Chapter 6

State Feedback

Intuitively, the state may be regarded as a kind of information storage or memory or accumulation of past causes. We must, of course, demand that the set of internal states Σ be sufficiently rich to carry all information about the past history of Σ to predict the effect of the past upon the future. We do not insist, however, that the state is the least *such information although this is often a convenient assumption.*

R. E. Kalman, P. L. Falb and M. A. Arbib, 1969 [KFA69].

This chapter describes how feedback of a system's state can be used shape the local behavior of a system. The concept of reachability is introduced and used to investigate how to "design" the dynamics of a system through assignment of its eigenvalues. In particular, it will be shown that under certain conditions it is possible to assign the system eigenvalues to arbitrary values by appropriate feedback of the system state.

6.1 Reachability

One of the fundamental properties of a control system is what set of points in the state space can be reached through the choice of a control input. It turns out that the property of "reachability" is also fundamental in understanding the extent to which feedback can be used to design the dynamics of a system.

Definition

We begin by disregarding the output measurements of the system and focusing on the evolution of the state, given by

$$\frac{dx}{dt} = Ax + Bu, \tag{6.1}$$

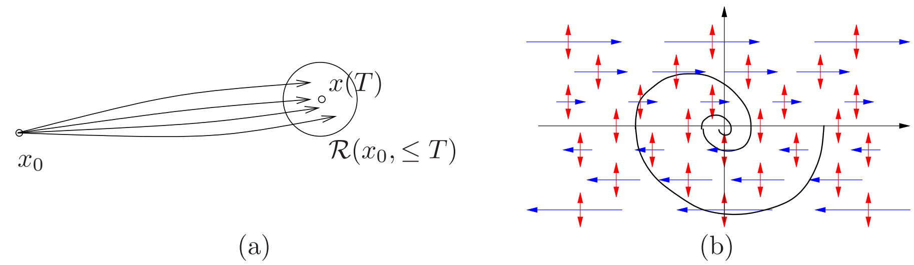


Figure 6.1: The reachable set for a control system: (a) the set $\mathcal{R}(x_0, \leq T)$ is the set of points reachable from x_0 in time less than T; (b) phase portrait for the double integrator showing the natural dynamics (horizontal arrows), the control inputs (vertical arrows) and a sample path to the origin.

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, A is an $n \times n$ matrix and B an $n \times 1$ matrix. A fundamental question is whether it is possible to find control signals so that any point in the state space can be reached through some choice of input. To study this, we define the reachable set $\mathcal{R}(x_0, \leq T)$ as the set of all points x_f such that there exists an input $u(t)$, $0 \leq t \leq T$ that steers the system from $x(0) = x_0$ to $x(T) = x_f$, as illustrated in Figure 6.1.

Definition 6.1 (Reachability)**.** A linear system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u\colon [0, T] \to \mathbb{R}$ such that the corresponding solution satisfies $x(0) = x_0$ and $x(T) = x_f$.

The set of points that we are most interested in reaching is the set of equilibrium points of the system (since we can remain at those points once we get there). The set of all possible equilibria for constant controls is given by

$$\mathcal{E} = \{x_e : Ax_e + bu_e = 0 \text{ for some } u_e \in \mathbb{R}\}.$$

This means that possible equilibria lie in a one (or possibly higher) dimensional subspace. If the matrix A is invertible this subspace is spanned by $A^{-1}B$.

In addition to reachability of equilibrium points, we can also ask whether it is possible to reach all points in the state space in a *transient* fashion. The following example provides some insight into the possibilities.

Example 6.1 (Double integrator)**.** Consider a linear system consisting of a double integrator, whose dynamics are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u. \end{aligned}$$

Figure 6.1b shows a phase portrait of the system. The open loop dynamics ($u = 0$) are shown as horizontal arrows pointed to the right for $x_2 > 0$ and the the left for $x_2 < 0$. The control input is represented by a double arrow in the vertical direction, corresponding to our ability to set the value of $\dot{x}_2$. The set of equilibrium points $\mathcal{E}$ corresponds to the x_1 axis, with $u_e = 0$.

Suppose first that we wish to reach the origin from an initial condition $(a, 0)$. We can directly move the state up and down in the phase plane, but we must rely on the natural dynamics to control the motion to the left and right. If $a > 0$, we can move the origin by first setting $u < 0$, which will case x_2 to become negative. Once $x_2 < 0$, the value of x_1 will begin to decrease and we will move to the left. After a while, we can set u_2 to be positive, moving x_2 back toward zero and slowing the motion in the x_1 direction. If we bring $x_2 > 0$, we can move the system state in the opposite direction.

Figure 6.1b shows a sample trajectory bringing the system to the origin. Note that if we steer the system to an equilibrium point, it is possible to remain there indefinitely (since $\dot{x}_1 = 0$ when $x_2 = 0$), but if we go to any other point in the state space, we can only pass through the point in a transient fashion. ∇

To find general conditions under which a linear system is reachable, we will first give a heuristic argument based on formal calculations with impulse functions. We note that if we can reach all points in the state space through some choice of input, then we can also reach all equilibrium points. Hence reachability of the entire state space implies reachability of all equilibrium points.

Testing for Reachability

When the initial state is zero, the response of the state to a unit step in the input is given by

$$x(t) = \int_0^t e^{A(t-\tau)} B d\tau = A^{-1}(e^{At} - I)B \tag{6.2}$$

The derivative of a unit step function is the impulse function, $\delta(t)$, defined in Section 5.2. Since derivatives are linear operations, it follows (see Exercise 7) that the response of the system to an impulse function is thus the derivative of equation (6.2) (i.e., the impulse response),

$$\frac{dx}{dt} = e^{At} B.$$

Similarly we find that the response to the derivative of a impulse function is

$$\frac{d^2x}{dt^2} = Ae^{At}B.$$

Continuing this process and using the linearity of the system, the input

$$u(t) = \alpha_1\delta(t) + \alpha_2\dot{\delta}(t) + \alpha\ddot{\delta}(t) + \cdots + \alpha_n\delta^{(n-1)}(t)$$

gives the state

$$x(t) = \alpha_1 e^{At}B + \alpha_2 Ae^{At}B + \alpha_3 A^2 e^{At}B + \cdots + \alpha_n A^{n-1}e^{At}B.$$

Hence, right after the initial time $t = 0$, denoted $t = 0+$, we have

$$x(0+) = \alpha_1 B + \alpha_2 AB + \alpha_3 A^2 B + \cdots + \alpha_n A^{n-1}B.$$

The right hand is a linear combination of the columns of the matrix

$$W_r = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}. \tag{6.3}$$

To reach an arbitrary point in the state space we thus require that there are n linear independent columns of the matrix W_r. The matrix is called the *reachability matrix*.

