}(s)e^{st}. \tag{8.5}$$

An important point in the derivation of the transfer function is the fact that we have restricted s so that $s \neq \lambda_i(A)$, $i = 1, \dots, n$, where $\lambda_i(A)$. At those values of s, we see that the response of the system is singular (since $sI - A$ will fail to be invertible). These correspond to "modes" of the system and are particularly problematic when $\operatorname{Re} s \geq 0$, since this can result in bounded inputs creating unbounded outputs. This situation can only happen when the system has eigenvalues with either positive or zero real part, and hence it relates to the stability of the system. In particular, if a linear system is asymptotically stable, then bounded inputs will always produce bounded outputs.

Coordinate Changes

The matrices A, B and C in equation (8.2) depend on the choice of coordinate system for the states. Since the transfer function relates input to outputs, it should be invariant to coordinate changes in the state space. To show this, consider the model (8.2) and introduce new coordinates z by the transformation $z = Tx$, where T is a nonsingular matrix. The system is then described by

$$\begin{aligned}\dot{z} &= T(Ax + Bu) = TAT^{-1}z + TBu = \tilde{A}z + \tilde{B}u \\ y &= Cx + DU = CT^{-1}z + Du = \tilde{C}z + Du\end{aligned}$$

This system has the same form as equation (8.2) but the matrices A, B and C are different:

$$\tilde{A} = TAT^{-1} \qquad \tilde{B} = TB \qquad \tilde{C} = CT^{-1} \qquad \tilde{D} = D. \tag{8.6}$$

Computing the transfer function of the transformed model we get

$$\begin{aligned}\tilde{G}(s) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + D \\ &= CT^{-1}T(sI - A)^{-1}T^{-1}TB + D \\ &= CT^{-1}(sI - TAT^{-1})^{-1}TB + D \\ &= C\big(T^{-1}(sI - TAT^{-1})T\big)^{-1}B + D \\ &= C(sI - A)^{-1}B + D = G(s),\end{aligned}$$

which is identical to the transfer function (8.4) computed from the system description (8.2). The transfer function is thus invariant to changes of the coordinates in the state space.

Another property of the transfer function is that it corresponds to the portion of the state space dynamics that are both reachable and observable. In particular, if we make use of the Kalman decomposition (Section 7.5), then the transfer function only depends on the dynamics on the reachable and observable subspace, $\mathcal{S}_{ro}$ (Exercise 2).

Transfer Functions for Linear Differential Equations

Consider a linear input/output system described by the differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u, \tag{8.7}$$

where u is the input and y is the output. This type of description arises in many applications, as described briefly in Section 2.2. Note that here we have generalized our previous system description to allow both the input and its derivatives to appear.

To determine the transfer function of the system (8.7), let the input be $u(t) = e^{st}$. Since the system is linear, there is an output of the system that is also an exponential function $y(t) = y_0 e^{st}$. Inserting the signals in equation (8.7) we find

$$(s^n + a_1 s^{n-1} + \cdots + a_n) y_0 e^{st} = (b_0 s^m + b_1 s^{m-1} \cdots + b_m) e^{-st}$$

and the response of the system can be completely described by two polynomials

$$\begin{aligned} a(s) &= s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \\ b(s) &= b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m. \end{aligned} \tag{8.8}$$

The polynomial $a(s)$ is the characteristic polynomial of the ordinary differential equation. If $a(s) \neq 0$ it follows that

$$y(t) = y_0 e^{st} = \frac{b(s)}{a(s)} e^{st} = G(s) u(t). \tag{8.9}$$

The transfer function of the system (8.7) is thus the rational function

$$G(s) = \frac{b(s)}{a(s)}, \tag{8.10}$$

where the polynomials $a(s)$ and $b(s)$ are given by equation (8.8). Notice that the transfer function for the system (8.7) can be obtained by inspection, since the coefficients of $a(s)$ and $b(s)$ are precisely the coefficients of the derivatives of u and y.

Equations (8.7)–(8.10) can be used to compute the transfer functions of many simple ODEs. The following table gives some of the more common forms:

Type	ODE	Transfer Function
Integrator	$\dot{y} = u$	$\dfrac{1}{s}$
Differentiator	$y = \dot{u}$	s
First order system	$\dot{y} + ay = u$	$\dfrac{1}{s+a}$
Double Integrator	$\ddot{y} = u$	$\dfrac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 = u$	$\dfrac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
PID controller	$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \dfrac{k_i}{s}$
Time delay	$y(t) = u(t-\tau)$	$e^{-\tau s}$

The first five of these follow directly from the analysis above. For the PID controller, we let the input be $u(t) = e^{st}$ and search for a solution $y(t) = e^{st}$. It follows that

$$y(t) = k_p e^{st} + k_d s e^{st} + \frac{k_i}{s} e^{st},$$

giving the indicated transfer function.

Time delays appear in many systems: typical examples are delays in nerve propagation, communication and mass transport. A system with a time delay has the input/output relation

$$y(t) = u(t - T). \tag{8.11}$$

As before the input be $u(t) = e^{st}$. Assuming that there is an output of the form $y(t) = y_0 e^{st}$ and inserting into equation (8.11) we get

$$y(t) = y_0 e^{st} = e^{s(t-T)} = e^{-sT} e^{st} = e^{-sT} u(t).$$

The transfer function of a time delay is thus $G(s) = e^{-sT}$ which is not a rational function, but is analytic except at infinity.

Example 8.1 (Operational amplifiers)**.** To further illustrate the use of exponential signals, we consider the operational amplifier circuit introduced in Section 3.3 and reproduced in Figure 8.3. The model introduced in Section 3.3 is a simplification because the linear behavior of the amplifier was modeled as a constant gain. In reality there is significant dynamics in the

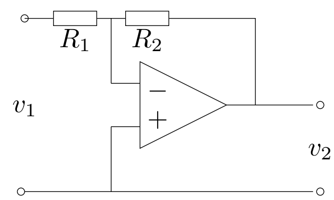


Figure 8.3: Schematic diagram of a stable amplifier based on negative feedback around an operational amplifier.

amplifier and the static model $v_{\text{out}} = -kv$ (equation (3.10)), should therefore be replaced by a dynamic model. In the linear range the amplifier, we can model the op amp as having a steady state frequency response

$$\frac{v_{\text{out}}}{v} = -\frac{k}{1+sT} =: G(s). \tag{8.12}$$

This response corresponds to a first order system with time constant T; typical parameter values are $k = 10^6$ and $T = 1$.

Since all of the elements of the circuit are modeled as being linear, if we drive the input v_1 with an exponential signal e^{st} then in steady state all signals will be exponentials of the same form. This allows us to manipulate the equations describing the system in an algebraic fashion. Hence we can write

$$\frac{v_1 - v}{R_1} = \frac{v - v_2}{R_2} \qquad \text{and} \qquad v_2 = G(s)v, \tag{8.13}$$

using the fact that the current into the amplifier is very small, as we did in Section 3.3. We can now "solve" for v_1 in terms of v by eliminating v_2 in the first equation:

$$v_1 = R_1\Big(\frac{v}{R_1} + \frac{v}{R_2} - \frac{v_2}{R_2}\Big) = R_1\Big(\frac{1}{R_1} + \frac{1}{R_2} - \frac{G(s)}{R_2}\Big)v.$$

Rewriting v in terms of v_1 and substituting into the second formula (8.13), we obtain

$$\frac{v_2}{v_1} = \frac{R_2 G(s)}{R_1 + R_2 - R_1 G(s)} = \frac{R_2 k}{(R_1 + R_2)(1 + sT) + kR_1}.$$

This model for the frequency response shows that when s is large in magnitude (very fast signals) the frequency response of the circuit drops off. Note also that if we take T to be very small (corresponding to an op amp with a very fast response time), our circuit performs well up to higher frequencies. In the limit that $T = 0$, we recover the responds that we derived in Section 3.3.

