Chapter 4

Dynamic Behavior

Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?

Talk given by Edward Lorenz, December 1972 meeting of the American Association for the Advancement of Science.

In this chapter we give a broad discussion of the behavior of dynamical systems, focused on systems modeled by nonlinear differential equations. This allows us to discuss equilibrium points, stability, limit cycles and other key concepts of dynamical systems. We also introduce some methods for analyzing global behavior of solutions.

4.1 Solving Differential Equations

In the last chapter, we saw that one of the methods of modeling dynamical systems is through the use of ordinary differential equations (ODEs). A state space, input/output system has the form

$$\begin{aligned} \frac{dx}{dt} &= f(x, u) \\ y &= h(x, u), \end{aligned} \tag{4.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and $y \in \mathbb{R}^q$ is the output. The smooth maps $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$ represent the dynamics and measurements for the system. We will focus in this text on single input, single output (SISO) systems, for which $p = q = 1$.

We begin by investigating systems in which the input has been set to a function of the state, $u = \alpha(x)$. This is one of the simplest types of feedback, in which the system regulates its own behavior. The differential equations in this case become

$$\frac{dx}{dt} = f(x, \alpha(x)) = F(x). \tag{4.2}$$

In order to understand the dynamic behavior of this system, we need to analyze the features of the solutions of equation (4.2). While in some simple situations we can write down the solutions in analytical form, more often we must rely on computational approaches. We begin by describing the class of solutions for this problem.

Initial Value Problems

We say that $x(t)$ is a *solution* of the differential equation (4.2) on the time interval $t_0 \in \mathbb{R}$ to $t_f \in \mathbb{R}$ if

$$\frac{dx(t)}{dt} = F(x(t)) \quad \text{for all } t_0 \leq t \leq t_f.$$

A given differential equation may have many solutions. We will most often be interested in the *initial value problem*, where $x(t)$ is prescribed at a given time $t_0 \in \mathbb{R}$ and we wish to find a solution valid for all *future* time, $t > t_0$.

We say that $x(t)$ is a solution of the differential equation (4.2) *with initial value $x_0 \in \mathbb{R}^n$ at $t_0 \in \mathbb{R}$* if

$$x(t_0) = x_0 \quad \text{and} \quad \frac{dx(t)}{dt} = F(x(t)) \quad \text{for all } t_0 \leq t \leq t_f.$$

For most differential equations we will encounter, there is a *unique* solution that is defined for $t_0 \leq t \leq t_f$. The solution may defined for all time $t \geq t_0$, in which case we take $t_f = \infty$. Because we will primarily be interested in solutions of the initial value problem for ODEs, we will often refer to this simply as the solution of an ODE.

We will usually assume that t_0 is equal to 0. In the case when F is independent of time (as in equation (4.2)), we can do so without loss of generality by choosing a new independent (time) variable, $\tau = t - t_0$ (Exercise 2).

Example 4.1 (Damped oscillator)**.** Consider a damped, linear oscillator, introduced in Example 2.4. The equations of motion for the system are

$$m\ddot{q} + c\dot{q} + kq = 0,$$

where q is the displacement of the oscillator from its rest position. We assume that $c^2 < 4km$, corresponding to a lightly damped system (the

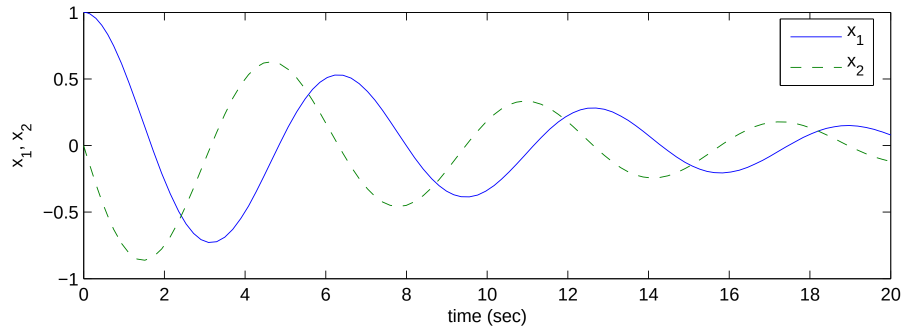


Figure 4.1: Response of the damped oscillator to the initial condition $x_0 = (1, 0)$.

reason for this particular choice will become clear later). We can rewrite this in state space form by setting $x_1 = q$ and $x_2 = \dot{q}$, giving

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2.\end{aligned}$$

In vector form, the right hand side can be written as

$$F(x) = \begin{pmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{pmatrix}.$$

The solution to the initial value problem can be written in a number of different ways and will be explored in more detail in Chapter 5. Here we simply assert that the solution can be written as

$$\begin{aligned}x_1(t) &= e^{-\frac{ct}{2m}}\left(x_{10}\cos\omega_d t + \left(\frac{cx_{10} + 2mx_{20}}{2m\omega_d}\right)\sin\omega_d t\right)\\ x_2(t) &= e^{-\frac{ct}{2m}}\left(x_{20}\cos\omega_d t - \left(\frac{2kx_{10} + cx_{20}}{2m\omega_d}\right)\sin\omega_d t\right),\end{aligned}$$

where $x_0 = (x_{10}, x_{20})$ is the initial condition and $\omega_d = \sqrt{4km - c^2}/2m$. This solution can be verified by substituting it into the differential equation. We see that the solution is explicitly dependent on the initial condition and it can be shown that this solution is unique. A plot of the initial condition response is shown in Figure 4.1. We note that this form of the solution only holds for $c^2 - 4km < 0$, corresponding to an "underdamped" oscillator. ∇

Numerical Solutions

One of the benefits of the computer revolution that is that it is very easy to obtain a numerical solution of a differential equation when the initial condition is given. A nice consequence of this is as soon as we have a model in the form of equation (4.2), it is straightforward to generate the behavior of x for different initial conditions, as we saw briefly in the previous chapter.

Modern computing environments such as LabVIEW, MATLAB and Mathematica allow simulation of differential equations as a basic operation. For example, these packages provides several tools for representing, simulating, and analyzing ordinary differential equations of the form in equation (4.2). To define an ODE in MATLAB or LabVIEW, we define a function representing the right hand side of equation (4.2):

```
function xdot = system(t, x)
  xdot(1) = F1(x);
  xdot(2) = F2(x);
  ...
```

Each expression `Fi(x)` takes a (column) vector `x` and returns the ith element of the differential equation. The second argument to the function `system`, `t`, represents the current time and allows for the possibility of time-varying differential equations, in which the right hand side of the ODE in equation (4.2) depends explicitly on time.

ODEs defined in this fashion can be simulated by using the `ode45` command:

```
ode45('file', [0,T], [x10, x20, ..., xn0])
```

The first argument is the name of the function defining the ODE, the second argument gives the time interval over which the simulation should be performed and the final argument gives the vector of initial conditions. Similar capabilities exist in other packages such as Octave and Scilab.

Example 4.2 (Balance system)**.** Consider the balance system given in Example 2.1 and reproduced in Figure 4.2a. Suppose that a coworker has designed a control law that will hold the position of the system steady in the upright position at $p = 0$. The form of the control law is

$$F = -Kx,$$

where $x = (p, \theta, \dot{p}, \dot{\theta}) \in \mathbb{R}^4$ is the state of the system, F is the input, and $K = (k_1, k_2, k_3, k_4)$ is the vector of "gains" for the control law.

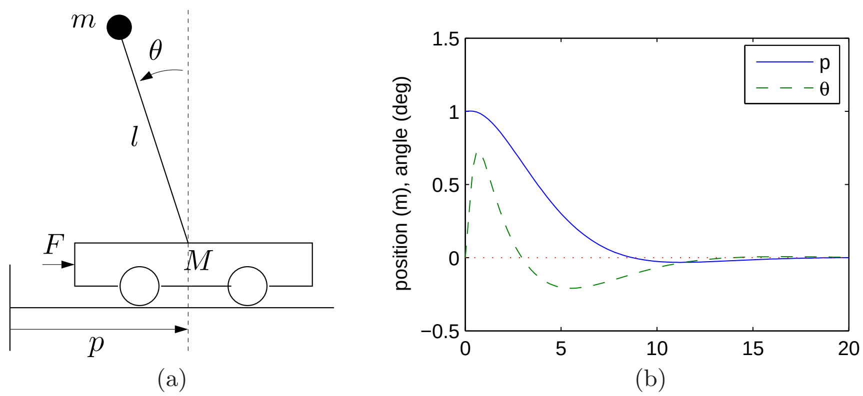


Figure 4.2: Balance system: (a) simplified diagram and (b) initial condition response.