An input consisting of a sum of impulse functions and their derivatives is a very violent signal. To see that an arbitrary point can be reached with smoother signals we can also argue as follows. Assuming that the initial condition is zero, the state of a linear system is given by

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t e^{A\tau}Bu(t-\tau)d\tau.$$

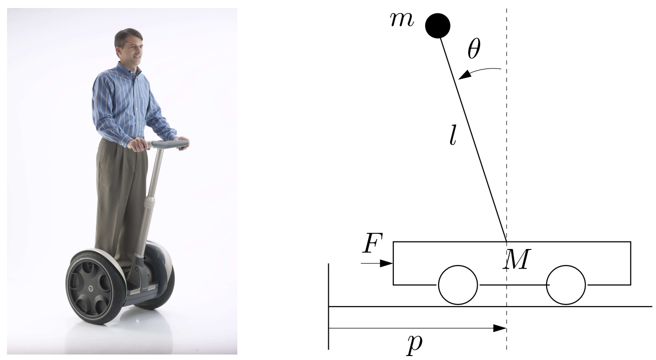
It follows from the theory of matrix functions, specifically the Cayley-Hamilton theorem [Str88] that

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau),$$

where $\alpha_i(\tau)$ are scalar functions, and we find that

$$\begin{aligned} x(t) = B\int_0^t \alpha_0(\tau)u(t-\tau)\,d\tau + AB\int_0^t \alpha_1(\tau)u(t-\tau)\,d\tau + \\ \cdots + A^{n-1}B\int_0^t \alpha_{n-1}(\tau)u(t-\tau)\,d\tau. \end{aligned}$$

Again we observe that the right hand side is a linear combination of the columns of the reachability matrix W_r given by equation (6.3). This basic approach leads to the following theorem.



(a) (b)

Figure 6.2: Balance system: (a) Segway human transportation system and (b) simplified diagram.

Theorem 6.1. *A linear system is reachable if and only the reachability matrix W_r is invertible.*

The formal proof of this theorem is beyond the scope of this text, but follows along the lines of the sketch above and can be found in most books on linear control theory, such as [CD91]. We illustrate the concept of reachability with the following example.

Example 6.2 (Reachability of balance systems)**.** Consider the balance system introduced in Example 2.1 and shown in Figure 6.2. Recall that this system is a model for a class of examples in which the center of mass is balanced above a pivot point. One example is the Segway transportation system shown in the left hand figure, in which a natural question to ask is whether we can move from one stationary point to another by appropriate application of forces through the wheels.

The nonlinear equations of motion for the system are given in equation (2.7) and repeated here:

$$
\begin{aligned}
(M+m)\ddot{p} - ml\cos\theta\,\ddot{\theta} &= -c\dot{p} + ml\sin\theta\,\dot{\theta}^2 + F \\
(J+ml^2)\ddot{\theta} - ml\cos\theta\,\ddot{p} &= -\gamma\dot{\theta} + mgl\sin\theta,
\end{aligned}
\tag{6.4}
$$

For simplicity, we take $c = \gamma = 0$. Linearizing around the equilibrium point

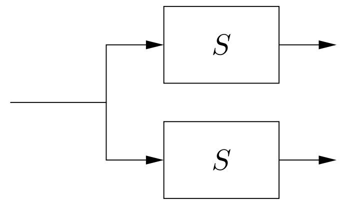


Figure 6.3: A non-reachable system.

$x_e = (p, 0, 0, 0)$, the dynamics matrix and the control matrix are

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{M_tJ_t - m^2l^2} & 0 & 0 \\ 0 & \frac{M_tmgl}{M_tJ_t - m^2l^2} & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \frac{J_t}{M_tJ_t - m^2l^2} \\ \frac{lm}{M_tJ_t - m^2l^2} \end{pmatrix},$$

The reachability matrix is

$$W_r = \begin{pmatrix} 0 & \frac{J_t}{M_tJ_t - m^2l^2} & 0 & \frac{gl^3m^3}{(M_tJ_t - m^2l^2)^2} \\ 0 & \frac{lm}{M_tJ_t - m^2l^2} & 0 & \frac{gl^2m^2(m+M)}{(M_tJ_t - m^2l^2)^2} \\ \frac{J_t}{M_tJ_t - m^2l^2} & 0 & \frac{gl^3m^3}{(M_tJ_t - m^2l^2)^2} & 0 \\ \frac{lm}{M_tJ_t - m^2l^2} & 0 & \frac{g^2l^2m^2(m+M)}{(M_tJ_t - m^2l^2)^2} & 0 \end{pmatrix}. \tag{6.5}$$

This matrix has determinant

$$\det(W_r) = \frac{g^2l^4m^4}{(M_tJ_t - m^2l^2)^4} \neq 0$$

and we can conclude that the system is reachable. This implies that we can move the system from any initial state to any final state and, in particular, that we can always find an input to bring the system from an initial state to an equilibrium point. ∇

Systems That Are Not Reachable

It is useful of have an intuitive understanding of the mechanisms that make a system unreachable. An example of such a system is given in Figure 6.3. The system consists of two identical systems with the same input. Clearly, we can not separately cause the first and second system to do something different since they have the same input. Hence we cannot reach arbitrary states and so the system is not reachable (Exercise 1).

More subtle mechanisms for non-reachability can also occur. For example, if there is a linear combination of states that always remains constant, then the system is not reachable. To see this, suppose that there exists a row vector H such that

$$0 = \frac{d}{dt}Hx = H(Ax + Bu) \quad \text{for all } u.$$

Then H is in the left null space of both A and B and it follows that

$$HW_r = H \begin{pmatrix} B AB \cdots A^{n-1}B \end{pmatrix} = 0.$$

Hence the reachability matrix is not full rank. In this case, if we have an initial condition x_0 and we wish to reach a state x_f for which $Hx_0 \neq Hx_f$, then since $Hx(t)$ is constant, no input u can move from x_0 to x_f.

Reachable Canonical Form

As we have already seen in previous chapters, it is often convenient to change coordinates and write the dynamics of the system in the transformed coordinates $z = Tx$. One application of a change of coordinates is to convert a system into a canonical form in which it is easy to perform certain types of analysis. Once such canonical form is called reachable canonical form.

Definition 6.2 (Reachable canonical form)**.** A linear state space system is in *reachable canonical form* if its dynamics are given by

$$\begin{aligned} \frac{dz}{dt} &= \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix} z. \end{aligned} \tag{6.6}$$

A block diagram for a system in reachable canonical form is shown in Figure 6.4. We see that the coefficients that appear in the A and B matrices show up directly in the block diagram. Furthermore, the output of the system is a simple linear combination of the outputs of the integration blocks.