Note that in solving this example, we bypassed explicitly writing the signals as $v = v_0 e^{st}$ and instead worked directly with v, assuming it was an exponential. This shortcut is very handy in solving problems of this sort. ∇

Although we have focused thus far on ordinary differential equations, transfer functions can also be used for other types of linear systems. We illustrate this via an example of a transfer function for a partial differential equation.

Example 8.2 (Transfer function for heat propagation)**.** Consider the one dimensional heat propagation in a semi-infinite metal rod. Assume that the input is the temperature at one end and that the output is the temperature at a point on the rod. Let θ be the temperature at time t and position x. With proper choice of length scales and units, heat propagation is described by the partial differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial^2 x}, \tag{8.14}$$

and the point of interest can be assumed to have $x = 1$. The boundary condition for the partial differential equation is

$$\theta(0, t) = u(t).$$

To determine the transfer function we choose the input as $u(t) = e^{st}$. Assume that there is a solution to the partial differential equation of the form $\theta(x, t) = \psi(x)e^{st}$, and insert this into equation (8.14) to obtain

$$s\psi(x) = \frac{d^2\psi}{dx^2},$$

with boundary condition $\psi(0) = e^{st}$. This ordinary differential equation (with independent variable x) has the solution

$$\psi(x) = Ae^{x\sqrt{s}} + Be^{-x\sqrt{s}}.$$

Matching the boundary conditions gives $A = 0$ and $B = e^{st}$, so the solution is

$$y(t) = \theta(1, t) = \psi(1)e^{st} = e^{-\sqrt{s}}e^{st} = e^{-\sqrt{s}}u(t).$$

The system thus has the transfer function $G(s) = e^{-\sqrt{s}}$. As in the case of a time delay, the transfer function is not a simple ratio of polynomials, but it is an analytic function. ∇

Transfer Function Properties

The transfer function has many useful physical interpretations and the features of a transfer function are often associated with important system properties.

The *zero frequency gain* of a system is given by the magnitude of the transfer function at $s = 0$. It represents the ratio of the steady state value of the output with respect to a step input (which can be represented as $u = e^{st}$ with $s = 0$). For a state space system, we computed the zero frequency gain in equation (5.20):

$$G(0) = D - CA^{-1}B.$$

For a system written as a linear ODE, as in equation (8.7), if we assume that the input and output of the system are constants y_0 and u_0, then we find that $a_n y_0 = b_m u_0$. Hence the zero frequency gain gain is

$$G(0) = \frac{y_0}{u_0} = \frac{b_m}{a_n}. \tag{8.15}$$

Next consider a linear system with the rational transfer function

$$G(s) = \frac{b(s)}{a(s)}.$$

The roots of the polynomial $a(s)$ are called *poles* of the system and the roots of $b(s)$ are called the *zeros* of the system. If p is a pole it follows that $y(t) = e^{pt}$ is a solution of equation (8.7) with $u = 0$ (the homogeneous solution). The function e^{pt} is called a *mode* of the system. The unforced motion of the system after an arbitrary excitation is a weighted sum of the modes. Since the pure exponential output corresponding to the input $u(t) = e^{st}$ with $a(s) \neq 0$ is $G(s)e^{st}$ it follows that the pure exponential output is zero if $b(s) = 0$. Zeros of the transfer function thus block the transmission of the corresponding exponential signals.

For a state space system with transfer function $G(s) = C(sI-A)^{-1}B+D$, the poles of the transfer function are the eigenvalues of the matrix A in the state space model. One easy way to see this is to notice that the value of $G(s)$ is unbounded when s is an eigenvalue of a system since this is precisely the set of points where the characteristic polynomial $\lambda(s) = \det(sI - A) = 0$ (and hence $sI - A$ is non-invertible). It follows that the poles of a state space system depend only on the matrix A, which represents the intrinsic dynamics of the system.

To find the zeros of a state space system, we observe that the zeros are complex numbers s such that the input $u(t) = e^{st}$ gives zero output. Inserting the pure exponential response $x(t) = x_0 e^{st}$ and $y(t) = 0$ in equation (8.2) gives

$$\begin{aligned} se^{st}x_0 &= Ax_0e^{st} + Bu_0e^{st} \\ 0 &= Ce^{st}x_0 + De^{st}u_0, \end{aligned}$$

which can be written as

$$\begin{pmatrix} sI - A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = 0.$$

This equation has a solution with nonzero x_0, u_0 only if the matrix on the left does not have full rank. The zeros are thus the values s such that

$$\det \begin{pmatrix} sI - A & B \\ C & D \end{pmatrix} = 0. \tag{8.16}$$

Since the zeros depend on A, B, C and D, they therefore depend on how the inputs and outputs are coupled to the states. Notice in particular that if the matrix B has full rank then the matrix has n linearly independent rows for all values of s. Similarly there are n linearly independent columns if the matrix C has full rank. This implies that systems where the matrices B or C are of full rank do not have zeros. In particular it means that a system has no zeros if it is fully actuated (each state can be controlled independently) or if the full state is measured.

A convenient way to view the poles and zeros of a transfer function is through a *pole zero diagram*, as shown in Figure 8.4. In this diagram, each pole is marked with a cross and each zero with a circle. If there are multiple poles or zeros at a fixed location, these are often indicated with overlapping crosses or circles (or other annotations). Poles in the left half plane correspond to stable models of the system and poles in the right half plane correspond to unstable modes. Notice that the gain must also be given to have a complete description of the transfer function.

8.3 Block Diagrams and Transfer Functions

The combination of block diagrams and transfer functions is a powerful way to represent control systems. Transfer functions relating different signals in the system can be derived by purely algebraic manipulations of the transfer

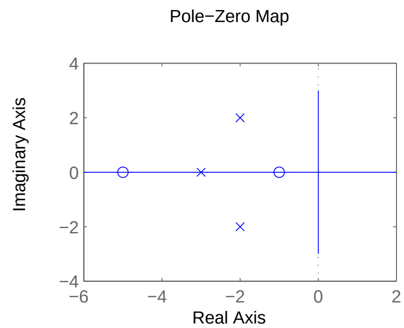


Figure 8.4: A pole zero digram for a transfer function with zeros at -5 and -1, and poles at -3 and $-2 \pm 2j$. The circles represent the locations of the zeros and the crosses the locations of the poles.

functions of the blocks using *block diagram algebra*. To show how this can be done, we will begin with simple combinations of systems.

Consider a system which is a cascade combination of systems with the transfer functions $G_1(s)$ and $G_2(s)$, as shown in Figure 8.5a. Let the input of the system be $u = e^{st}$. The pure exponential output of the first block is the exponential signal $G_1 u$, which is also the input to the second system. The pure exponential output of the second system is

$$y = G_2(G_1 u) = (G_2 G_1)u.$$

The transfer function of the system is thus $G = G_2 G_1$, i.e. the product of the transfer functions. The order of the individual transfer functions is due to the fact that we place the input signal on the right hand side of this

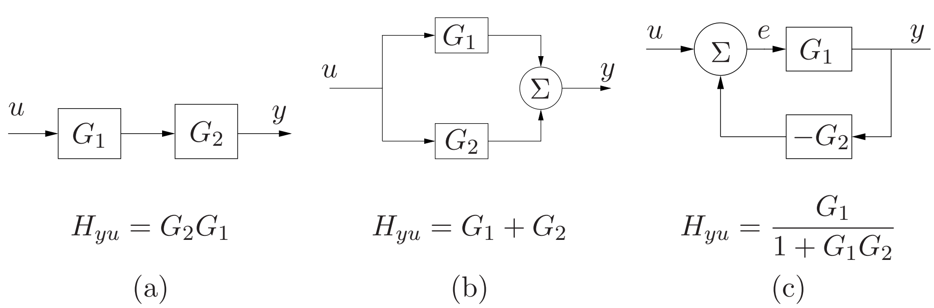


Figure 8.5: Interconnections of linear systems: (a) series, (b) parallel and (c) feedback connections.

expression, hence we first multiply by G_1 and then by G_2. Unfortunately, this has the opposite ordering from the diagrams that we use, where we typically have the signal flow from left to right, so one needs to be careful. The ordering is important if either G_1 or G_2 is a vector-valued transfer function, as we shall see in some examples.