The equations of motion for the system, in state space form, are

$$
\frac{d}{dt}\begin{pmatrix} p \\ \theta \\ \dot p \\ \dot\theta \end{pmatrix} = \begin{pmatrix} \dot p \\ \dot\theta \\ \dfrac{-ml\sin\theta\dot\theta^2 + mg(ml^2/J_t)\sin\theta\cos\theta - c\dot p + u}{M_t - m(ml^2/J_t)\cos^2\theta} \\ \dfrac{-ml^2\sin\theta\cos\theta\dot\theta^2 + M_t gl\sin\theta + cl\cos\theta\dot p + \gamma\dot\theta + l\cos\theta u}{J_t(M_t/m) - m(l\cos\theta)^2} \end{pmatrix}
$$

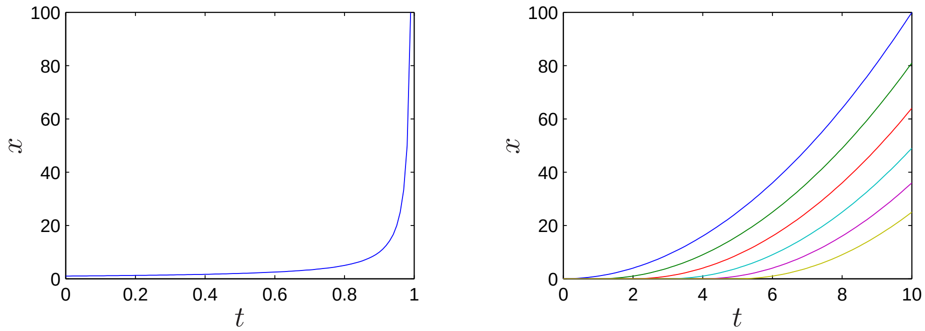
$$
y = \begin{pmatrix} p \\ \theta \end{pmatrix},
$$

where $M_t = M + m$ and $J_t = J + ml^2$. We use the following parameters for the system (corresponding roughly to a human being balanced on a stabilizing cart):

$$
\begin{aligned}
M &= 10 \text{ kg} & m &= 80 \text{ kg} & c &= 0.1 \text{ Ns/m} \\
J &= 100 \text{ kg m}^2/\text{s}^2 & l &= 1 \text{ m} & g &= 9.8 \text{ m/s}^2
\end{aligned}
$$

$$
K = \begin{pmatrix} -1 & 120 & -4 & 20 \end{pmatrix}
$$

This system can now be simulated using MATLAB or a similar numerical tool. The results are shown in Figure 4.2b, with initial condition $x_0 = (1, 0, 0, 0)$. We see from the plot that after an initial transient, the angle and position of the system return to zero (and remain there). ∇



(a) (b)

Figure 4.3: Solutions to the differential equations (4.3) and (4.4).

Existence and Uniqueness

Without imposing some conditions on the function F, the differential equation (4.2) may not have a solution for all t, and there is no guarantee that the solution is unique. We illustrate these possibilities with two examples.

Example 4.3 (Finite escape time)**.** Let $x \in \mathbb{R}$ and consider the differential equation

$$\frac{dx}{dt} = x^2 \tag{4.3}$$

with initial condition $x(0) = 1$. By differentiation we can verify that the function

$$x(t) = \frac{1}{1-t} \tag{4.4}$$

satisfies the differential equation and it also satisfies the initial condition. A graph of the solution is given in Figure 4.3a; notice that the solution goes to infinity as t goes to 1. Thus the solution only exists in the time interval $0 \leq t < 1$. ∇

Example 4.4 (No unique solution)**.** Let $x \in \mathbb{R}$ and consider the differential equation

$$\frac{dx}{dt} = \sqrt{x}$$

with initial condition $x(0) = 0$. We can show that the function

$$x(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a \\ \frac{1}{4}(t-a)^2 & \text{if } t > a \end{cases}$$

satisfies the differential equation for all values of the parameter $a \geq 0$. To see this, we differentiate $x(t)$ to obtain

$$\frac{dx}{dt} = \begin{cases} 0 & \text{if } 0 \leq t \leq a \\ \frac{1}{2}(t-a) & \text{if } t > a \end{cases}$$

and hence $\dot{x} = \sqrt{x}$ for all $t \geq 0$ with $x(0) = 0$. A graph of some of the possible solutions is given in Figure 4.3b. Notice that in this case there are many solutions to the differential equation. ∇

These simple examples show that there may be difficulties even with simple differential equations. Existence and uniqueness can be guaranteed by requiring that the function F has the property that for some fixed $c \in \mathbb{R}$

$$\|F(x) - F(y)\| < c\|x - y\| \quad \text{for all } x, y,$$

which is called *Lipschitz continuity*. A sufficient condition for a function to be Lipschitz is that the Jacobian, $\partial F/\partial x$, is uniformly bounded for all x. The difficulty in Example 4.3 is that the derivative $\partial F/\partial x$ becomes large for large x and the difficulty in Example 4.4 is that the derivative $\partial F/\partial x$ is infinite at the origin.

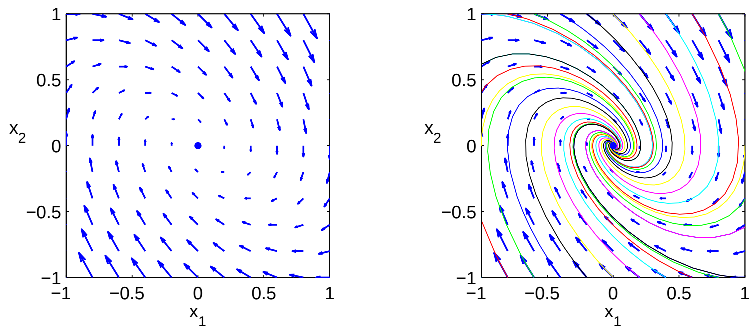
4.2 Qualitative Analysis

The qualitative behavior of nonlinear systems is important for understanding some of the key concepts of stability in nonlinear dynamics. We will focus on an important class of systems known as planar dynamical systems. These systems have two state variables $x \in \mathbb{R}^2$, allowing their solutions to be plotted in the (x_1, x_2) plane. The basic concepts that we describe hold more generally and can be used to understand dynamical behavior in higher dimensions.

Phase Portraits

A convenient way to understand the behavior of dynamical systems with state $x \in \mathbb{R}^2$ is to plot the *phase portrait* of the system, briefly introduced in Chapter 2. We start by introducing the concept of a vector field. For a system of ordinary differential equations

$$\frac{dx}{dt} = F(x),$$



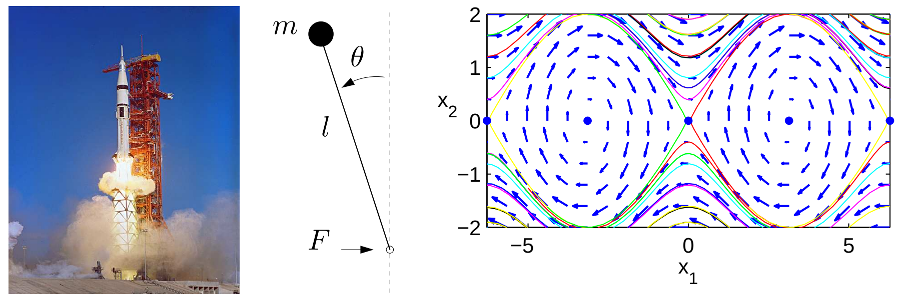
(a) (b)

Figure 4.4: Vector field plot (a) and phase portrait (b) for a damped oscillator. This plots were produced using the `phaseplot` command in MATLAB.

the right hand side of the differential equation defines at every $x \in \mathbb{R}^n$ a velocity $F(x) \in \mathbb{R}^n$. This velocity tells us how x changes and can be represented as a vector $F(x) \in \mathbb{R}^n$. For planar dynamical systems, we can plot these vectors on a grid of points in the plane and obtain a visual image of the dynamics of the system, as shown in Figure 4.4a.

A phase portrait is constructed by plotting the flow of the vector field corresponding to the planar dynamical system. That is, for a set of initial conditions, we plot the solution of the differential equation in the plane $\mathbb{R}^2$. This corresponds to following the arrows at each point in the phase plane and drawing the resulting trajectory. By plotting the resulting trajectories for several different initial conditions, we obtain a phase portrait, as show in Figure 4.4b.

Phase portraits give us insight into the dynamics of the system by showing us the trajectories plotted in the (two dimensional) state space of the system. For example, we can see whether all trajectories tend to a single point as time increases or whether there are more complicated behaviors as the system evolves. In the example in Figure 4.4, corresponding to a damped oscillator, we see that for all initial conditions the system approaches the origin. This is consistent with our simulation in Figure 4.1 (also for a damped oscillator), but it allows us to infer the behavior for all initial conditions rather than a single initial condition. However, the phase portrait does not readily tell us the rate of change of the states (although this can be inferred from the length of the arrows in the vector field plot).



(a) (b) (c)

Figure 4.5: An inverted pendulum: (a) motivating application, a Saturn rocket; (b) a simplified diagram of the model; (c) phase portrait. In the phase portrait, the equilibrium points are marked by solid dots along the $x_2 = 0$ line.