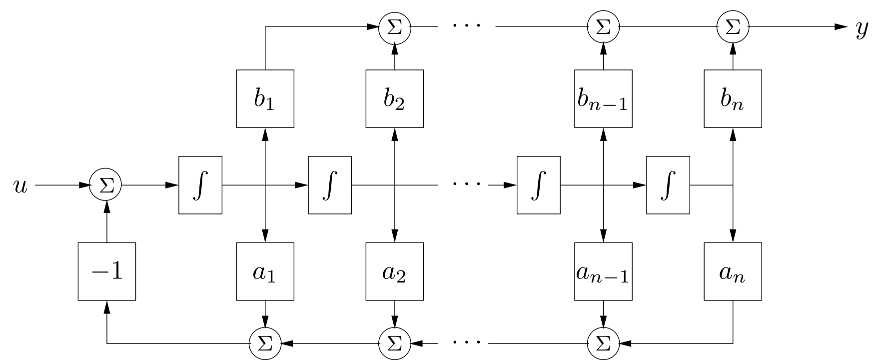


Figure 6.4: Block diagram for a system in reachable canonical form.

The characteristic polynomial for a system in reachable canonical form is given by

$$\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n. \tag{6.7}$$

The reachability matrix also has a relatively simple structure:

$$W_r = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} = \begin{pmatrix} 1 & -a_1 & a_1^2 - a_2 & \cdots & * \\ 0 & 1 & -a_1 & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $*$ indicates a possibly nonzero term. This matrix is clearly full rank since no column can be written as a linear combination of the others due to the triangular structure of the matrix.

We now consider the problem of changing coordinates such that the dynamics of a system can be written in reachable canonical form. Let A, B represent the dynamics of a given system and $\tilde{A}, \tilde{B}$ be the dynamics in reachable canonical form. Suppose that we wish to transform the original system into reachable canonical form using a coordinate transformation $z = Tx$. As shown in the last chapter, the dynamics matrix and the control matrix for the transformed system are

$$\begin{aligned} \tilde{A} &= TAT^{-1} \\ \tilde{B} &= TB. \end{aligned}$$

The reachability matrix for the transformed system then becomes

$$\tilde{W}_r = \begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix}.$$

Transforming each element individually, we have

$$
\begin{aligned}
\tilde{A}\tilde{B} &= TAT^{-1}TB = TAB \\
\tilde{A}^2\tilde{B} &= (TAT^{-1})^2TB = TAT^{-1}TAT^{-1}TB = TA^2B \\
&\vdots \\
\tilde{A}^n\tilde{B} &= TA^nB.
\end{aligned}
$$

and hence the reachability matrix for the transformed system is

$$
\tilde{W}_r = T\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = TW_r. \tag{6.8}
$$

Since W_r is invertible, we can thus solve for the transformation T that takes the system into reachable canonical form:

$$
T = \tilde{W}_r W_r^{-1}.
$$

The following example illustrates the approach.

Example 6.3. Consider a simple two dimensional system of the form

$$
\dot{x} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.
$$

We wish to find the transformation that converts the system into reachable canonical form:

$$
\tilde{A} = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

The coefficients a_1 and a_2 can be determined by looking at the characteristic equation for the original system:

$$
\lambda(s) = \det(sI - A) = s^2 - 2\alpha s + (\alpha^2 + \omega^2) \qquad \Longrightarrow \qquad \begin{aligned} a_1 &= -2\alpha \\ a_2 &= \alpha^2 + \omega^2. \end{aligned}
$$

The reachability matrix for each system is

$$
W_r = \begin{pmatrix} 0 & \omega \\ 1 & \alpha \end{pmatrix} \qquad \tilde{W}_r = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix}.
$$

The transformation T becomes

$$
T = \tilde{W}_r W_r^{-1} \begin{pmatrix} -\frac{a_1+\alpha}{\omega} & 1 \\ \frac{1}{\omega} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\omega} & 1 \\ \frac{1}{\omega} & 0 \end{pmatrix}
$$

and hence the coordinates

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tx = \begin{pmatrix} \frac{\alpha}{\omega} x_1 + x_2 \\ x_2 \end{pmatrix}$$

put the system in reachable canonical form. ∇

We summarize the results of this section in the following theorem.

Theorem 6.2. *Let (A, B) be the dynamics and control matrices for a reachable system. Then there exists a transformation $z = Tx$ such that in the transformed coordinates the dynamics and control matrices are in reachable canonical form* (6.6) *and the characteristic polynomial for A is given by*

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n.$$

One important implication of this theorem is that for any reachable system, we can always assume without loss of generality that the coordinates are chosen such that the system is in reachable canonical form. This is particularly useful for proofs, as we shall see later in this chapter.

6.2 Stabilization by State Feedback

The state of a dynamical system is a collection of variables that permits prediction of the future development of a system. We now explore the idea of designing the dynamics a system through feedback of the state. We will assume that the system to be controlled is described by a linear state model and has a single input (for simplicity). The feedback control will be developed step by step using one single idea: the positioning of closed loop eigenvalues in desired locations.

Figure 6.5 shows a diagram of a typical control system using state feedback. The full system consists of the process dynamics, which we take to be linear, the controller elements, K and k_r, the reference input, r, and processes disturbances, d. The goal of the feedback controller is to regulate the output of the system, y, such that it tracks the reference input in the presence of disturbances and also uncertainty in the process dynamics.

An important element of the control design is the performance specification. The simplest performance specification is that of stability: in the absence of any disturbances, we would like the equilibrium point of the system to be asymptotically stable. More sophisticated performance specifications typically involve giving desired properties of the step or frequency

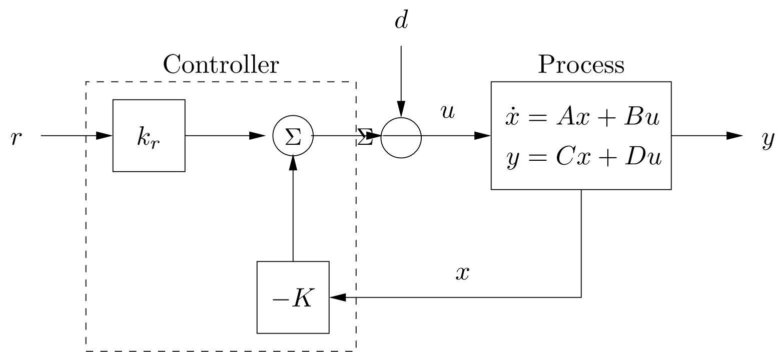


Figure 6.5: A feedback control system with state feedback.

response of the system, such as specifying the desired rise time, overshoot and settling time of the step response. Finally, we are often concerned with the disturbance rejection properties of the system: to what extent can we tolerate disturbance inputs d and still hold the output y near the desired value.