Consider next a parallel connection of systems with the transfer functions G_1 and G_2, as shown in Figure 8.5b. Letting $u = e^{st}$ be the input to the system, the pure exponential output of the first system is then $y_1 = G_1 u$ and the output of the second system is $y_2 = G_2 u$. The pure exponential output of the parallel connection is thus

$$y = G_1 u + G_2 u = (G_1 + G_2)u$$

and the transfer function for a parallel connection $G = G_1 + G_2$.

Finally, consider a feedback connection of systems with the transfer functions G_1 and G_2, as shown in Figure 8.5c. Let $u = e^{st}$ be the input to the system, y the pure exponential output, and e be the pure exponential part of the intermediate signal given by the sum of u and the output of the second block. Writing the relations for the different blocks and the summation unit we find

$$y = G_1 e \qquad e = u - G_2 y.$$

Elimination of e gives

$$y = G_1(u - G_2 y),$$

hence

$$(1 + G_1 G_2)y = G_1 u,$$

which implies

$$y = \frac{G_1}{1 + G_1 G_2} u.$$

The transfer function of the feedback connection is thus

$$G = \frac{G_1}{1 + G_1 G_2}.$$

These three basic interconnections can be used as the basis for computing transfer functions for more complicated systems, as shown in the following examples.

Example 8.3 (Control system transfer functions)**.** Consider the system in Figure 8.6, which was given already at the beginning of the chapter. The system has three blocks representing a process P, a feedback controller C and

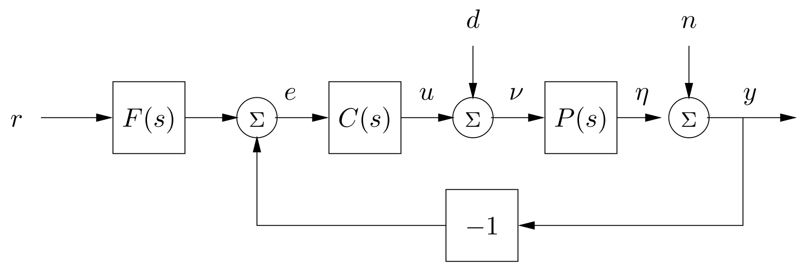


Figure 8.6: Block diagram of a feedback system.

a feedforward controller F. There are three external signals, the reference r, the load disturbance d and the measurement noise n. A typical problem is to find out how the error e is related to the signals r, d and n.

To derive the transfer function we simply assume that all signals are exponential functions, drop the arguments of signals and transfer functions and trace the signals around the loop. We begin with the signal in which we are interested, in this case the error e, given by

$$e = Fr - y.$$

The signal y is the sum of n and η, where η is the output of the process and u is the output of the controller:

$$y = n + \eta \qquad \eta = P(d + u) \qquad u = Ce.$$

Combining these equations gives

$$\begin{aligned} e &= Fr - y = Fr - (n + \eta) = Fr - \big(n + P(d + u)\big) \\ &= Fr - \big(n + P(d + Ce)\big) \end{aligned}$$

and hence

$$e = Fr - \big(n + P(d + Ce)\big) = Fr - n - Pd - PCe.$$

Finally, solving this equation for e gives

$$e = \frac{F}{1 + PC} r - \frac{1}{1 + PC} n - \frac{P}{1 + PC} d = G_{er} r + G_{en} n + G_{ed} d \qquad (8.17)$$

and the error is thus the sum of three terms, depending on the reference r, the measurement noise n and the load disturbance d. The functions

$$G_{er} = \frac{F}{1 + PC} \qquad G_{en} = \frac{-1}{1 + PC} \qquad G_{ed} = \frac{-P}{1 + PC} \qquad (8.18)$$

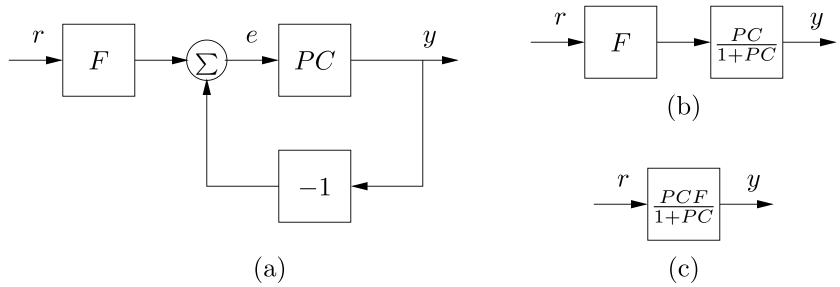


Figure 8.7: Example of block diagram algebra.

are the transfer functions from reference r, noise n and disturbance d to the error e.

We can also derive transfer functions by manipulating the block diagrams directly, as illustrated in Figure 8.7. Suppose we wish to compute the transfer function between the reference r and the output y. We begin by combining the process and controller blocks in Figure 8.6 to obtain the diagram in Figure 8.7a. We can now eliminate the feedback look (Figure 8.7b) and then use the series interconnection rule to obtain

$$G_{yr} = \frac{PCF}{1+PC}. \tag{8.19}$$

Similar manipulations can be used to obtain other transfer functions. ∇

The example illustrates an effective way to manipulate the equations to obtain the relations between inputs and outputs in a feedback system. The general idea is to start with the signal of interest and to trace signals around the feedback loop until coming back to the signal we started with. With a some practice, equations (8.17) and (8.18) can be written directly by inspection of the block diagram. Notice that all terms in equation (8.17) and (8.18) have the same denominators. There may, however, be factors that cancel due to the form of the numerator.

Example 8.4 (Vehicle steering)**.** Consider the linearized model for vehicle steering introduced in Example 2.8. In Examples 6.4 and 7.3 we designed a state feedback compensator and state estimator. A block diagram for the resulting control system is given in Figure 8.8. Note that we have split the estimator into two components, $G_{\hat{x}u}(s)$ and $G_{\hat{x}y}(s)$, corresponding to its inputs u and y. The controller can be described as the sum of two (open loop) transfer functions

$$u = G_{uy}(s)y + G_{ur}(s)r.$$

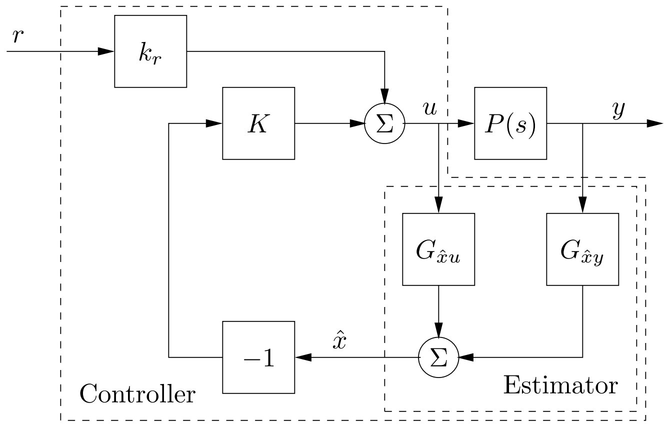


Figure 8.8: Block diagram for the steering control system.

The first transfer function, $G_{uy}(s)$, describes the feedback term and the second, $G_{ur}(s)$, describes the feedforward term. We call these "open loop" transfer functions because they represent the relationships between the signals without considering the dynamics of the process (e.g., removing $P(s)$ from the system description). To derive these functions, we compute the the transfer functions for each block and then use block diagram algebra.