Equilibrium Points

An *equilibrium point* of a dynamical system represents a stationary condition for the dynamics. We say that a state x_e is an equilibrium point for a dynamical system

$$\frac{dx}{dt} = F(x)$$

if $F(x_e) = 0$. If a dynamical system has an initial condition $x(0) = x_e$ then it will stay at the equilibrium point: $x(t) = x_e$ for all $t \geq 0$.[1]

Equilibrium points are one of the most important features of a dynamical system since they define the states corresponding to constant operating conditions. A dynamical system can have zero, one or more equilibrium points.

Example 4.5 (Inverted pendulum)**.** Consider the inverted pendulum in Figure 4.5, which is a portion of the balance system we considered in Chapter 2. The inverted pendulum is a simplified version of the problem of stabilizing a rocket: by applying forces at the base of the rocket, we seek to keep the rocket stabilized in the upright position. The state variables are the angle $\theta = x_1$ and the angular velocity $d\theta/dt = x_2$, the control variable is the acceleration u of the pivot, and the output is the angle θ.

For simplicity we ignore any damping ($\gamma = 0$) and assume that $mgl/J_t = 1$ and $ml/J_t = 1$, where $J_t = J + ml^2$, so that the dynamics (equation (2.8))

[1] We take $t_0 = 0$ from here on.

become

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} x_2 \\ \sin x_1 + u\cos x_1 \end{pmatrix} \\ y &= x_1. \end{aligned} \tag{4.5}$$

This is a nonlinear time-invariant system of second order.

The equilibrium points for the system are given by

$$x_e = \begin{pmatrix} 0 \\ \pm n\pi \end{pmatrix}$$

where $n = 0, 1, 2, \ldots$. The equilibrium points for n even correspond to the pendulum pointing up and those for n odd correspond to the pendulum hanging down. A phase portrait for this system (without corrective inputs) is shown in Figure 4.5c. The phase plane shown in the figure is $\mathbb{R} \times \mathbb{R}$, which results in our model having an infinite number of equilibria, corresponding to 0, $\pm\pi$, $\pm 2\pi$, ... ∇

Limit Cycles

Nonlinear systems can exhibit very rich behavior. Consider the differential equation

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2 - x_1(1 - x_1^2 - x_2^2) \\ \frac{dx_2}{dt} &= x_1 - x_2(1 - x_1^2 - x_2^2). \end{aligned} \tag{4.6}$$

The phase portrait and time domain solutions are given in Figure 4.6. The figure shows that the solutions in the phase plane converge to a circular trajectory. In the time domain this corresponds to an oscillatory solution. Mathematically the circle is called a *limit cycle*. More formally, we call a solution $x(t)$ a limit cycle of period $T > 0$ if $x(t + T) = x(t)$ for all $t \in \mathbb{R}$.

Example 4.6 (Predator prey)**.** Consider the predator prey example introduced in Section 3.7. The dynamics for the system are given by

$$\begin{aligned} \frac{dH}{dt} &= r_h H\left(1 - \frac{H}{K}\right) - \frac{aHL}{1 + aHT_h} \qquad & H \geq 0 \\ \frac{dL}{dt} &= r_l L\left(1 - \frac{L}{kH}\right) & L \geq 0. \end{aligned}$$

The phase portrait for this system is shown in Figure 4.7. In addition to the two equilibrium points, we see a limit cycle in the diagram. This limit cycle

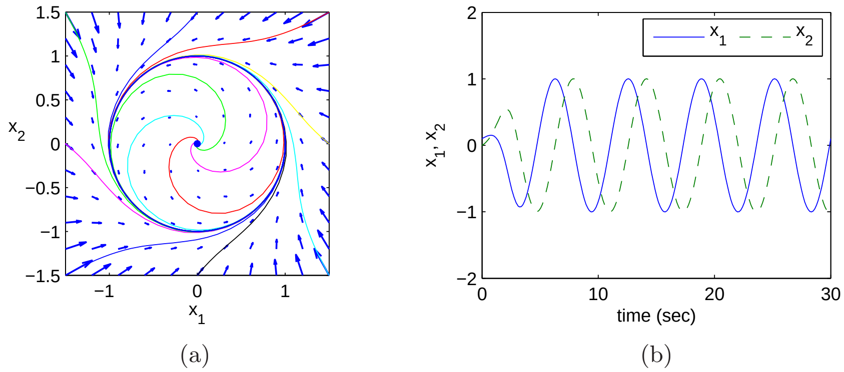


Figure 4.6: Phase portrait and time domain simulation for a system with a limit cycle.

is *attracting* or *stable* since initial conditions near the limit cycle approach it as time increases. It divides the phase space into two different regions: one inside the limit cycle in which the size of the population oscillations growth with time (until they rich the limit cycle) and one outside the limit cycle in which they decay. ∇

There are methods for determining limit cycles for second order systems, but for general higher order systems we have to resort to computational analysis. Computer algorithms find limit cycles by searching for periodic trajectories in state space that satisfy the dynamics of the system. In many situations, stable limit cycles can be found by simulating the system with different initial conditions.

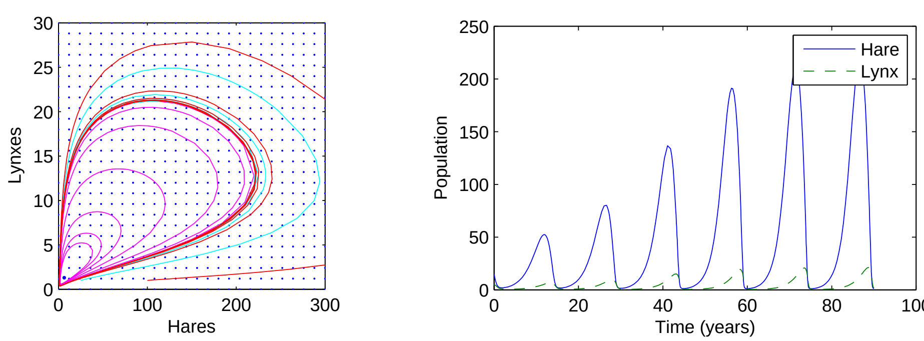
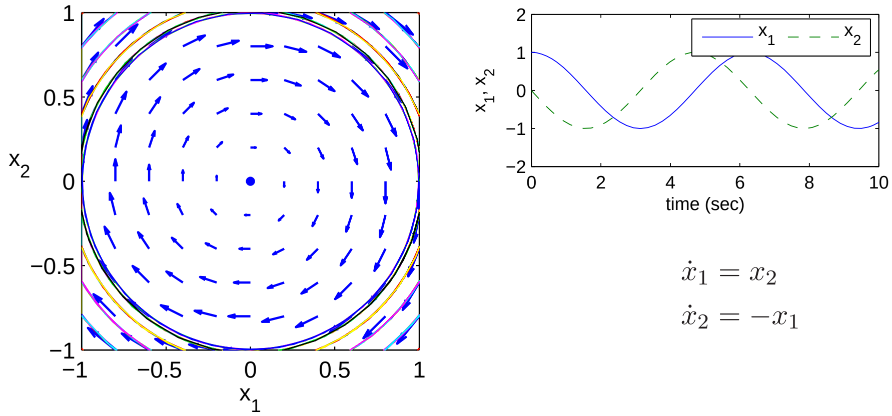


Figure 4.7: Phase portrait and time domain simulation for the predator prey system.



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

Figure 4.8: Phase portrait and time domain simulation for a system with a single stable equilibrium point.

4.3 Stability

The stability of an equilibrium point determines whether or not solutions nearby the equilibrium point remain nearby, get closer, or move further away.

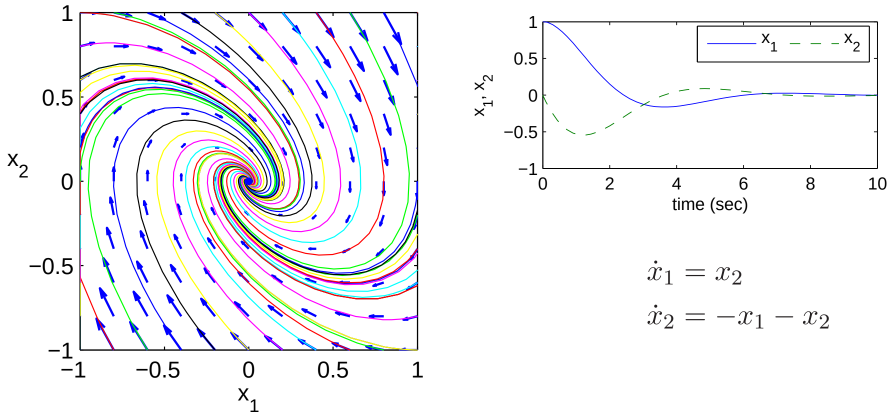
Definitions

An equilibrium point is *stable* if initial conditions that start near an equilibrium point stay near that equilibrium point. Formally, we say that an equilibrium point x_e is stable if for all $\epsilon > 0$, there exists an $\delta > 0$ such that

$$\|x(0) - x_e\| < \delta \quad \Longrightarrow \quad \|x(t) - x_e\| < \epsilon \quad \text{for all } t > 0.$$

Note that this definition does not imply that $x(t)$ gets closer to x_e as time increases, but just that it stays nearby. Furthermore, the value of δ may depend on ϵ, so that if we wish to stay very close to the equilibrium point, we may have to start very, very close ($\delta \ll \epsilon$). This type of stability is sometimes called stability "in the sense of Lyapunov". If a system is stable in the sense of Lyapunov and the trajectories don't converge to the equilibrium point, we say that the equilibrium point is *neutrally stable*.