Consider a system described by the linear differential equation

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx,\end{aligned} \tag{6.9}$$

where we have taken $D = 0$ for simplicity and ignored the disturbance signal d for now. Our goal is to drive the output y to a given reference value, r, and hold it there.

We begin by assuming that all components of the state vector are measured. Since the state at time t contains all information necessary to predict the future behavior of the system, the most general time invariant control law is a function of the state and the reference input:

$$u = \alpha(x, r).$$

If the feedback is restricted to be a linear, it can be written as

$$u = -Kx + k_r r \tag{6.10}$$

where r is the reference value, assumed for now to be a constant.

This control law corresponds to the structure shown in Figure 6.5. The negative sign is simply a convention to indicate that negative feedback is the normal situation. The closed loop system obtained when the feedback (6.9) is applied to the system (6.10) is given by

$$\frac{dx}{dt} = (A - BK)x + Bk_r r \tag{6.11}$$

We attempt to determine the feedback gain K so that the closed loop system has the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n \tag{6.12}$$

This control problem is called the eigenvalue assignment problem or "pole placement" problem (we will define "poles" more formally in a later chapter).

Note that the k_r does not affect the stability of the system (which is determined by the eigenvalues of $A - BK$), but does affect the steady state solution. In particular, the equilibrium point and steady state output for the closed loop system are given by

$$x_e = -(A - BK)^{-1}Bk_r r \qquad y_e = Cx_e,$$

hence k_r should be chosen such that $y_e = r$ (the desired output value). Since k_r is a scalar, we can easily solve to show

$$k_r = -1/\big(C(A - BK)^{-1}B\big). \tag{6.13}$$

Notice that k_r is exactly the inverse of the zero frequency gain of the closed loop system.

Using the gains K and k_r, we are thus able to design the dynamics of the closed loop system to satisfy our goal. To illustrate how to such construct a state feedback control law, we begin with a few examples that provide some basic intuition and insights.

Examples

Example 6.4 (Vehicle steering)**.** In Example 5.12 we derived a normalized linear model for vehicle steering. The dynamics describing the lateral deviation where given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad D = 0.$$

The reachability matrix for the system is thus

$$W_r = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

The system is reachable since $\det W_r = -1 \neq 0$.

We now want to design a controller that stabilizes the dynamics and tracks a given reference value r of the lateral position of the vehicle. To do this we introduce the feedback

$$u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r,$$

and the closed loop system becomes

$$\begin{aligned} \frac{dx}{dt} &= (A - BK)x + Bk_r r = \begin{pmatrix} -\alpha k_1 & 1 - \alpha k_2 \\ -k_1 & -k_2 \end{pmatrix} x + \begin{pmatrix} \alpha k_r \\ k_r \end{pmatrix} r \\ y &= Cx + Du = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \end{aligned} \tag{6.14}$$

The closed loop system has the characteristic polynomial

$$\det(sI - A + BK) = \det \begin{pmatrix} s + \alpha k_1 & \alpha k_2 - 1 \\ k_1 & s + k_2 \end{pmatrix} = s^2 + (\alpha k_1 + k_2)s + k_1.$$

Suppose that we would like to use feedback to design the dynamics of the system to have a characteristic polynomial

$$p(s) = s^2 + 2\zeta_c\omega_c s + \omega_c^2.$$

Comparing this with the characteristic polynomial of the closed loop system we see that the feedback gains should be chosen as

$$k_1 = \omega_c^2, \quad k_2 = 2\zeta_c\omega_c - \alpha\omega_c^2.$$

To have $x_1 = r$ in the steady state it must be required that the parameter k_r equal to $k_1 = \omega_c^2$. The control law can thus be written as

$$u = k_1(r - x_1) - k_2 x_2 = \omega_c^2(r - x_1) - (2\zeta_c\omega_c - \alpha\omega_c^2)x_2.$$

∇

The example of the vehicle steering system illustrates how state feedback can be used to set the eigenvalues of the closed loop system to arbitrary values. The next example demonstrates that this is not always possible.

Example 6.5 (An unreachable system)**.** Consider the system

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} x + \begin{pmatrix}1\\ 0\end{pmatrix} u\\ y &= \begin{pmatrix}1 & 0\end{pmatrix} x\end{aligned}$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_r r.$$

The closed loop system is

$$\frac{dx}{dt} = \begin{pmatrix}-k_1 & 1-k_2\\ 0 & 0\end{pmatrix} x + \begin{pmatrix}k_r\\ 0\end{pmatrix} r.$$

This system has the characteristic polynomial

$$\det \begin{pmatrix}s+k_1 & -1+k_2\\ 0 & s\end{pmatrix} = s^2 + k_1 s = s(s+k_1),$$

which has zeros at $s = 0$ and $s = -k_1$. Since one closed loop eigenvalue is always equal to $s = 0$, independently of our choice of gains, it is not possible to obtain an arbitrary characteristic polynomial.

A visual inspection of the equations of motion shows that this system also has the property that it is not reachable. In particular, since $\dot{x}_2 = 0$, we can never steer x_2 between one value and another. Computation of the reachability matrix W_r verifies that the system is not reachable. ∇

The reachable canonical form has the property that the parameters of the system are the coefficients of the characteristic equation. It is therefore natural to consider systems on this form when solving the eigenvalue assignment problem. In the next example we investigate the case when the system is in reachable canonical form.

Example 6.6 (System in reachable canonical form)**.** Consider a system in reachable canonical form, i.e,

$$\begin{aligned}\frac{dz}{dt} &= \tilde{A}z + \tilde{B}u = \begin{pmatrix}-a_1 & -a_2 & -a_3 & \dots & -a_n\\ 1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\\ \vdots & & \ddots & \ddots & \vdots\\ 0 & & & 1 & 0\end{pmatrix} z + \begin{pmatrix}1\\ 0\\ \vdots\\ 0\\ 0\end{pmatrix} u \\ y &= \tilde{C}z = \begin{pmatrix}b_1 & b_2 & \cdots & b_n\end{pmatrix} z.\end{aligned} \tag{6.15}$$

The open loop system has the characteristic polynomial

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n,$$

as we saw in Example 6.6.

Before making a formal analysis we will investigate the block diagram of the system shown in Figure 6.4. The characteristic polynomial is given by the parameters a_k in the figure. Notice that the parameter a_k can be changed by feedback from state x_k to the input u. It is thus straight forward to change the coefficients of the characteristic polynomial by state feedback.