We begin with the estimator, which takes u and y as its inputs and produces an estimate $\hat{x}$. The dynamics for this process was derived in Example 7.3 and is given by

$$\begin{aligned}\frac{d\hat{x}}{dt} &= (A - LC)\hat{x} + Ly + Bu \\ \hat{x} &= \underbrace{\big(sI - (A - LC)\big)^{-1}B}_{G_{\hat{x}u}}\, u + \underbrace{\big(sI - (A - LC)\big)^{-1}L}_{G_{\hat{x}y}}\, y.\end{aligned}$$

Using the expressions for A, B, C and L from Example 7.3, we obtain

$$G_{\hat{x}u} = \begin{pmatrix} \dfrac{\alpha s + 1}{s^2 + l_1 s + l_2} \\ \dfrac{s + l_1 - \alpha l_2}{s^2 + l_1 s + l_2} \end{pmatrix} \qquad G_{\hat{x}y} = \begin{pmatrix} \dfrac{l_1 s + l_2}{s^2 + l_1 s + l_2} \\ \dfrac{l_2 s}{s^2 + l_1 s + l_2} \end{pmatrix}.$$

We can now proceed to compute the transfer function for the overall control system. Using block diagram algebra, we have

$$G_{uy} = \frac{-KG_{\hat{x}y}}{1 + KG_{\hat{x}u}} = -\frac{s(k_1 l_1 + k_2 l_2) + k_1 l_2}{s^2 + s(\alpha k_1 + k_2 + l_1) + k_1 + l_2 + k_2 l_1 - \alpha k_2 l_2}$$

and

$$G_{ur} = \frac{k_r}{1 + KG_{\hat{x}u}} = k_1 \frac{s^2 + l_1 s + l_2}{s^2 + s(\alpha k_1 + k_2 + l_1) + k_1 + l_2 + k_2 l_1 - \alpha k_2 l_2}.$$

Finally, we compute the full closed loop dynamics. We begin by deriving the transfer function for the process, $P(s)$. We can compute this directly from the state space description of the dynamics, which was given in Example 6.4. Using that description, we have

$$P = G_{yu} = C(sI - A)^{-1}B + D = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \frac{\alpha s + 1}{s^2}.$$

The transfer function for the full closed loop system between the input r and the output y is then given by

$$G_{yr} = \frac{k_r P(s)}{1 - P(s)G_{yu}(s)} = \frac{k_1(\alpha s + 1)}{s^2 + (k_1\alpha + k_2)s + k_1}.$$

Note that the observer gains do not appear in this equation. This is because we are considering steady state analysis and, in steady state, the estimated state exactly tracks the state of the system if we assume perfect models. We will return to this example in Chapter 12 to study the robustness of this particular approach. ∇

The combination of block diagrams and transfer functions is a powerful tool because it is possible both to obtain an overview of a system and find details of the behavior of the system.

Pole/Zero Cancellations

Because transfer functions are often polynomials in s, it can sometimes happen that the numerator and denominator have a common factor, which can be canceled. Sometimes these cancellations are simply algebraic simplifications, but in other situations these cancellations can mask potential fragilities in the model. In particular, if a pole/zero cancellation occurs due to terms in separate blocks that just happen to coincide, the cancellation may not occur if one of the systems is slightly perturbed. In some situations this can result in severe differences between the expected behavior and the actual behavior, as illustrated in this section.

To illustrate when we can have pole/zero cancellations, consider the block diagram shown in Figure 8.6 with $F = 1$ (no feedforward compensation) and C and P given by

$$C = \frac{n_c(s)}{d_c(s)} \qquad P = \frac{n_p(s)}{d_p(s)}.$$

The transfer function from r to e is then given by

$$G_{er} = \frac{1}{1+PC} = \frac{d_c(s)d_p(s)}{d_c(s)d_p(s) + n_c(s)n_p(s)}.$$

If there are common factors in the numerator and denominator polynomials, then these terms can be factored out and eliminated from both the numerator and denominator. For example, if the controller has a zero at $s = a$ and the process has a pole at $s = a$, then we will have

$$G_{er} = \frac{(s+a)d'_c(s)d_p(s)}{(s+a)d_c(s)d'_p(s) + (s+a)n'_c(s)n_p(s)} = \frac{d'_c(s)d_p(s)}{d_c(s)d'_p(s) + n'_c(s)n_p(s)},$$

where $n'_c(s)$ and $d'_p(s)$ represent the relevant polynomials with the term $s+a$ factored out.

Suppose instead that we compute the transfer function from d to e, which represents the effect of a disturbance on the error between the reference and the output. This transfer function is given by

$$G_{ed} = \frac{d'_c(s)n_p(s)}{(s+a)d_c(s)d'_p(s) + (s+a)n'_c(s)n_p(s)}.$$

Notice that if $a < 0$ then the pole is in the right half plane and the transfer function G_{ed} is *unstable*. Hence, even though the transfer function from r to e appears to be OK (assuming a perfect pole/zero cancellation), the transfer function from d to e can exhibit unbounded behavior. This unwanted behavior is typical of an *unstable pole/zero cancellation*.

It turns out that the cancellation of a pole with a zero can also be understood in terms of the state space representation of the systems. Reachability or observability is lost when there are cancellations of poles and zeros (Exercise 11). A consequence is that the transfer function only represents the dynamics in the reachable and observable subspace of a system (see Section 7.5).

8.4 The Bode Plot

The frequency response of a linear system can be computed from its transfer function by setting $s = j\omega$, corresponding to a complex exponential

$$u(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t).$$

The resulting output has the form

$$y(t) = Me^{j\omega t+\varphi} = M\cos(\omega t+\varphi) + jM\sin(\omega t+\varphi)$$

where M and φ are the gain and phase of G:

$$M = |G(j\omega)| \qquad \varphi = \angle G(j\omega) = \arctan \frac{\operatorname{Im} G(j\omega)}{\operatorname{Re} G(j\omega)}.$$

The phase of G is also called the *argument* of G, a term that comes from the theory of complex variables.

It follows from linearity that the response to a single sinusoid (sin or cos) is amplified by M and phase shifted by φ. Note that $\varphi \in [0, 2\pi)$, so the arctangent must be taken respecting the signs of the numerator and denominator. It will often be convenient to represent the phase in degrees rather than radians. We will use the notation $\angle G(j\omega)$ for the phase in degrees and $\arg G(j\omega)$ for the phase in radians. In addition, while we always take $\arg G(j\omega)$ to be in the range $[0, 2\pi)$, we will take $\angle G(j\omega)$ to be continuous, so that it can take on values outside of the range of 0 to 360°.

The frequency response $G(j\omega)$ can thus be represented by two curves: the gain curve and the phase curve. The gain curve gives gain $|G(j\omega)|$ as a function of frequency ω and the phase curve gives phase $\angle G(j\omega)$ as a function of frequency ω. One particularly useful way of drawing these curves is to use a log/log scale for the magnitude plot and a log/linear scale for the phase plot. This type of plot is called a *Bode plot* and is shown in Figure 8.9.

Part of the popularity of Bode plots is that they are easy to sketch and to interpret. Consider a transfer function which is a ratio of polynomial terms $G(s) = (b_1(s)b_2(s))/(a_1(s)a_2(s))$. We have

$$\log |G(s)| = \log |b_1(s)| + \log |b_2(s)| - \log |a_1(s)| - \log |a_2(s)|$$

and hence we can compute the gain curve by simply adding and subtracting gains corresponding to terms in the numerator and denominator. Similarly

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s)$$

and so the phase curve can be determined in an analogous fashion. Since a polynomial is a product of terms of the type

$$k, \quad s, \quad s + a, \quad s^2 + 2\zeta a s + a^2,$$

it suffices to be able to sketch Bode diagrams for these terms. The Bode plot of a complex system is then obtained by adding the gains and phases of the terms.