An example of a neutrally stable equilibrium point is shown in Figure 4.8. From the phase portrait, we see that if we start near the equilibrium then we stay near the equilibrium. Indeed, for this example, given any ϵ that



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

Figure 4.9: Phase portrait and time domain simulation for a system with a single asymptotically stable equilibrium point.

defines the range of possible initial conditions, we can simply choose $\delta = \epsilon$ to satisfy the definition of stability.

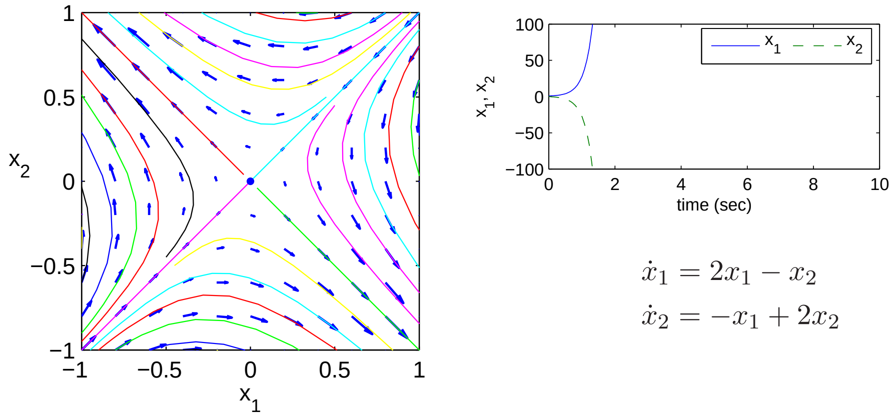
An equilibrium point x_e is (locally) *asymptotically stable* if it is stable in the sense of Lyapunov and also $x(t) \to x_e$ as $t \to \infty$ for $x(t)$ sufficiently close to x_e. This corresponds to the case where all nearby trajectories converge to the equilibrium point for large time. Figure 4.9 shows an example of an asymptotically stable equilibrium point. Note from the phase portraits that not only do all trajectories stay near the equilibrium point at the origin, but they all approach the origin as t gets large (the directions of the arrows on the phase plot show the direction in which the trajectories move).

An equilibrium point is *unstable* if it is not stable. More specifically, we say that an equilibrium point is unstable if given some $\epsilon > 0$, there does not exist a $\delta > 0$ such that if $\|x(0) - x_e\| < \delta$ then $\|x(t) - x_e\| < \epsilon$ for all t. An example of an unstable equilibrium point is shown in Figure 4.10.

The definitions above are given without careful description of their domain of applicability. More formally, we define an equilibrium point to be *locally* stable (or asymptotically stable) if it is stable for all initial conditions $x \in B_r(x_e)$ where

$$B_r(x_e) = \{x : \|x - x_e\| < \delta\}$$

is a ball of radius r around x_e and $r > 0$. A system is globally stable if it stable for all $r > 0$. Systems whose equilibrium points are only locally stable can have interesting behavior away from equilibrium points, as we explore in the next section.



$$\dot{x}_1 = 2x_1 - x_2$$
$$\dot{x}_2 = -x_1 + 2x_2$$

Figure 4.10: Phase portrait and time domain simulation for a system with a single unstable equilibrium point.

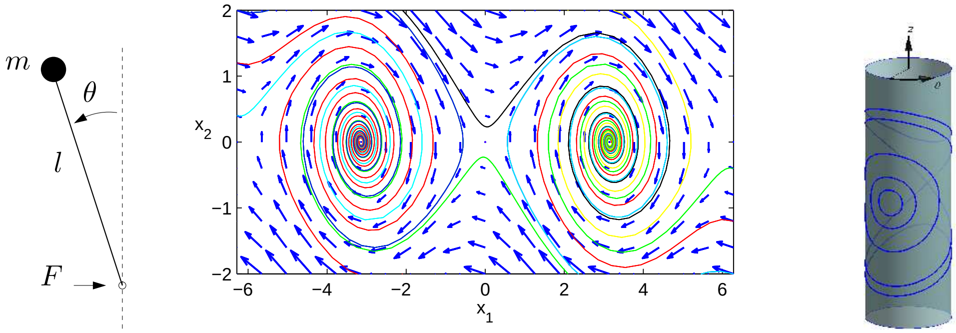
For planar dynamical systems, equilibrium points have been assigned names based on their stability type. An asymptotically stable equilibrium point is called a *sink* or sometimes an *attractor*. An unstable equilibrium point can either be a *source*, if all trajectories lead away from the equilibrium point, or a *saddle*, if some trajectories lead to the equilibrium point and others move away (this is the situation pictured in Figure 4.10). Finally, an equilibrium point which is stable but not asymptotically stable (such as the one in Figure 4.8) is called a *center*.

Example 4.7 (Damped inverted pendulum)**.** Consider the damped inverted pendulum introduced Example 2.2. The equations of motion are

$$\frac{d}{dt}\begin{pmatrix}\theta \\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\dot{\theta} \\ \frac{mgl}{J_t}\sin\theta - \frac{\gamma}{J_t}\dot{\theta} + \frac{l}{J_t}\cos\theta\, u\end{pmatrix} \tag{4.7}$$

A phase diagram for the system is shown in Figure 4.11. The equilibrium point at $x = (0,0)$ is a locally unstable equilibrium point (corresponding to the inverted position). The equilibrium points at $x = (\pm\pi, 0)$ correspond to locally asymptotically stable equilibrium points. An example of locally stable (but not asymptotically) stable points is the undamped pendulum, shown in Figure 4.5 on page 119.

It is much more natural to describe the pendulum in terms of an angle φ and an angular velocity. The phase space is then a manifold $S^1 \times \mathbb{R}$, where S^1 represents the unit circle. Using this description, the dynamics evolve on a cylinder and there are only two equilibria, as shown in Figure 4.11c. ∇



(a) (b) (c)

Figure 4.11: Phase portrait for a damped inverted pendulum: (a) diagram of the inverted pendulum system; (b) phase portrait with $\theta \in [2\pi, 2\pi]$; (c) phase portrait with θ periodic.

Example 4.8 (Congestion control)**.** The model for congestion control in a network consisting of a single computer connected to a router, introduced in Example 2.12, is given by

$$\begin{aligned} \frac{dx}{dt} &= -b\frac{x^2}{2} + (b_{\max} - b) \\ \frac{db}{dt} &= x - c, \end{aligned}$$

where x is the transmission rate from the source and b is the buffer size of the router. The phase portrait is shown in Figure 4.12 for two different parameter values. In each case we see that the system converges to an equilibrium point in which the full capacity of the link is used and the router buffer is not at capacity. The horizontal and vertical lines on the plots correspond to the router buffer limit and link capacity limits. When the system is operating outside these bounds, packets are being lost.

We see from the phase portrait that the equilibrium point at

$$x^* = c \qquad b^* = \frac{2b_{\max}}{2 + c^2},$$

is stable, since all initial conditions result in trajectories that converge to this point. Note also that some of the trajectories cross outside of the region where $x > 0$ and $b > 0$, which is not possible in the actual system; this shows some of the limits of this model away from the equilibrium points. A more accurate model would use additional nonlinear elements in the model to insure that the quantities in the model always stayed positive. ∇

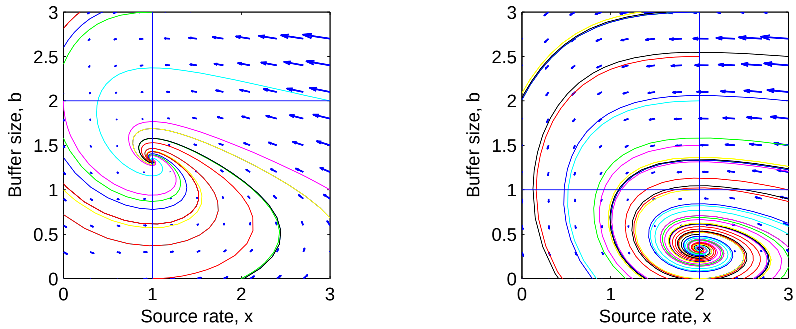


Figure 4.12: Phase portraits for a congestion control protocol running with a single source computer: (a) with router buffer size $b_{\max} = 2$ Mb and link capacity $c = 1$ Mb/sec and (b) router buffer size $b_{\max} = 1$ Mb and link capacity $c = 2$ Mb/sec.