Having developed some intuition we will now proceed formally. Introducing the control law

$$u = -\tilde{K}z + k_r r = -\tilde{k}_1 z_1 - \tilde{k}_2 z_2 - \cdots - \tilde{k}_n z_n + k_r r, \tag{6.16}$$

the closed loop system becomes

$$\begin{aligned}\frac{dz}{dt} &= \begin{pmatrix} -a_1 - \tilde{k}_1 & -a_2 - \tilde{k}_2 & -a_3 - \tilde{k}_3 & \dots & -a_n - \tilde{k}_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} k_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} r \\ y &= \begin{pmatrix} b_n & \cdots & b_2 & b_1 \end{pmatrix} z. \end{aligned} \tag{6.17}$$

The feedback changes the elements of the first row of the A matrix, which corresponds to the parameters of the characteristic equation. The closed loop system thus has the characteristic polynomial

$$s^n + (a_l + \tilde{k}_1)s^{n-1} + (a_2 + \tilde{k}_2)s^{n-2} + \cdots + (a_{n-1} + \tilde{k}_{n-1})s + a_n + \tilde{k}_n.$$

Requiring this polynomial to be equal to the desired closed loop polynomial (6.12) we find that the controller gains should be chosen as

$$\begin{aligned}\tilde{k}_1 &= p_1 - a_1 \\ \tilde{k}_2 &= p_2 - a_2 \\ &\vdots \\ \tilde{k}_n &= p_n - a_n.\end{aligned}$$

This feedback simply replaces the parameters a_i in the system (6.17) by p_i. The feedback gain for a system in reachable canonical form is thus

$$\tilde{K} = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix}. \tag{6.18}$$

To have zero frequency gain equal to unity, the parameter k_r should be chosen as

$$k_r = \frac{a_n + \tilde{k}_n}{b_n} = \frac{p_n}{b_n}. \tag{6.19}$$

Notice that it is essential to know the precise values of parameters a_n and b_n in order to obtain the correct zero frequency gain. The zero frequency gain is thus obtained by precise calibration. This is very different from obtaining the correct steady state value by integral action, which we shall see in later sections. We thus find that it is easy to solve the eigenvalue assignment problem when the system has the structure given by equation (6.15). ∇

The General Case

We have seen through the examples how feedback can be used to design the dynamics of a system through assignment of its eigenvalues. To solve the problem in the general case, we simply change coordinates so that the system is in reachable canonical form. Consider the system (6.9). Change the coordinates by a linear transformation

$$z = Tx$$

so that the transformed system is in reachable canonical form (6.15). For such a system the feedback is given by equation (6.16), where the coefficients are given by equation (6.18). Transforming back to the original coordinates gives the feedback

$$u = -\tilde{K}z + k_r r = -\tilde{K}Tx + k_r r.$$

The results obtained can be summarized as follows.

Theorem 6.3 (Eigenvalue assignment by state feedback)**.** *Consider the system given by equation* (6.9),

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx,\end{aligned}$$

with one input and one output. Let $\lambda(s) = s^n + d_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ *be the characteristic polynomial of* A*. If the system is reachable then there exists a feedback*

$$u = -Kx + k_r r$$

that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1} s + p_n$$

and unity zero frequency gain between r and y. The feedback gain is given by

$$K = \tilde{K}T = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} \tilde{W}_r W_r^{-1} \tag{6.20}$$

$$k_r = \frac{p_n}{a_n}, \tag{6.21}$$

where a_i are the coefficients of the characteristic polynomial of the matrix A and the matrices W_r and $\tilde{W}_r$ are given by

$$W_r = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \qquad \tilde{W}_r = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1}.$$

We have thus obtained a solution to the problem and the feedback has been described by a closed form solution.

For simple problems, the eigenvalue assignment problem can be solved by introducing the elements k_i of K as unknown variables. We then compute the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)$$

and equate coefficients of equal powers of s to the coefficients of the desired characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1} + p_n.$$

This gives a system of linear equations to determine k_i. The equations can always be solved if the system is observable, exactly as we did in Example 6.4.

For systems of higher order it is more convenient to use equation (6.21), which can also be used for numeric computations. However, for large systems this is not numerically sound, because it involves computation of the characteristic polynomial of a matrix and computations of high powers of matrices. Both operations lead to loss of numerical accuracy. For this reason there are other methods that are better numerically. In MATLAB the state feedback can be computed by the procedure `place` or `acker`.

Example 6.7 (Predator prey)**.** To illustrate how state feedback might be applied, consider the problem of regulating the population of an ecosystem by modulating the food supply. We use the predator prey model introduced in Section 3.7. The dynamics for the system are given by

$$\begin{aligned} \frac{dH}{dt} &= (r_h + u)H\left(1 - \frac{H}{K}\right) - \frac{aHL}{1 + aHT_h} & H \geq 0 \\ \frac{dL}{dt} &= r_l L\left(1 - \frac{L}{kH}\right) & L \geq 0 \end{aligned}$$

We choose the following nominal parameters for the system, which correspond to the values used in previous simulations:

$$\begin{aligned} r_h &= 0.02 \quad K = 500 \quad a = 0.03 \\ r_l &= 0.01 \quad k = 0.2 \quad T_h = 5 \end{aligned}$$

We take the parameter r_h, corresponding to the growth rate for hares, as the input to the system, which we might modulate by controlling a food source for the hares. This is reflected in our model by the term $(r_h + u)$ in the first equation.

To control this system, we first linearize the system around the equilibrium point of the system, (H_e, L_e), which can be determined numerically to be $H \approx (6.5, 1.3)$. This yields a linear dynamical system

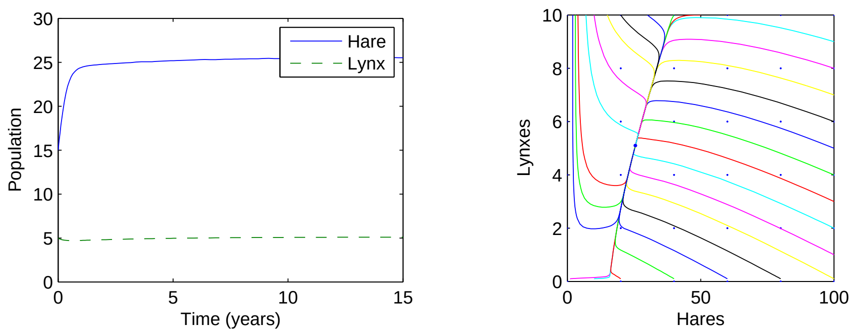
$$\frac{dd}{ddt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.001 & -0.01 \\ 0.002 & -0.01 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 6.4 \\ 0 \end{pmatrix} v$$

where $z_1 = L - L_e$, $z_2 = H - H_e$ and $v = u$. It is easy to check that the system is reachable around the equilibrium $(z, v) = (0, 0)$ and hence we can assign the eigenvalues of the system using state feedback.