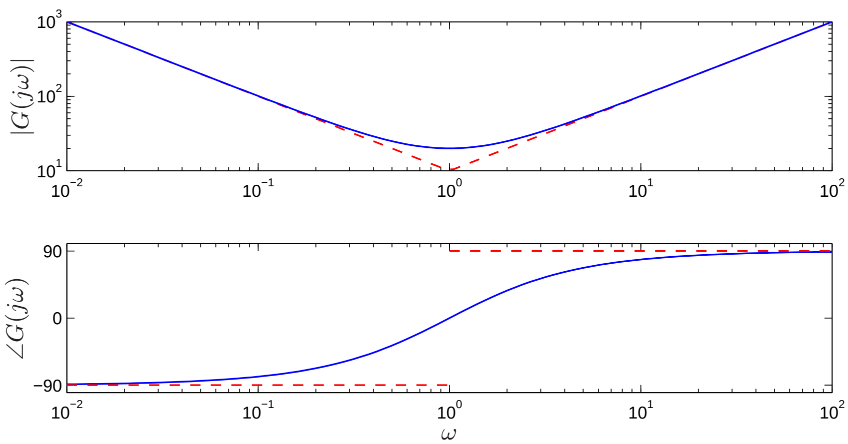


Figure 8.9: Bode plot of the transfer function $C(s) = 20 + 10/s + 10s$ of an ideal PID controller. The top plot is the gain curve and bottom plot is the phase curve. The dashed lines show straight line approximations of the gain curve and the corresponding phase curve.

The simplest term in a transfer function is a power of s, s^k, where $k > 0$ if the term appears in the numerator and $k < 0$ if the term is in the denominator. The magnitude and phase of the term are given by

$$\log|G(j\omega)| = k\log\omega, \quad \angle G(j\omega) = 90k.$$

The gain curve is thus a straight line with slope k and the phase curve is a constant at $90^\circ \times k$. The case when $k = 1$ corresponds to a differentiator and has slope 1 with phase 90°. The case when $k = -1$ corresponds to an integrator and has slope 1 with phase 90°. Bode plots of the various powers of k are shown in Figure 8.10.

Consider next the transfer function of a first order system, given by

$$G(s) = \frac{a}{s+a}.$$

We have

$$\log G(s) = \log a - \log s + a$$

and hence

$$\log|G(j\omega)| = \log a - \frac{1}{2}\log\left(\omega^2 + a^2\right)), \quad \angle G(j\omega) = -\frac{180}{\pi}\arctan\omega/a.$$

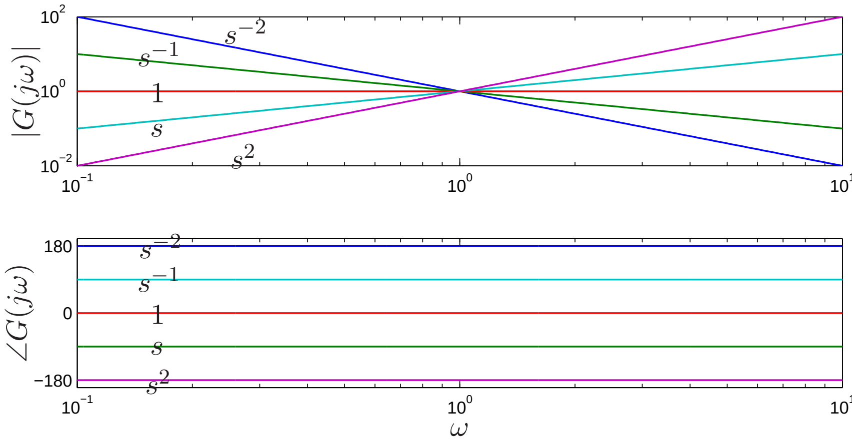


Figure 8.10: Bode plot of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$.

The Bode plot is shown in Figure 8.11a, with the magnitude normalized by the zero frequency gain. Both the gain curve and the phase curve can be approximated by the following straight lines

$$\log |G(j\omega)| \approx \begin{cases} \log a & \text{if } \omega < a \\ -\log \omega & \text{if } \omega > a \end{cases}$$

$$\angle G(j\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10 \\ -45 - 45(\log \omega - \log a) & a/10 < \omega < 10a \\ -180 & \text{if } \omega > 10a. \end{cases}$$

Notice that a first order system behaves like a constant for low frequencies and like an integrator for high frequencies. Compare with the Bode plot in Figure 8.10.

Finally, consider the transfer function for a second order system

$$G(s) = \frac{\omega_0^2}{s^2 + 2a\zeta s + \omega_0^2}.$$

We have

$$\log |G(j\omega)| = 2 \log |\omega_0| - \log |(-\omega^2 + 2j\omega_0\zeta\omega + \omega_0^2)|$$

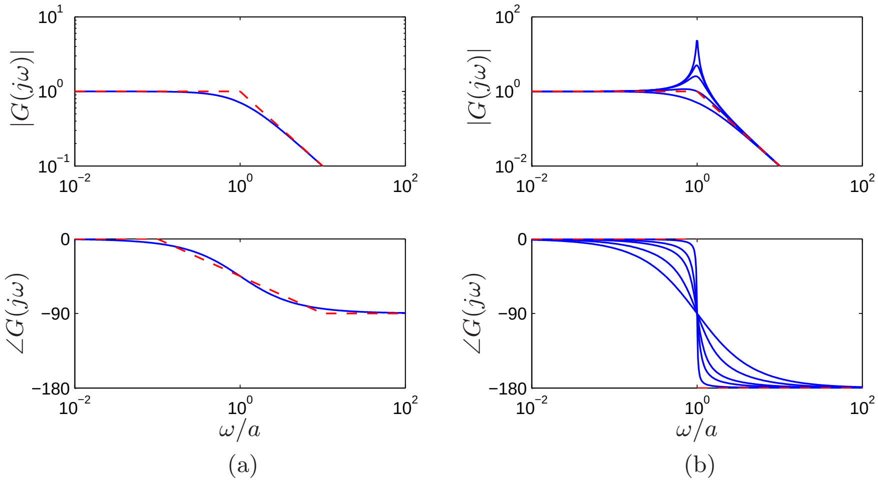


Figure 8.11: Bode plots of the systems $G(s) = a/(s+a)$ (left) and $G(s) = \omega_0^2/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$ (right). The full lines show the Bode plot and the dashed lines show the straight line approximations to the gain curves and the corresponding phase curves. The plot for second order system has $\zeta = 0.02$, 0.1, 0.2, 0.5 and 1.0.

and hence

$$\log |G(j\omega)| = 2\log\omega_0 - \frac{1}{2}\log\left(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2 - 1) + \omega_0^4\right)$$
$$\angle G(j\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2}$$

The gain curve has an asymptote with zero slope for $\omega \ll \omega_0$. For large values of ω the gain curve has an asymptote with slope -2. The largest gain $Q = \max_\omega |G(j\omega)| \approx 1/(2\zeta)$, called the *Q value*, is obtained for $\omega \approx \omega_0$. The phase is zero for low frequencies and approaches 180° for large frequencies. The curves can be approximated with the following piece-wise linear expressions

$$\log |G(j\omega)| \approx \begin{cases} 0 & \textit{if } \omega \ll \omega_0, \\ -2\log\omega & \textit{if } \omega \gg \omega_0 \end{cases} \qquad \angle G(j\omega) \approx \begin{cases} 0 & \textit{if } \omega \ll \omega_0, \\ -180 & \textit{if } \omega \gg \omega_0 \end{cases}.$$

The Bode plot is shown in Figure 8.11b. Note that the asymptotic approximation is poor near $\omega = a$ and the Bode plot depends strongly on ζ near this frequency.