Stability Analysis via Linear Approximation

An important feature of differential equations is that it is often possible to determine the local stability of an equilibrium point by approximating the system by a linear system. We shall explore this concept in more detail later, but the following examples illustrates the basic idea.

Example 4.9 (Inverted pendulum)**.** Consider again the inverted pendulum, whose dynamics are given by

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ \frac{mgl}{J_t}\sin x_1 - \frac{\gamma}{J_t}x_2 + \frac{l}{J_t}\cos x_1\, u \end{pmatrix}$$
$$y = x_1,$$

where we have defined the state as $x = (\theta, \dot\theta)$. We first consider the equilibrium point at $x = (0, 0)$, corresponding to the straight up position. If we assume that the angle $\theta = x_1$ remains small, then we can replace $\sin x_1$ with x_1 and $\cos x_1$ with 1, which gives the approximate system

$$\frac{dx}{dt} = \begin{pmatrix} x_2 \\ \frac{mgl}{J_t}x_1 - \frac{\gamma}{J_t}x_2 + \frac{l}{J_t}u \end{pmatrix} \qquad (4.8)$$
$$y = x_1.$$

Intuitively, this system should behave similarly to the more complicated model as long as x_1 is small. In particular, it can be verified that the system (4.5) is unstable by plotting the phase portrait or computing the eigenvalues of the system matrix (as described in the next chapter).

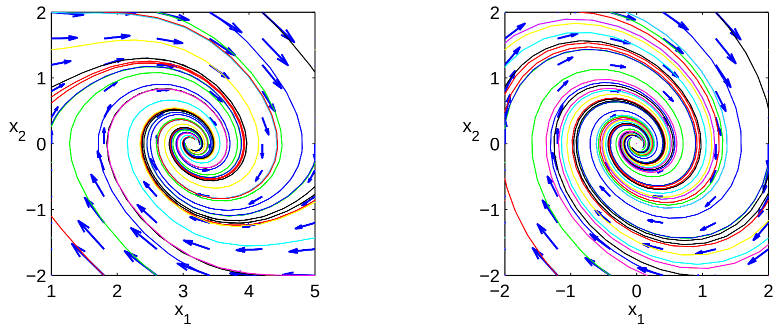


Figure 4.13: Comparison between the phase portraits for the full nonlinear systems (left) and its linear approximation around the origin (right).

We can also approximate the system around the stable equilibrium point at $x = (\pi, 0)$. In this case we have to expand $\sin x_1$ and $\cos x_1$ around $x_1 = \pi$, according to the expansions

$$\sin(\pi + \theta) = -\sin\theta \approx -\theta \qquad \cos(\pi + \theta) = \cos(\theta) \approx 1.$$

If we define $z_1 = x_1 - \pi$ and $z_2 = x_2$, the resulting approximate dynamics are given by

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} z_2 \\ -\frac{mgl}{J_t} z_1 - \frac{\gamma}{J_t} z_2 + \frac{l}{J_t} u \end{pmatrix} \\ y &= z_1. \end{aligned} \tag{4.9}$$

Note that $z = (0, 0)$ is the equilibrium point for this system and that it has the same basic form as the dynamics shown in Figure 4.9. Figure 4.13 shows the phase portraits for the original system and the approximate system around the corresponding equilibrium points. Note that they are very similar (although not exactly the same). More generally, it can be shown that if a linear approximation has either asymptotically stable or unstable equilibrium point, then the local stability of the original system must be the same. ∇

The fact that a linear model can sometimes be used to study the behavior of a nonlinear system near an equilibrium point is a powerful one. Indeed, we can take this even further and use local linear approximations of a nonlinear system to design a feedback law that keeps the system near its equilibrium point (design of dynamics). By virtue of the fact that the closed loop dynamics have been chosen to stay near the equilibrium, we can even use the linear approximation to design the feedback that ensures this condition is true!

Lyapunov Functions

A powerful tool for determining stability is the use of Lyapunov functions. A *Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ is an energy-like function that can be used to determine stability of a system. Roughly speaking, if we can find a non-negative function that always decreases along trajectories of the system, we can conclude that the minimum of the function is a stable equilibrium point (locally).

To describe this more formally, we start with a few definitions. We say that a continuous function $V(x)$ is *positive definite* if $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$. We will often write this as $V(x) \succ 0$. Similarly, a function is *negative definite* if $V(x) < 0$ for all $x \neq 0$ and $V(0) = 0$. We say that a function $V(x)$ is *positive semidefinite* if $V(x)$ can be zero at points other than $x = 0$ but otherwise $V(x)$ is strictly positive. We write this as $V(x) \succeq 0$ and define negative semi-definite functions analogously.

To illustrate the difference between a positive definite function and a positive semi-definite function, suppose that $x \in \mathbb{R}^2$ and let

$$V_1(x) = x_1^2 \qquad V_2(x) = x_1^2 + x_2^2.$$

Both V_1 and V_2 are always non-negative. However, it is possible for V_1 to be zero even if $x \neq 0$. Specifically, if we set $x = (0, c)$ where $c \in \mathbb{R}$ is any non-zero number, then $V_1(x) = 0$. On the other hand, $V_2(x) = 0$ if and only if $x = (0, 0)$. Thus $V_1(x) \succeq 0$ and $V_2(x) \succ 0$.

We can now characterize the stability of a system

$$\frac{dx}{dt} = F(x) \qquad x \in \mathbb{R}^n.$$

Theorem 4.1. *Let $V(x)$ be a non-negative function on $\mathbb{R}^n$ and let $\dot{V}$ represent the time derivative of V along trajectories of the system dynamics:*

$$\frac{dV(x)}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}F(x).$$

Let $B_r = B_r(0)$ be a ball of radius r around the origin. If there exists $r > 0$ such that $\dot{V} \preceq 0$ for all $x \in B_r$, then $x = 0$ is locally stable in the sense of Lyapunov. If $\dot{V} \prec 0$ in B_r, then $x = 0$ is locally asymptotically stable.

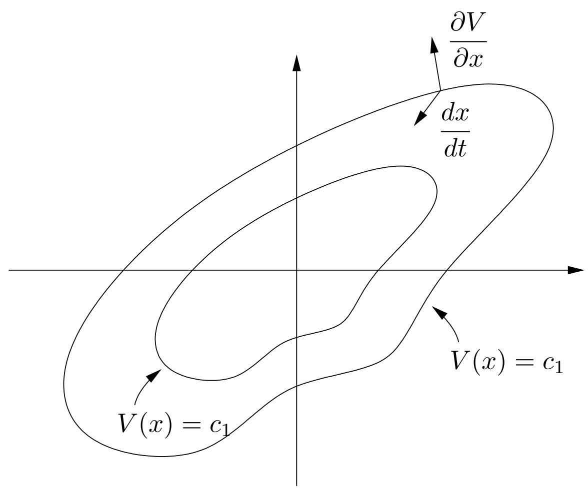


Figure 4.14: Geometric illustration of Lyapunov's stability theorem. The dashed ellipses correspond to level sets of the Lyapunov function; the solid line is a trajectory of the system.

If V satisfies one of the conditions above, we say that V is a (local) *Lyapunov function* for the system. These results have a nice geometric interpretation. The level curves for a positive definite function are closed contours as shown in Figure 4.14. The condition that $\dot{V}(x)$ is negative simply means that the vector field points towards lower level curves. This means that the trajectories move to smaller and smaller values of V and, if $\dot{V} \prec 0$, then x must approach 0.

A slightly more complicated situation occurs if $\dot{V}(x) \preceq 0$. In this case it is possible that $\dot{V}(x) = 0$ when $x \neq 0$ and hence x could stop decreasing in value. The following example illustrates these two cases.

Example 4.10. Consider the second order system

$$\begin{aligned}\frac{dx_1}{dt} &= -ax_1 \\ \frac{dx_2}{dt} &= -bx_1 - cx_2.\end{aligned}$$

Suppose first that $a, b, c > 0$ and consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2.$$

Taking the derivative of V and substituting the dynamics, we have

$$\frac{dV(x)}{dt} = -ax_1^2 - bx_1x_2 - cx_2^2.$$

To check whether this is negative definite, we complete the square by writing

$$\frac{dV}{dt} = -a(x_1 + \frac{b}{a}x_2)^2 - (c - \frac{b^2}{a})x_2^2.$$

Clearly $\dot{V} \prec 0$ if $a > 0$ and $(c - \frac{b^2}{a}) > 0$.

Suppose now that $a, b, c > 0$ and $c = b^2/a$. Then the derivative of the Lyapunov function becomes

$$\frac{dV}{dt} = -a(x_1 + \frac{b}{a}x_2)^2 \leq 0.$$

This function is not negative definite since if $x_1 = -\frac{b}{a}x_2$ then $\dot{V} = 0$ but $x \neq 0$. Hence we cannot include asymptotic stability, but we *can* say the system is stable (in the sense of Lyapunov).