Determining the eigenvalues of the closed loop system requires balancing the ability to modulate the input against the natural dynamics of the system. This can be done by the process of trial and error or by using some of the more systematic techniques discussed in the remainder of the text. For now, we simply choose the desired closed loop poles to be at $\lambda = \{-0.01, -0.02\}$. We can then solve for the feedback gains using the techniques described earlier, which results in

$$K = \begin{pmatrix} 0.005 & -0.15 \end{pmatrix}.$$

Finally, we choose the reference number of hares to be $r = 20$ and solve for the reference gain, k_r, using equation 6.13 to obtain $k_r = 0.003$.



(a) (b)

Figure 6.6: Simulation results for the controlled predatory prey system: (a) population of lynxes and hares as a function of time; (b) phase portrait for the controlled system.

Putting these steps together, our control law becomes

$$v = -Kz + k_r r.$$

In order to implement the control law, we must rewrite it using the original coordinates for the system, yielding

$$u = u_e + K(x - x_e) + k_r r = \begin{pmatrix} 0.005 & -0.15 \end{pmatrix} \begin{pmatrix} H - 6.5 \\ L - 1.3 \end{pmatrix} + 0.003\, r.$$

This rule tells us how much we should modulate r_h as a function of the current number of lynxes and hares in the ecosystem. Figure 6.6a shows a simulation of the resulting closed loop system using the parameters defined above and starting an initial population of 15 hares and 5 lynxes. Note that the system quickly stabilizes the population of lynxes at the reference value ($r = 20$). A phase portrait of the system is given in Figure 6.6b, showing how other initial conditions converge to the stabilized equilibrium population. Notice that the dynamics are very different than the natural dynamics (shown in Figure 4.6 on page 120). ∇

6.3 State Feedback Design Issues

The location of the eigenvalues determines the behavior of the closed loop dynamics and hence where we place the eigenvalue is the main design decision to be made. As with all other feedback design problems, there are tradeoffs between the magnitude of the control inputs, the robustness of the system to perturbations and the closed loop performance of the system, including step response, disturbance attenuation and noise injection. For simple systems, there are some basic guidelines that can be used and we briefly summarize them in this section.

We start by focusing on the case of second order systems, for which the closed loop dynamics have a characteristic polynomial of the form

$$\lambda(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2. \tag{6.22}$$

Since we can solve for the step and frequency response of such a system analytically, we can compute the various metrics described in Sections 5.3 and 5.3 in closed form and write the formulas for these metrics in terms of ζ and ω_0.

As an example, consider the step response for a control system with characteristic polynomial (6.22). This was derived in Section 5.4 and has the form

$$\begin{aligned}
y(t) &= \frac{k}{\omega_0^2}\left(1 - e^{-\zeta\omega_0 t}\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\omega_d t\right) && \zeta < 1\\
y(t) &= \frac{k}{\omega_0^2}\left(1 - e^{\omega_0 t} - \omega_0 t\right) && \zeta = 1\\
y(t) &= \frac{k}{\omega_0^2}\left(1 - e^{-\omega_0 t} - \frac{1}{2(1+\zeta)}e^{\omega_0(1-2\zeta)t}\right) && \zeta \geq 1.
\end{aligned}$$

We focus on the case of $0 < \zeta < 1$ and leave the other cases as an exercise for the reader.

To compute the maximum overshoot, we rewrite the output as

$$y(t) = \frac{k}{\omega_0^2}\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin(\omega_d t + \varphi)\right) \tag{6.23}$$

where $\varphi = \arccos\zeta$. The maximum overshoot will occur at the first time in which the derivative of y is zero, and hence we look for the time t_p at which

$$0 = \frac{k}{\omega_0^2}\left(\frac{\zeta\omega_0}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin(\omega_d t + \varphi) - \frac{\omega_d}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\cos(\omega_d t + \varphi)\right). \tag{6.24}$$

Eliminating the common factors, we are left with

$$\tan(\omega_d t_p + \varphi) = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$

Table 6.1: Properties of the response to reference values of a second order system for $|\zeta\| < 1$. The parameter $\varphi = \arccos \zeta$.

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Steady state error	$1/\omega_0^2$	$1/\omega_0^2$	$1/\omega_0^2$	$1/\omega_0^2$
Rise time	$T_r = 1/\omega_0 \ \cdot e^{\varphi/\tan\varphi}$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8/\omega_0$	$5.7/\omega_0$	$4/\omega_0$

Since $\varphi = \arccos \zeta$, it follows that we must have $\omega_d t_p = \pi$ (for the first non-trivial extremum) and hence $t_p = \pi/\omega_d$. Substituting this back into equation (6.23), subtracting off the steady state value and normalizing, we have

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}.$$

Similar computations can be done for the other characteristics of a step response. Table 6.1 summarizes the calculations.

One way to visualize the effect of the closed loop eigenvalues on the dynamics is to use the eigenvalue plot in Figure 6.7. This charge shows representative step and frequency responses as a function of the location of the eigenvalues. The diagonal lines in the left half plane represent the damping ratio $\zeta = \sqrt{2} \approx 0.707$, a common value for many designs.

One important consideration that is missing from the analysis so far is the amount of control authority required to obtain the desired dynamics.

Example 6.8 (Drug administration)**.** To illustrate the usage of these formulas, consider the two compartment model for drug administration, described in Section 3.6. The dynamics of the system is

$$\begin{aligned}
\frac{dc}{dt} &= \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u \\
y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x,
\end{aligned}$$

where c_1 and c_2 are the concentrations of the drug in each compartment, $k_i, i = 0, \dots, 2$ and b are parameters of the system, u is the flow rate of the drug into compartment 1 and y is the concentration of the drug in