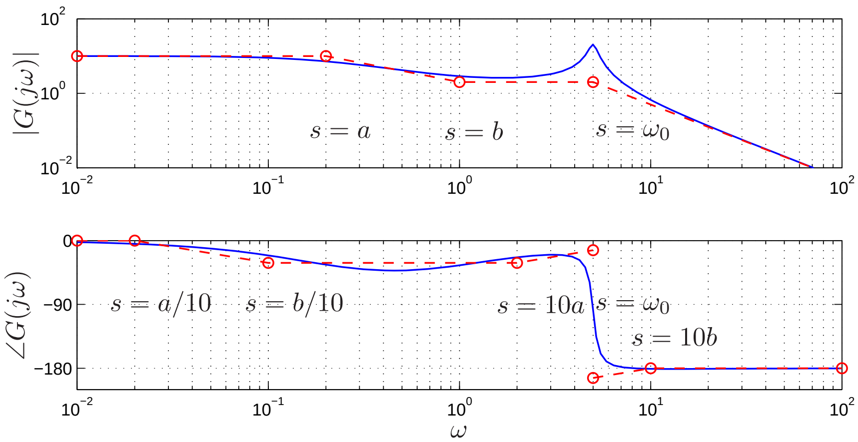


Figure 8.12: Sample Bode plot with asymptotes that give approximate curve.

Given the Bode plots of the basic functions, we can now sketch the frequency response for a more general system. The following example illustrates the basic idea.

Example 8.5. Consider the transfer function given by

$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)} \qquad a \ll b \ll \omega_0.$$

The Bode plot for this transfer function is shown in Figure 8.12, with the complete transfer function shown in blue (solid) and a sketch of the Bode plot shown in red (dashed).

We begin with the magnitude curve. At low frequency, the magnitude is given by

$$G(0) = \frac{kb}{a\omega^2}.$$

When we hit the pole at $s = a$, the magnitude begins to decrease with slope -1 until it hits the zero at $s = b$. At that point, we increase the slope by 1, leaving the asymptote with net slope 0. This slope is used until we reach the second order pole at $s = \omega_c$, at which point the asymptote changes to slope -2. We see that the magnitude curve is fairly accurate except in the region of the peak of the second order pole (since for this case ζ is reasonably small).

The phase curve is more complicated, since the effect of the phase stretches out much further. The effect of the pole begins at $s = a/10$, at which point we change from phase 0 to a slope of $-45°$/decade. The zero begins to affect the phase at $s = b/10$, giving us a flat section in the phase. At $s = 10a$ the phase contributions from the pole end and we are left with a slope of $+45°$/decade (from the zero). At the location of the second order pole, $s \approx j\omega_c$, we get a jump in phase of $-180°$. Finally, at $s = 10b$ the phase contributions of the zero end and we are left with phase -180 degrees. We see that the straight line approximation for the phase is not as accurate as it was for the gain curve, but it does capture the basic features of the phase changes as a function of frequency. ∇

The Bode plot gives a quick overview of a system. Many properties can be read from the plot and because logarithmic scales are used the plot gives the properties over a wide range of frequencies. Since any signal can be decomposed into a sum of sinusoids it is possible to visualize the behavior of a system for different frequency ranges. Furthermore when the gain curves are close to the asymptotes, the system can be approximated by integrators or differentiators. Consider for example the Bode plot in Figure 8.9. For low frequencies the gain curve of the Bode plot has the slope -1 which means that the system acts like an integrator. For high frequencies the gain curve has slope +1 which means that the system acts like a differentiator.

8.5 Transfer Functions from Experiments

The transfer function of a system provides a summary of the input/output response and is very useful for analysis and design. However, modeling from first principles can be difficult and time consuming. Fortunately, we can often build an input/output model for a given application by directly measuring the frequency response and fitting a transfer function to it. To do so, we perturb the input to the system using a sinusoidal signal at a fixed frequency. When steady state is reached, the amplitude ratio and the phase lag gives the frequency response for the excitation frequency. The complete frequency response is obtained by sweeping over a range of frequencies.

By using correlation techniques it is possible to determine the frequency response very accurately and an analytic transfer function can be obtained from the frequency response by curve fitting. The success of this approach has led to instruments and software that automate this process, called *spectrum analyzers*. We illustrate the basic concept through two examples.

Example 8.6 (Atomic force microscope)**.** To illustrate the utility of spectrum analysis, we consider the dynamics of the atomic force microscope,

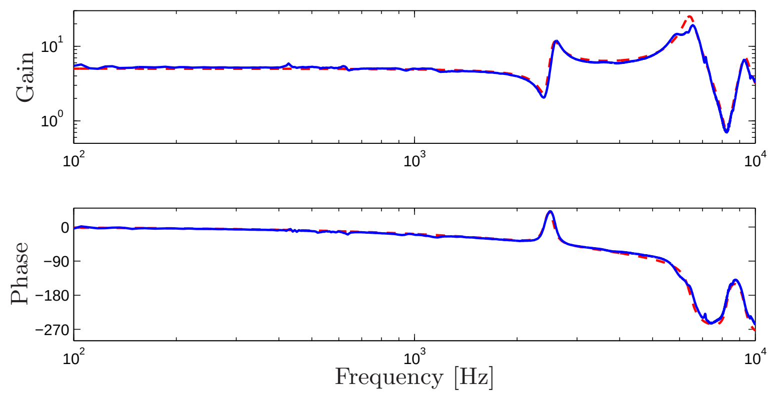


Figure 8.13: Frequency response of a piezoelectric drive for an atomic force microscope. The input is the voltage to the drive amplifier and the output is the output of the amplifier that measures beam deflection.

introduced in Section 3.5. Experimental determination of the frequency response is particularly attractive for this system because its dynamics are very fast and hence experiments can be done quickly. A typical example is given in Figure 8.13, which shows an experimentally determined frequency response (solid line). In this case the frequency response was obtained in less than a second. The transfer function

$$G(s) = \frac{k\omega_2^2\omega_3^2\omega_5^2(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_4\omega_4 s + \omega_4^2)e^{-sT}}{\omega_1^2\omega_4^2(s^2 + 2\zeta_2\omega_2 s + \omega_2^2)(s^2 + 2\zeta_3\omega_3 s + \omega_3^2)(s^2 + 2\zeta_5\omega_5 s + \omega_5^2)}$$

with $\omega_1 = 2420$, $\zeta_1 = 0.03$, $\omega_2 = 2550$, $\zeta_2 = 0.03$, $\omega_3 = 6450$, $\zeta_3 = 0.042$, $\omega_4 = 8250$, $\zeta_4 = 0.025$, $\omega_5 = 9300$, $\zeta_5 = 0.032$, $T = 10^{-4}$, and $k = 5$. was fit to the data (dashed line). The frequencies associated with the zeros are located where the gain curve has minima and the frequencies associated with the poles are located where the gain curve has local maxima. The relative damping are adjusted to give a good fit to maxima and minima. When a good fit to the gain curve is obtained the time delay is adjusted to give a good fit to the phase curve. ∇

Experimental determination of frequency response is less attractive for systems with slow dynamics because the experiment takes a long time.

Example 8.7 (Pupillary light reflex dynamics)**.** The human eye is an organ that is easily accessible for experiments. It has a control system that adjusts

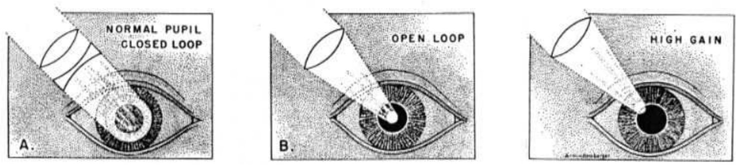


Figure 8.14: Light stimulation of the eye. In A the light beam is so large that it always covers the whole pupil, giving the closed loop dynamics. In B the light is focused into a beam which is so narrow that it is not influenced by the pupil opening, giving the open loop dynamics. In C the light beam is focused on the edge of the pupil opening, which has the effect of increasing the gain of the system since small changes in the pupil opening have a large effect on the amount of light entering the eye. From [Sta59].