The fact that $\dot{V}$ is not negative definite does not mean that this system is not asymptotically stable. As we shall see in Chapter 5, we can check stability of a linear system by looking at the eigenvalues of the dynamics matrix for the model

$$\frac{dx}{dt} = \begin{pmatrix} -a & 0 \\ -b & -c \end{pmatrix} x.$$

By inspection (since the system is lower triangular), the eigenvalues are $\lambda_1 = -a < 0$ and $\lambda_2 = -c < 0$, and hence the system can be shown to be asymptotically stable.

To demonstrate asymptotic stability using Lyapunov functions, we must try a different Lyapunov function candidate. Suppose we try

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \frac{b}{c-a}x_1)^2.$$

It is easy to show that $V(x) \succ 0$ since $V(x) \geq 0$ for all x and $V(x) = 0$ implies that $x_1 = 0$ and $x_2 - \frac{b}{c-a}x_1 = x_2 = 0$. We now check the time derivative of V:

$$\begin{aligned}
\frac{dV(x)}{dt} &= x_1\dot{x}_1 + (x_2 - \frac{b}{c-a}x_1)(\dot{x}_2 - \frac{b}{c-a}\dot{x}_1) \\
&= -ax_1^2 + (x_2 - \frac{b}{c-a}x_1)(-bx_1 - cx_2 + \frac{b}{c-a}x_1) \\
&= -ax_1^2 - c(x_2 - \frac{b}{c-a}x_1)^2.
\end{aligned}$$

We see that $\dot{V} \prec 0$ as long as $c \neq a$ and hence we can show stability except for this case (explored in more detail in the exercises). ∇

As this example illustrates, Lyapunov functions are not unique and hence we can use many different methods to find one. It turns out that Lyapunov functions can always be found for any stable system (under certain conditions) and hence one knows that if a system is stable, a Lyapunov function exists (and vice versa). Recent results using "sum of squares" methods have provided systematic approaches for finding Lyapunov systems [PPP02]. Sum of squares techniques can be applied to a broad variety of systems, including systems whose dynamics are described by polynomial equations as well as "hybrid" systems, which can have different models for different regions of state space.

Lyapunov Functions for Linear Systems

For a linear dynamical system of the form

$$\dot{x} = Ax$$

it is possible to construct Lyapunov functions in a systematic manner. To do so, we consider quadratic functions of the form

$$V(x) = x^T P x$$

where $P \in \mathbb{R}^{n \times x}$ is a symmetric matrix ($P = P^T$). The condition that $V \succ 0$ is equivalent to the condition that P is a *positive definite* matrix:

$$x^T P x > 0 \qquad \text{for all } x \neq 0,$$

which we write as $P > 0$. It can be shown that if P is symmetric and positive definite then all of its eigenvalues are real and positive.

Given a candidate Lyapunov function, we can now compute its derivative along flows of the system:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = x^T(A^T P + PA)x.$$

The requirement that $\dot{V} \prec 0$ (for asymptotic stability) becomes a condition that the matrix $Q = A^T P + PA$ be *negative definite*:

$$x^T Q x < 0 \qquad \text{for all } x \neq 0.$$

Thus, to find a Lyapunov function for a linear system it is sufficient to choose a $Q < 0$ and solve the *Lyapunov equation*:

$$A^T P + PA = Q.$$

This is a linear equation in the entries of P and hence it can be solved using linear algebra. The following examples illustrate its use.

Example 4.11. Consider the linear system from Example 4.10, for which we have

$$A = \begin{pmatrix} -a & 0 \\ -b & -c \end{pmatrix} \qquad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

We choose $Q = -I \in \mathbb{R}^{2\times 2}$ and the corresponding Lyapunov equation is

$$\begin{pmatrix} -a & -b \\ 0 & -c \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} -a & 0 \\ -b & -c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and solving for the elements of P yields

$$P = \begin{pmatrix} \frac{b^2+ac+c^2}{2a^2c+2ac^2} & \frac{-b}{2c(a+c)} \\ \frac{-b}{2c(a+c)} & \frac{1}{2} \end{pmatrix}$$

or

$$V(x) = \frac{b^2+ac+c^2}{2a^2c+2ac^2}x_1^2 - \frac{b}{c(a+c)}x_1x_2 + \frac{1}{2}x_2^2.$$

It is easy to verify that $P > 0$ (check its eigenvalues) and by construction $\dot{P} = -I < 0$. Hence the system is asymptotically stable. ∇

This same technique can also be used for searching for Lyapunov functions for nonlinear systems. If we write

$$\frac{dx}{dt} = f(x) =: Ax + \tilde{f}(x),$$

where $\tilde{f}(x)$ contains terms that are second order and higher in the elements of x, then we can find a Lyapunov function for the linear portion of the system and check to see if this is a Lyapunov function for the full nonlinear system. The following example illustrates the approach.

Example 4.12 (Congestion control)**.** Consider the congestion control problem described in Example 4.8, where we used phase portraits to demonstrate stability of the equilibrium points under different parameter values. We now wish to consider the general set of equations (from Example 2.12):

$$\begin{aligned} \frac{dx_i}{dt} &= -b\frac{x_i^2}{2} + (b_{\max} - b) \\ \frac{db}{dt} &= \sum_{i=1}^{N} x_i - c, \end{aligned}$$

The equilibrium points are given by

$$x_i^* = \frac{c}{N} \quad \text{for all } i \qquad b^* = \frac{2N^2 b_{\max}}{2N^2 + c^2},$$

To check for stability, we search for an appropriate Lyapunov function. For notational simplicity, we choose $N = 3$. It will also be convenient to rewrite the dynamics about the equilibrium point by choosing variables

$$z = \begin{pmatrix} z_1 \\ z_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_1^* \\ x_2 - x_2^* \\ b - b^* \end{pmatrix}.$$

The dynamics written in terms of z become

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{b^*(z_1+c)z_1}{N^2} - \left(1 + \frac{(2c+Nz_1)^2}{2N^2}\right) \\ -\frac{b^*(z_2+c)z_2}{2} - \left(1 + \frac{(2c+Nz_2)^2}{2N^2}\right) \\ z_1 + z_2 \end{pmatrix} =: F(z)$$

and $z = 0$ is an equilibrium point for the transformed system.

We now write $F(z)$ as a linear portion plus higher order terms:

$$\begin{aligned} F(z) &= \begin{pmatrix} -\frac{b^*c}{N} z_1 - \frac{c^2+2N^2}{2N^2} z_3 \\ -\frac{b^*c}{N} z_2 - \frac{c^2+2N^2}{2N^2} z_3 \\ z_1 + z_2 \end{pmatrix} + \begin{pmatrix} -\frac{b^*}{2} z_1^2 \frac{z_1(2c+Nz_1)z_3}{2N} \\ -\frac{b^*}{2} z_2^2 \frac{z_2(2c+Nz_2)z_3}{2N} \\ z_1 + z_2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{b^*c}{N} & 0 & -\frac{c^2+2N^2}{2N^2} \\ 0 & -\frac{b^*c}{N} & -\frac{c^2+2N^2}{2N^2} \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} -\frac{b^*}{2} z_1^2 \frac{z_1(2c+Nz_1)z_3}{2N} \\ -\frac{b^*}{2} z_2^2 \frac{z_2(2c+Nz_2)z_3}{2N} \\ z_1 + z_2 \end{pmatrix}. \end{aligned}$$

To find a candidate Lyapunov function, we solve the equation

$$A^T P + PA = Q$$

where A is the linear portion of F and $Q < 0$. Choosing $Q = -I \in \mathbb{R}^{3\times 3}$, we obtain

$$P = \begin{pmatrix} \frac{c^2N+3N^3}{2b^*c^3+4b^*cN^2} & \frac{N^3}{2b^*c^3+4b^*cN^2} & \frac{N^2}{2c^2+4N^2} \\ \frac{N^3}{2b^*c^3+4b^*cN^2} & \frac{c^2N+3N^3}{2b^*c^3+4b^*cN^2} & \frac{N^2}{2c^2+4N^2} \\ \frac{N^2}{2c^2+4N^2} & \frac{N^2}{2c^2+4N^2} & \frac{c^2N+4N^3}{4b^*cN} + \frac{b^*cN}{2c^2+4N^2}. \end{pmatrix}$$

We now check to see if this is a Lyapunov function for the original system:

$$\begin{aligned}\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} &= (z^T A^T + \tilde{F}^T(z))Pz + z^T P(Az + \tilde{F}(z)) \\ &= z^T(A^T P + PA)z + \tilde{F}^T(z)Pz + z^T P\tilde{F}(z).\end{aligned}$$

Note that all terms in $\tilde{F}$ are quadratic or higher order in z and hence it follows that $\tilde{F}^T(z)Pz$ and $z^T P\tilde{F}(z)$ consist of terms that are at least third order in z. It follows that if z is sufficiently close to zero then the cubic and higher order terms will be smaller than the quadratic terms. Hence, sufficiently close to $z = 0$, $\dot{V} \prec 0$. ∇

This technique for proving local stability of a nonlinear system by looking at the linearization about an equilibrium point is a general one that we shall return to in Chapter 5.