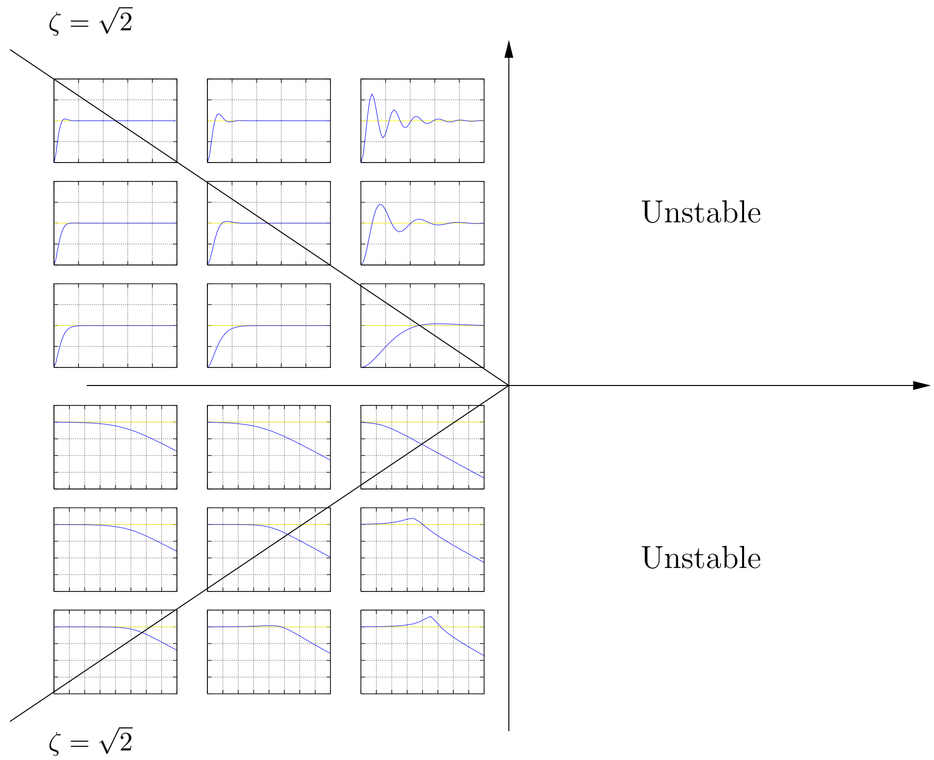


Figure 6.7: Representative step and frequency responses for second order systems. Step responses are shown in the upper half of the plot, with the location of the origin of the step response indicating the value of the eigenvalues. Frequency reponses are shown in the lower half of the plot.

compartment 2. We assume that we can measure the concentrations of the drug in each compartment and we would like to design a feedback law to maintain the output at a given reference value r.

We choose $\zeta = 0.9$ to minimize the overshoot and choose the rise time to be $T_r = 10$ min. This gives a value for $\omega_0 = 0.22$ using the formulas in Table 6.1. We then compute the gain to place the eigenvalues at this location. The response of the controller is shown in Figure 6.8 and compared with an "open loop" strategy involving administering periodic doses of the drug. ∇

Our emphasis so far has only considered second order systems. For higher order systems, eigenvalue assignment is considerably more difficult, especially when trying to account for the many tradeoffs that are present in a feedback design. We illustrate some of the main ideas using the balance

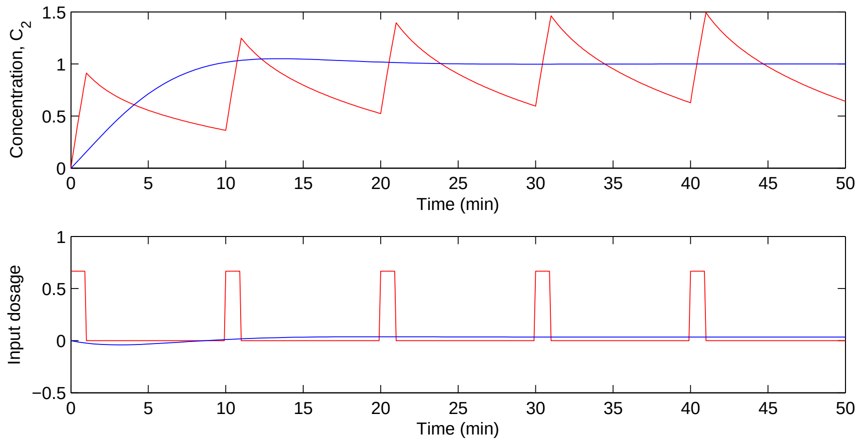


Figure 6.8: Comparison between drug administration using a sequence of doses versus continuously monitoring the concentrations and adjusting the dosage continuously.

system as an example.

To design state feedback controllers for more complicated systems, more sophisticated tools are needed. Optimal control techniques, such as the linear quadratic regular problem discussed below, are one approach that is available. One can also focus on the frequency response for performing the design, which is the subject of Chapters 8–12.

6.4 Integral Action

The controller based on state feedback achieves the correct steady state response to reference signals by careful calibration of the gain k_r. However, one of the primary uses of feedback is to allow good performance in the presence of uncertainty, and hence requiring that we have an *exact* model of the process is undesirable. An alternative to calibration is to make use of integral feedback, in which the controller uses an integrator to provide zero steady state error. The basic concept of integral feedback was already given in Section 1.5 and in Section 3.1; here we provide a more complete description and analysis.

The basic approach in integral feedback is to create a state within the controller that computes the integral of the error signal, which is then used as a feedback term. We do this by augmenting the description of the system with a new state z:

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} Ax + Bu \\ y - r \end{pmatrix} = \begin{pmatrix} Ax + Bu \\ Cx - r \end{pmatrix}$$

The state z is seen to be the integral of the error between the desired output, r, and the actual output, y. Note that if we find a compensator that stabilizes the system then necessarily we will have $\dot{z} = 0$ in steady state and hence $y = r$ in steady state.

Given the augmented system, we design a state space controller in the usual fashion, with a control law of the form

$$u = -Kx - k_i z + k_r r$$

where K is the usual state feedback term, k_i is the integral term and k_r is used to set the nominal input for the desired steady state. The resulting equilibrium point for the system is given as

$$x_e = -(A - BK)^{-1} B (k_r r - k_i z_e)$$

Note that the value of z_e is not specified, but rather will automatically settle to the value that makes $\dot{z} = y - r = 0$, which implies that at equilibrium the output will equal the reference value. This holds independently of the specific values of A, B and K, as long as the system is stable (which can be done through appropriate choice of K and k_i).

The final compensator is given by

$$\begin{aligned} u &= -Kx - k_i z + k_r r \\ \dot{z} &= y - r, \end{aligned}$$

where we have now included the dynamics of the integrator as part of the specification of the controller. This type of compensator is known as a *dynamic compensator* since it has its own internal dynamics. The following example illustrates the basic approach.

Example 6.9 (Cruise control)**.** Consider the speed control example introduced in Section 3.1 and considered further in Example 5.10.

The linearized dynamics of the process around an equilibrium point v_e, u_e are given by

$$\begin{aligned} \dot{\tilde{v}} &= a\tilde{v} - b_g g\theta + b\tilde{u} \\ y &= v = \tilde{v} + v_e, \end{aligned}$$

where $\tilde{v} = v - v_e$, $\tilde{u} = u - u_e$, m is the mass of the car and θ is the angle of the road. The constant a depends on the throttle characteristic and is given in Example 5.10.