the pupil opening to regulate the light intensity at the retina. This control system was explored extensively by Stark in the late 1960s [Sta68]. To determine the dynamics, light intensity on the eye was varied sinusoidally and the pupil opening was measured. A fundamental difficulty is that the closed loop system is insensitive to internal system parameters, so analysis of a closed loop system thus gives little information about the internal properties of the system. Stark used a clever experimental technique that allowed him to investigate both open and closed loop dynamics. He excited the system by varying the intensity of a light beam focused on the eye and he measured pupil area; see Figure 8.14. By using a wide light beam that covers the whole pupil the measurement gives the closed loop dynamics. The open loop dynamics were obtained by using a narrow beam, which is small enough that it is not influenced by the pupil opening. The result of one experiment for determining open loop dynamics is given in Figure 8.15. Fitting a transfer function to the gain curves gives a good fit for $G(s) = 0.17/(1+0.08s)^3$. This curve gives a poor fit to the phase curve as shown by the dashed curve in Figure 8.15. The fit to the phase curve is improved by adding a time delay, which leaves the gain curve unchanged while substantially modifying the phase curve. The final fit gives the model

$$G(s) = \frac{0.17}{(1+0.08)^3} e^{-0.2s}.$$

The Bode plot of this is shown with dashed curves in Figure 8.15. ∇

Notice that for both the AFM drive and the pupillary dynamics it is not easy to derive appropriate models from first principles. In practice, it is

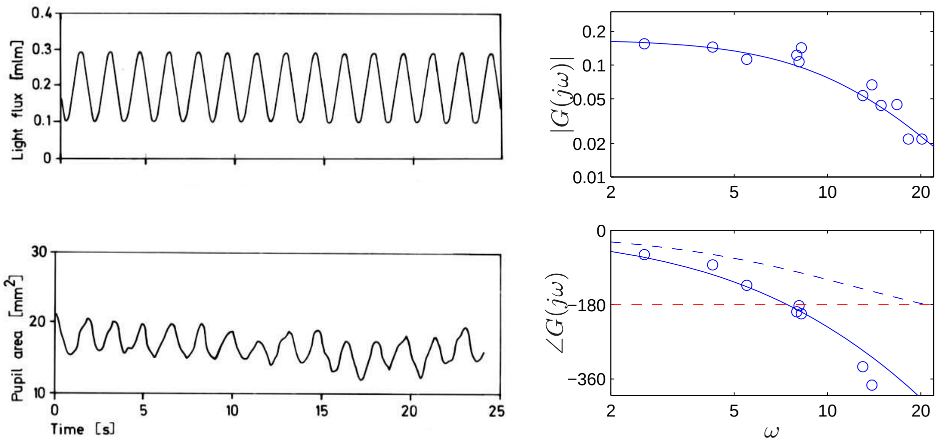


Figure 8.15: Sample curves from open loop frequency response of the eye (left) and Bode plot for the open loop dynamics (right). Redrawn from the data of [Sta59]. The dashed curve in the Bode plot is the minimum phase curve corresponding to the gain curve.

often fruitful to use a combination of analytical modeling and experimental identification of parameters.

8.6 Laplace Transforms

Transfer functions are typically introduced using Laplace transforms and in this section we derive the transfer function using this formalism. We assume basic familiarity with Laplace transforms; students who are not familiar with them can safely skip this section.

Traditionally, Laplace transforms were also used to compute responses of linear system to different stimuli. Today we can easily generate the responses using computers. Only a few elementary properties are needed for basic control applications. There is, however, a beautiful theory for Laplace transforms that makes it possible to use many powerful tools of the theory of functions of a complex variable to get deep insights into the behavior of systems.

Definitions and Properties

Consider a time function $f : \mathbb{R}^+ \to \mathbb{R}$ which is integrable and grows no faster than $e^{s_0 t}$ for some finite $s_0 \in \mathbb{R}$ and large t. The Laplace transform maps f

to a function $F = \mathcal{L}f : \mathbb{C} \to \mathbb{C}$ of a complex variable. It is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad \operatorname{Re} s > s_0. \tag{8.20}$$

The transform has some properties that makes it very well suited to deal with linear systems.

First we observe that the transform is linear because

$$\begin{aligned}\mathcal{L}(af + bg) &= \int_0^\infty e^{-st}(af(t) + bg(t))\,dt \\ &= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt = a\mathcal{L}f + b\mathcal{L}g.\end{aligned} \tag{8.21}$$

Next we will calculate the Laplace transform of the derivative of a function. We have

$$\mathcal{L}\frac{df}{dt} = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\mathcal{L}f,$$

where the second equality is obtained by integration by parts. We thus obtain the following important formula for the transform of a derivative

$$\mathcal{L}\frac{df}{dt} = s\mathcal{L}f - f(0) = sF(s) - f(0). \tag{8.22}$$

This formula is particularly simple if the initial conditions are zero because it follows that differentiation of a function corresponds to multiplication of the transform with s.

Since differentiation corresponds to multiplication with s we can expect that integration corresponds to division by s. This is true, as can be seen by calculating the Laplace transform of an integral. We have

$$\begin{aligned}\mathcal{L}\int_0^t f(\tau)\,d\tau &= \int_0^\infty \Big(e^{-st}\int_0^t f(\tau)\,d\tau\Big)\,dt \\ &= -\frac{e^{-st}}{s}\int_0^t e^{-s\tau} f(\tau)\,d\tau\Big|_0^\infty + \int_0^\infty \frac{e^{-s\tau}}{s} f(\tau)\,d\tau = \frac{1}{s}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau,\end{aligned}$$

hence

$$\mathcal{L}\int_0^t f(\tau)\,d\tau = \frac{1}{s}\mathcal{L}f = \frac{1}{s}F(s). \tag{8.23}$$

Integration of a time function thus corresponds to dividing the Laplace transform by s.

The Laplace Transform of a Convolution

Consider a linear time-invariant system with zero initial state. The relation between the input u and the output y is given by the convolution integral

$$y(t) = \int_0^\infty h(t-\tau)u(\tau)\,d\tau,$$

where $h(t)$ is the impulse response for the system. We will now consider the Laplace transform of such an expression. We have

$$\begin{aligned} Y(s) &= \int_0^\infty e^{-st}y(t)\,dt = \int_0^\infty e^{-st}\int_0^\infty h(t-\tau)u(\tau)\,d\tau\,dt \\ &= \int_0^\infty \int_0^t e^{-s(t-\tau)}e^{-s\tau}h(t-\tau)u(\tau)\,d\tau\,dt \\ &= \int_0^\infty e^{-s\tau}u(\tau)\,d\tau \int_0^\infty e^{-st}h(t)\,dt = H(s)U(s) \end{aligned}$$

The result can be written as $Y(s) = H(s)U(s)$ where H, U and Y are the Laplace transforms of h, u and y. The system theoretic interpretation is that the Laplace transform of the output of a linear system is a product of two terms, the Laplace transform of the input $U(s)$ and the Laplace transform of the impulse response of the system $H(s)$. A mathematical interpretation is that the Laplace transform of a convolution is the product of the transforms of the functions that are convolved. The fact that the formula $Y(s) = H(s)U(s)$ is much simpler than a convolution is one reason why Laplace transforms have become popular in control.

The Transfer Function

The properties (8.21) and (8.22) makes the Laplace transform ideally suited for dealing with linear differential equations. The relations are particularly simple if all initial conditions are zero.

Consider for example a linear state space system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

Taking Laplace transforms under the assumption that all initial values are zero gives

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s). \end{aligned}$$

Elimination of $X(s)$ gives

$$Y(s) = \Big(C(sI - A)^{-1}B + D\Big)U(s). \tag{8.24}$$

The transfer function is thus $G(s) = C(sI - A)^{-1}B + D$ (compare with equation (8.4)).

The formula (8.24) has a strong intuitive interpretation because it tells that the Laplace transform of the output is the product of the transfer function of the system and the transform of the input. In the transform domain the action of a linear system on the input is simply a multiplication with the transfer function. The transfer function is a natural generalization of the concept of gain of a system.

8.7 Further Reading

Heaviside, who introduced the idea to characterize dynamics by the response to a unit step function, also introduced a formal operator calculus for analyzing linear systems. This was a significant advance because it gave the possibility to analyze linear systems algebraically. Unfortunately it was difficult to formalize Heaviside's calculus properly. This was not done until the the mathematician Laurent Schwartz developed the *distribution theory* in the late 1940s. Schwartz was given the Fields Medal in 1950. The idea of characterizing a linear system by its steady state response to sinusoids was introduced by Fourier in his investigation of heat conduction in solids [Fou07]. Much later it was used by Steinmetz when he introduced the $j\omega$ method to develop a theory for alternating currents.