Krasovskii-Lasalle Invariance Principle

For general nonlinear systems, especially those in symbolic form, it can be difficult to find a function $V \succ 0$ whose derivative is strictly negative definition ($\dot{V} \prec 0$). The Krasovskii-Lasalle theorem enables us to conclude asymptotic stability of an equilibrium point under less restrictive conditions, namely in the case that $\dot{V} \preceq 0$, which is often much easier to construct. However, it applies only to time-invariant or periodic systems.

We will deal with the time-invariant case and begin by introducing a few more definitions. We denote the solution trajectories of the time-invariant system

$$\frac{dx}{dt} = F(x) \tag{4.10}$$

as $x(t; x_0, t_0)$, which is the solution of equation (4.10) at time t starting from x_0 at t_0. We write $x(\,\cdot\,; x_0, t_0)$ for the set of all points lying along the trajectory.

Definition 4.1 (ω limit set)**.** The ω *limit set* of a trajectory $x(\,\cdot\,; x_0, t_0)$ is the set of all points $z \in \mathbb{R}^n$ such that there exists a strictly increasing sequence of times t_n such that

$$s(t_n; x_0, t_0) \to z$$

as $n \to \infty$.

Definition 4.2 (Invariant set)**.** The set $M \subset \mathbb{R}^n$ is said to be an *invariant set* if for all $y \in M$ and $t_0 \geq 0$, we have

$$x(t; y, t_0) \in M \quad \text{for all } t \geq t_0.$$

It may be proved that the ω limit set of every trajectory is closed and invariant. We may now state the Krasovskii-Lasalle principle.

Theorem 4.2 (Krasovskii-Lasalle principle)**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally positive definite function such that on the compact set $\Omega_r = \{x \in \mathbb{R}^n : V(x) \leq r\}$ we have $\dot{V}(x) \leq 0$. Define*

$$S = \{x \in \Omega_r : \dot{V}(x) = 0\}.$$

As $t \to \infty$, the trajectory tends to the largest invariant set inside S; i.e., its ω limit set is contained inside the largest invariant set in S. In particular, if S contains no invariant sets other than $x = 0$, then 0 is asymptotically stable.

A global version of the preceding theorem may also be stated. An application of the Krasovskii-Lasalle principle is given in the following example.

Example 4.13 (Damped spring mass system)**.** Consider a damped spring mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = 0.$$

A natural candidate for a Lyapunov function is the total energy of the system, given by

$$V = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2.$$

The derivative of this function along trajectories of the system is

$$\dot{V} = m\dot{q}\ddot{q} + kq\dot{q} = -c\dot{q}.$$

This function is only negative semi-definite and hence we cannot conclude asymptotic stability using Theorem 4.1. However, note that $\dot{V} = 0$ implies that $\dot{q} = 0$. If we define

$$S = \{(q, \dot{q}) : \dot{q} = 0\}$$

then we can compute the largest invariant set inside S. For this set, we must have $\dot{q}(t) = 0$ for all t and hence $\ddot{q}(t) = 0$ as well.

Using the dynamics of the system, we see that if $\dot{q}(t) = 0$ and $\ddot{q}(t) = 0$ then $\dot{q}(t) = 0$ as well. hence the largest invariant set inside S is $(q, \dot{q}) = 0$

and we can use the Krasovskii-Lasalle principle to conclude that the origin is asymptotically stable. Note that we have not made use of Ω_r in this argument; for this example we have $\dot{V}(x) \leq 0$ for any state and hence we can choose r arbitrarily large. ∇

4.4 Parametric and Non-Local Behavior

Most of the tools that we have explored are focused on the local behavior of a fixed system near an equilibrium point. In this section we briefly introduce some concepts regarding the global behavior of nonlinear systems and the dependence of the behavior on parameters in the system model.

Regions of attraction

To get some insight into the behavior of a nonlinear system we can start by finding the equilibrium points. We can then proceed to analyze the local behavior around the equilibria. The behavior of a system near an equilibrium point is called the *local* behavior of the system.

The solutions of the system can be very different far away from this equilibrium point. This is seen, for example, in the inverted pendulum in Example 4.7. The downward hanging equilibrium point is stable, with small oscillations that eventually converge to the origin. But far away from this equilibrium point there are trajectories for which the pendulum swings around the top multiple times, giving very long oscillations that are topologically different than those near the origin.

To better understand the dynamics of the system, we can examine the set of all initial conditions that converge to a given asymptotically stable equilibrium point. This set is called the *region of attraction* for the equilibrium point. An example is shown in Figure 4.15. In general, computing regions of attraction is extremely difficult. However, even if we cannot determine the region of attraction, we can often obtain patches around the stable equilibria that are attracting. This gives partial information about the behavior of the system.

One method for approximating the region of attraction is through the use of Lyapunov functions. Suppose that V is a local Lyapunov function for a system around an equilibrium point x_0. Let Γ_r be set on which $V(x)$ has value less than c,

$$\Gamma_r = \{x \in \mathbb{R}^n : V(x) \leq r\},$$

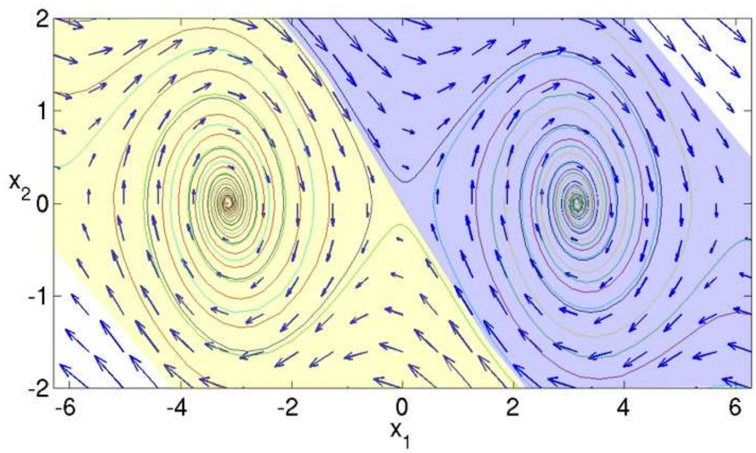


Figure 4.15: Phase portrait for an inverted pendulum with damping. Shaded regions indicate the regions of attraction for the two stable equilibrium points.

and suppose that $\dot{V}(x) \leq 0$ for all $x \in \Gamma_r$, with equality only at the equilibrium point x_0. Then Γ_r is inside the region of attraction of the equilibrium point. Since this approximation depends on the Lyapunov function and the choice of Lyapunov function is not unique, it can sometimes be a very conservative estimate.

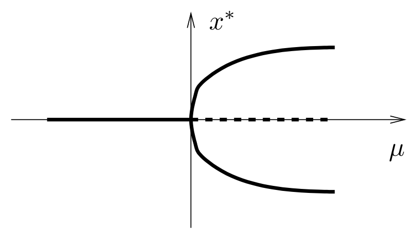
The Lyapunov tests that we derived for checking stability were local in nature. That is, we asked that a Lyapunov function satisfy $V \succ 0$ and $\dot{V} \prec 0$ for $x \in B_r$. If it turns out that the conditions on the Lyapunov function are satisfied for all $x \in \mathbb{R}^n$, then it can be shown that the region of attraction for the equilibrium point is the entire state space and the equilibrium point is said to be *globally* stable.

Bifurcations

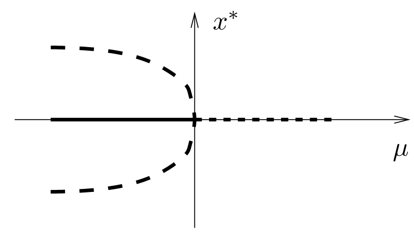
Another very important property of nonlinear systems is how their behavior changes as the parameters governing the dynamics change. We can study this in the context of models by exploring how the location of equilibrium points and their stability, regions of attraction and other dynamic phenomena such as limit cycles vary based on the values of the parameters in the model.

Consider a family of differential equations

$$\frac{dx}{dt} = F(x, \mu), \quad x \in \mathbb{R}^n,\, \mu \in \mathbb{R}^k, \tag{4.11}$$



(a) supercritical pitchfork



(b) subcritical pitchfork

Figure 4.16: Pitchfork bifurcation.

where x is the state and μ is a set of parameters that describe the family of equations. The equilibrium solutions satisfy

$$F(x, \mu) = 0$$

and as μ is varied, the corresponding solutions $x_e(\mu)$ vary. We say that the system (4.11) has a *bifurcation* at $\mu = \mu^*$ if the behavior of the system changes qualitatively at μ^*. This can occur either due to a change in stability type or a change in the number of solutions at a given value of μ. The following examples illustrate some of the basic concepts.