If we augment the system with an integrator, the process dynamics become

$$\begin{aligned}\dot{\tilde{v}} &= a\tilde{v} - g\theta + b\tilde{u}\\ \dot{z} &= r - y = (r - v_e) - \tilde{v},\end{aligned}$$

or, in state space form,

$$\frac{d}{dt}\begin{pmatrix}\tilde{v}\\ z\end{pmatrix} = \begin{pmatrix}a & 0\\ -1 & 0\end{pmatrix}\begin{pmatrix}\tilde{v}\\ z\end{pmatrix} + \begin{pmatrix}b\\ 0\end{pmatrix}u + \begin{pmatrix}-g\\ 0\end{pmatrix}\theta + \begin{pmatrix}0\\ r - v_e\end{pmatrix}.$$

Note that when the system is at equilibrium we have that $\dot{z} = 0$ which implies that the vehicle speed, $v = v_e + \tilde{v}$, should be equal to the desired reference speed, r. Our controller will be of the form

$$\begin{aligned}\dot{z} &= r - y\\ u &= -k_p\tilde{v} - k_i z + k_r r\end{aligned}$$

and the gains k_p, k_i and k_r will be chosen to stabilize the system and provide the correct input for the reference speed.

Assume that we wish to design the closed loop system to have characteristic polynomial

$$\lambda(s) = s^2 + a_1 s + a_2.$$

Setting the disturbance $\theta = 0$, the characteristic polynomial of the closed loop system is given by

$$\det\bigl(sI - (A - BK)\bigr) = s^2 + (bK - a)s - bk_i$$

and hence we set

$$K = \frac{a_1 + a}{b} \qquad k_i = -\frac{a_2}{b}.$$

The resulting controller stabilizes the system and hence brings $\dot{z} = y - r$ to zero, resulting in perfect tracking. Notice that even if we have a small error in the values of the parameters defining the system, as long as the closed loop poles are still stable then the tracking error will approach zero. Thus the exact calibration required in our previous approach (using k_r) is not required. Indeed, we can even choose $k_r = 0$ and let the feedback controller do all of the work (Exercise 5).

Integral feedback can also be used to compensate for constant disturbances. Suppose that we choose $\theta \neq 0$, corresponding to climbing a (linearized) hill. The stability of the system is not affected by this external disturbance and so we once again see that the car's velocity converges to the reference speed.

This ability to handle constant disturbances is a general property of controllers with integral feedback and is explored in more detail in Exercise 6. ∇

6.5 Linear Quadratic Regulators

In addition to selecting the closed loop eigenvalue locations to accomplish a certain objective, another way that the gains for a state feedback controller can be chosen is by attempting to optimize a cost function.

The infinite horizon, linear quadratic regulator (LQR) problem is one of the most common optimal control problems. Given a multi-input linear system

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m,$$

we attempt to minimize the quadratic cost function

$$\tilde{J} = \int_0^\infty \left(x^T Q_x x + u^T Q_u u\right)\, dt$$

where $Q_x \geq 0$ and $Q_u > 0$ are symmetric, positive (semi-) definite matrices of the appropriate dimension. This cost function represents a tradeoff between the distance of the state from the origin and the cost of the control input. By choosing the matrices Q_x and Q_u, described in more detail below, we can balance the rate of convergence of the solutions with the cost of the control.

The solution to the LQR problem is given by a linear control law of the form

$$u = -Q_u^{-1} B^T P x$$

where $P \in \mathbb{R}^{n \times n}$ is a positive definite, symmetric matrix that satisfies the equation

$$PA + A^T P - PBQ_u^{-1} B^T P + Q_x = 0. \tag{6.25}$$

Equation (6.25) is called the *algebraic Riccati equation* and can be solved numerically (for example, using the `lqr` command in MATLAB).

One of the key questions in LQR design is how to choose the weights Q_x and Q_u. In order to guarantee that a solution exists, we must have $Q_x \geq 0$ and $Q_u > 0$. In addition, there are certain "observability" conditions on Q_x that limit its choice. We assume here $Q_x > 0$ to insure that solutions to the algebraic Riccati equation always exists.

To choose specific values for the cost function weights Q_x and Q_u, we must use our knowledge of the system we are trying to control. A particularly simple choice of weights is to use diagonal weights

$$Q_x = \begin{pmatrix} q_1 & 0 & \cdots \\ & \ddots & \\ 0 & \cdots & q_n \end{pmatrix} \qquad Q_u = \rho \begin{pmatrix} r_1 & 0 & \cdots \\ & \ddots & \\ \cdots & 0 & r_n \end{pmatrix}.$$

For this choice of Q_x and Q_u, the individual diagonal elements describe how much each state and input (squared) should contribute to the overall cost. Hence, we can take states that should remain very small and attach higher weight values to them. Similarly, we can penalize an input versus the states and other inputs through choice of the corresponding input weight.

6.6 Further Reading

The importance of state models and state feedback was discussed in the seminal paper by Kalman [Kal60] where the state feedback gain was obtained by solving an optimization problem that minimized a quadratic loss function. The notions of reachability and observability (next chapter) are also due to Kalman [Kal61b];see also [Gil63, KHN63]. We note that in most textbooks the term "controllability" is used instead of "reachability", but we prefer the latter term because it is more descriptive of the fundamental property of being able to reach arbitrary states.

Most undergraduate textbooks on control will contain material on state space systems, including, for example, Franklin, Powell and Emami-Naeini [FPEN05] and Ogata [Oga01]. Friedland's textbook [Fri04] covers the material in the previous, current and next chapter in considerable detail, including the topic of optimal control.

6.7 Exercises

1. Consider the system shown in Figure 6.3. Write the dynamics of the two systems as

$$\frac{dx}{dt} = Ax + Bu$$
$$\frac{dz}{dt} = Az + Bu.$$

Observe that if x and z have the same initial condition, they will always have the same state, regardless of the input that is applied. Show that this violates the definition of reachability and further show that the reachability matrix W_r is not full rank.

2. Show that the characteristic polynomial for a system in reachable canonical form is given by equation (6.7).

3. Consider a system on reachable canonical form. Show that the inverse of the reachability matrix is given by

$$\tilde{W}_r^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & 1 & a_1 & \cdots & a_{n-1} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{6.26}$$

4. Extend the argument in Section 6.1 to show that if a system is reachable from an initial state of zero, it is reachable from a non-zero initial state.

5. Build a simulation for the speed controller designed in Example 6.9 and show that with $k_r = 0$, the system still achieves zero steady state error.

6. Show that integral feedback can be used to compensate for a constant disturbance by giving zero steady state error even when $d \neq 0$.

7. Show that if $y(t)$ is the output of a linear system corresponding to input $u(t)$, then the output corresponding to an input $\dot{u}(t)$ is given by $\dot{y}(t)$. (Hint: use the definition of the derivative: $\dot{y}(t) = \lim_{\epsilon \to 0}\big(y(t+\epsilon) - y(t)\big)/\epsilon$.)