The concept of transfer functions was an important part of classical control theory; see [JNP47]. It was introduced via the Laplace transform by Gardner Barnes [GB42], who also used it to calculate response of linear systems. The Laplace transform was very important in the early phase of control because it made it possible to find transients via tables. The Laplace transform is of less importance today when responses to linear systems can easily be generated using computers. For a mathematically inclined audience it is still a very convenient to introduce the transfer function via the Laplace transform, which is an important part of applied mathematics. For an audience with less background in mathematics it may be preferable to introduce the transfer function via the particular solution generated by the input e^{st} as was done in Section 8.2.

There are many excellent books on the use of Laplace transforms and transfer functions for modeling and analysis of linear input/output systems.

Traditional texts on control, such as [FPEN05] and [DB04], are representative examples.

8.8 Exercises

1. Let $G(s)$ be the transfer function for a linear system. Show that if we apply an input $u(t) = A\sin(\omega t)$ then the steady state output is given by $y(t) = |G(j\omega)|A\sin(\omega t + \arg G(j\omega))$.

2. Show that the transfer function of a system only depends on the dynamics in the reachable and observable subspace of the Kalman decomposition.

3. The linearized model of the pendulum in the upright position is characterized by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0.$$

   Determine the transfer function of the system.

4. Compute the frequency response of a PI controller using an op amp with frequency response given by equation (8.12).

5. Consider the speed control system given in Example 6.9. Compute the transfer function between the throttle position u, angle of the road θ and the speed of the vehicle v assuming a nominal speed v_e with corresponding throttle position u_e.

6. Consider the differential equation

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + a_2\frac{d^{n-2}y}{dt^{n-2}} + \cdots + a_n y = 0$$

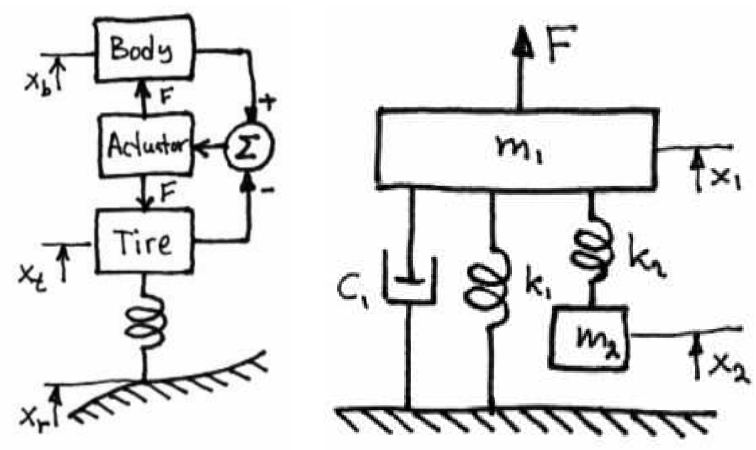
   Let λ be a root of the polynomial

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0.$$

   Show that the differential equation has the solution $y(t) = e^{\lambda t}$.

7. Consider the system

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_1\frac{d^{n-1}u}{dt^{n-1}} + b_2\frac{d^{n-2}u}{dt^{n-2}} + \cdots + b_n u,$$



(a) (b)

Figure 8.16: Schematic diagram of the *quarter car model* (a) and of a vibration absorber right (b).

Let λ be a zero of the polynomial

$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n$$

Show that if the input is $u(t) = e^{\lambda t}$ then there is a solution to the differential equation that is identically zero.

8. Active and passive damping is used in cars to give a smooth ride on a bumpy road. A schematic diagram of a car with a damping system in shown in Figure 8.16(a). The car is approximated with two masses, one represents a quarter of the car body and the other a wheel. The actuator exerts a force F between the wheel and the body based on feedback from the distance between body and the center of the wheel (the *rattle space*). A simple model of the system is given by Newton's equations for body and wheel

$$m_b \ddot{x}_b = F, \qquad m_w \ddot{x}_w = -F + k_t(x_r - x_w),$$

where m_b is a quarter of the body mass, m_w is the effective mass of the wheel including brakes and part of the suspension system (the *unsprung mass*), and k_t is the tire stiffness. Furthermore x_b, x_w and x_r represent the heights of body, wheel, and road, measured from their equilibria. For a conventional damper consisting of a spring and a damper we have $F = k(x_w - x_b) + c(\dot{x}_w - \dot{x}_b)$, for an active damper the force F can be more general and it can also depend on riding conditions. Rider comfort can be characterized by the transfer function G_{ax_r} from road height x_r to body acceleration $a = \ddot{x}_b$. Show that this transfer function has the property $G_{ax_r}(i\omega_t) = k_t/m_b$, where $\omega_t = \sqrt{k_t/m_w}$ (the *tire hop frequency*). The equation implies that there are fundamental limitations to the comfort that can be achieved with any damper. More details are given in [HB90].

9. Damping vibrations is a common engineering problem. A schematic diagram of a damper is shown in Figure 8.16(b). The disturbing vibration is a sinusoidal force acting on mass m_1 and the damper consists of mass m_2 and the spring k_2. Show that the transfer function from disturbance force to height x_1 of the mass m_1 is

$$G_{x_1F} = \frac{m_2s^2 + k_2}{m_1m_2s^4 + m_2c_1s^3 + (m_1k_2 + m_2(k_1 + k_2))s^2 + k_2c_1s + k_1k_2}$$

   How should the mass m_2 and the stiffness k_2 be chosen to eliminate a sinusoidal oscillation with frequency ω_0. More details are given on pages 87–93 in the classic text on vibrations [DH85].

10. Consider the linear state space system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx.\end{aligned}$$

   Show that the transfer function is

$$G(s) = \frac{b_1s^{n-1} + b_2s^{n-2} + \cdots + b_n}{s^n + a_1s^{n-1} + \cdots + a_n}$$

   where

$$\begin{aligned}b_1 &= CB\\ b_2 &= CAB + a_1CB\\ b_3 &= CA^2B + a_1CAB + a_2CB\\ &\vdots\\ b_n &= CA^{n-1}B + a_1CA^{n-1}B + \cdots + a_{n-1}CB\end{aligned}$$

   and $\lambda(s) = s^n + a_1s^{n-1} + \cdots + a_n$ is the characteristic polynomial for A.

11. Consider a closed loop system of the form of Figure 8.6 with $F = 1$ and P and C having a common pole. Show that if each system is written in state space form, the resulting closed loop system is not reachable and not observable.

12. The Physicist Ångström, who is associated with the length unit Å, used frequency response to determine thermal diffusivity of metals [Ång]. Heat propagation in a metal rod is described by the partial differential equation

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} - \mu T, \tag{8.25}$$

where $a = \frac{\lambda}{\rho C}$ is the thermal diffusivity, and the last term represents thermal loss to the environment. Show that the transfer function relating temperatures at points with the distance ℓ is

$$G(s) = e^{-\ell\sqrt{(s+\mu)/a}}, \tag{8.26}$$

and the frequency response is given by

$$\begin{aligned}\log|G(i\omega)| &= -\ell\sqrt{\frac{\mu + \sqrt{\omega^2+\mu^2}}{2a}}\\ \arg G(i\omega) &= -\ell\sqrt{\frac{-\mu + \sqrt{\omega^2+\mu^2}}{2a}}.\end{aligned}$$

Also derive the following equation

$$\log|G(i\omega)| \ \arg G(i\omega) = \frac{\ell^2\omega}{2a}.$$

This remarkably simple formula shows that diffusivity can be determined from the value of the transfer function at one frequency. It was the key in Ångström's method for determining thermal diffusivity. Notice that the parameter μ which represents the thermal losses does not appear in the formula.