Example 4.14 (Simple exchange of stability)**.** Consider the scalar dynamical system

$$\dot{x} = \mu x.$$

This system has a bifurcation at $\mu = 0$ since the stability of the system changes from asymptotically stable (for $\mu < 0$) to neutrally stable ($\mu = 0$) to unstable (for $\mu > 0$). ∇

This type of bifurcation is very common in control systems when a system changes from being stable to unstable when a parameter is changed.

Example 4.15 (Pitchfork bifurcation)**.** Consider the scalar dynamical system

$$\dot{x} = \mu x - x^3.$$

The equilibrium values of x are plotted in Figure 4.16a, with solid lines representing stable equilibria and dashed lines representing unstable equilibria. As illustrated in the figure, the number and type of the solutions changes at $\mu = 0$ and hence we say there is a bifurcation at $\mu = 0$.

Note that the sign of the cubic term determines whether the bifurcation generates a stable branch (called a *supercritical* bifurcation and shown in Figure 4.16a) or a unstable branch (called a *subcritical* bifurcation and shown in Figure 4.16b). ∇

Bifurcations provide a tool for studying how systems evolve as operating parameters change and are particularly useful in the study of stability of differential equations. To illustrate how bifurcations arise in the context of feedback systems, we consider the predator prey system introduced earlier.

Example 4.16 (Predator prey)**.** Consider the predator prey system described in Section 3.7. The dynamics of the system is given by

$$\begin{aligned}\frac{dH}{dt} &= r_h H\left(1-\frac{H}{K}\right) - \frac{aHL}{1+aHT_h}\\ \frac{dL}{dt} &= r_l L\left(1-\frac{L}{kH}\right),\end{aligned} \tag{4.12}$$

where H and L are the number of hares (prey) and lynxes (predators), and r_h, r_l, K, k, a and T_h are parameters that model a given predator prey system (described in more detail in Section 3.7). The system has an equilibrium point at $H_e > 0$ and $L_e > 0$ that can be solved for numerically.

To explore how the parameters of the model affect the behavior of the system, we choose to focus on two specific parameters of interest: r_l, the growth rate of the lynxes, and T_h, the time constant for prey consumption. Figure 4.17a is a numerically computed *parametric stability diagram* showing the regions in the chosen parameter space for which the equilibrium point is stable (leaving the other parameters at their nominal values). We see from this figure that for certain combinations of r_l and T_h we get a stable equilibrium point while at other values this equilibrium point is unstable.

Figure 4.17b shows a numerically computed *bifurcation diagram* for the system. In this plot, we choose one parameter to vary (T_h) and then plot the equilibrium value of one of the states (L) on the vertical axis. The remaining parameters are set to their nominal values. A solid line indicates that the equilibrium point is stable; a dashed line indicates that the equilibrium point is unstable. Note that the stability in the bifurcation diagram matches that in the parametric stability diagram for $r_l = 0.01$ (the nominal value) and T_h varying from 0 to 20. For the predator prey system, when the equilibrium point is unstable, the solution converges to a stable limit cycle. The amplitude of this limit cycle is shown using the dot-dashed line in Figure 4.17b. ∇

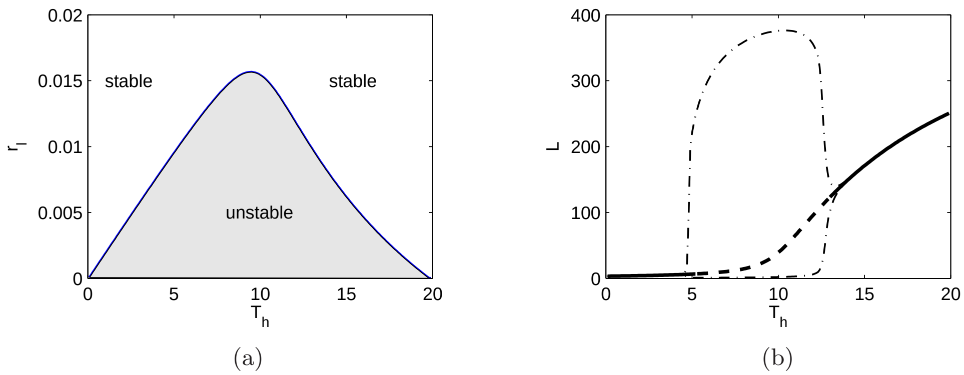


Figure 4.17: Bifurcation analysis of the predator prey system: (a) parametric stability diagram showing the regions in parameter space for which the system is stable; (b) bifurcation diagram showing the location and stability of the equilibrium point as a function of T_h. The dotted lines indicate the upper and lower bounds for the limit cycle at that parameter value (computed via simulation). The nominal values of the parameters in the model are $r_h = 0.02$, $K = 500$, $a = 0.03$, $T_h = 5$, $r_l = 0.01$ and $k = 0.2$.

Parametric stability diagrams and bifurcation diagrams can provide valuable insights into the dynamics of a nonlinear system. It is usually necessary to careful choose the parameters that one plots, including combining the natural parameters of the system to eliminate extra parameters when possible.

Control of bifurcations via feedback

Now consider a family of control systems

$$\dot{x} = F(x, u, \mu), \qquad x \in \mathbb{R}^n,\, u \in \mathbb{R}^m,\, \mu \in \mathbb{R}^k, \tag{4.13}$$

where u is the input to the system. We have seen in the previous sections that we can sometimes alter the stability of the system by choice of an appropriate feedback control, $u = \alpha(x)$. We now investigate how the control can be used to change the bifurcation characteristics of the system. As in the previous section, we rely on examples to illustrate the key points. A more detailed description of the use of feedback to control bifurcations can be found in the work of Abed and co-workers [LA96].

A simple case of bifurcation control is when the system can be *stabilized* near the bifurcation point through the use of feedback. In this case, we can completely eliminate the bifurcation through feedback, as the following simple example shows.

Example 4.17 (Stabilization of the pitchfork bifurcation)**.** Consider the subcritical pitchfork example from the previous section, with a simple additive control:

$$\dot{x} = \mu x + x^3 + u.$$

Choosing the control law $u = -kx$, we can stabilize the system at the nominal bifurcation point $\mu = 0$ since $\mu - k < 0$ at this point. Of course, this only shifts the bifurcation point and so k must be chosen larger than the maximum value of μ that can be achieved.

Alternatively, we could choose the control law $u = -kx^3$ with $k > 1$. This changes the sign of the cubic term and changes the pitchfork from a subcritical bifurcation to a supercritical bifurcation. The stability of the $x = 0$ equilibrium point is not changed, but the system operating point moves slowly away from zero after the bifurcation rather than growing without bound. ∇

4.5 Further Reading

The field of dynamical systems has a rich literature that characterizes the possible features of dynamical systems and describes how parametric changes in the dynamics can lead to topological changes in behavior. A very readable introduction to dynamical systems is given by Strogatz [Sto94]. More technical treatments include Guckenheimer and Holmes [GH83] and Wiggins [Wig90]. For students with a strong interest in mechanics, the text by Marsden and Ratiu [MR94] provides a very elegant approach using tools from differential geometry. Finally, very nice treatments of dynamical systems methods in biology are given by Wilson [Wil99] and Ellner and Guckenheimer [EG05].

There is a large literature on Lyapunov stability theory. We highly recommend the very comprehensive treatment by Khalil [Kha92].

4.6 Exercises

1. Consider the cruise control system described in Section 3.1. Plot the phase portrait for the combined vehicle dynamics and PI compensator with $k = 1$ and $k_i = 0.5$.

2. Show that if we have a solution of the differential equation (4.1) given by $x(t)$ with initial condition $x(t_0) = x_0$, then $\tilde{x}(\tau) = x(t - t_0) - x_0$ is a solution of the differential equation

$$\frac{d\tilde{x}}{d\tau} = F(\tilde{x})$$

with initial condition $\tilde{x}(0) = 0$.

3. We say that an equilibrium point $x^* = 0$ is an *exponentially stable* equilibrium point of (4.2) if there exist constants $m, \alpha > 0$ and $\epsilon > 0$ such that

$$\|x(t)\| \leq m e^{-\alpha(t-t_0)} \|x(t_0)\| \tag{4.14}$$

for all $\|x(t_0)\| \leq \epsilon$ and $t \geq t_0$. Prove that an equilibrium point is exponentially stable if and only if there exists an $\epsilon > 0$ and a function $V(x, t)$ that satisfies

$$\begin{aligned} \alpha_1 \|x\|^2 \leq V(x,t) &\leq \alpha_2 \|x\|^2 \\ \left.\frac{dV}{dt}\right|_{\dot{x}=f(x,t)} &\leq -\alpha_3 \|x\|^2 \\ \left\|\frac{\partial V}{\partial x}(x,t)\right\| &\leq \alpha_4 \|x\| \end{aligned}$$

for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\|x\| \leq \epsilon$.

4. Consider the asymptotically stable system

$$\frac{dx}{dt} = \begin{pmatrix} -\lambda & 0 \\ b & -\lambda \end{pmatrix} x,$$

where $\lambda > 0$. Find a Lyapunov function for the system that proves asymptotic stability.