CDS 101: Lecture 6.1
Transfer Functions

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Goals:

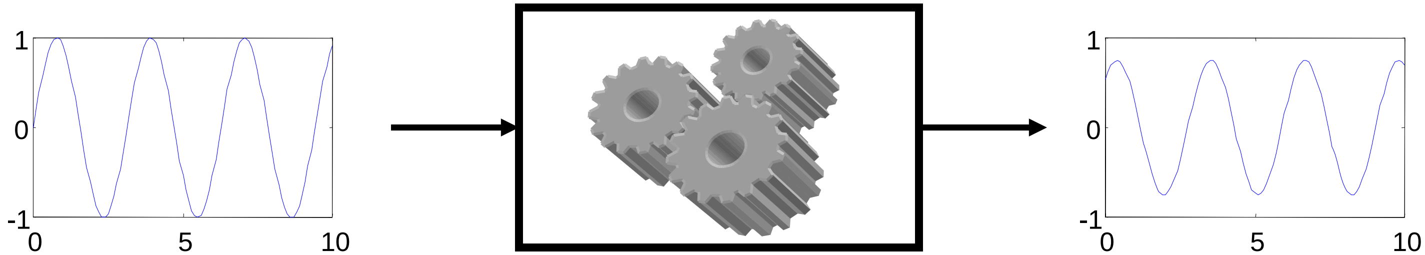
- Motivate and define the input/output transfer function of a linear system
- Understand the relationships among frequency response (Bode plot), transfer function, and state-space model
- Introduce block diagram algebra for transfer functions of interconnected systems

Reading:

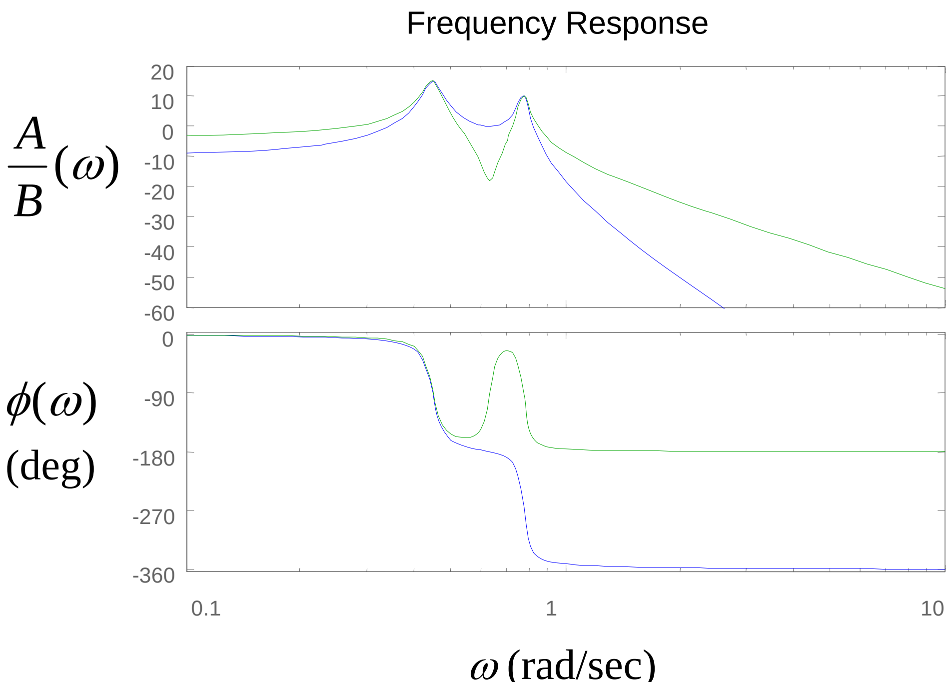
- Packard, Poola, Horowitz, Chapters 5-6
- *Optional*: Astrom, Section 5.1-5.3
- *Advanced*: Lewis, Chapters 3-4

Review: Frequency Response and Bode Plots

Defn. The *frequency response* of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



$u = A \sin(\omega t)$ $\qquad$ $y = B \sin(\omega t + \phi)$

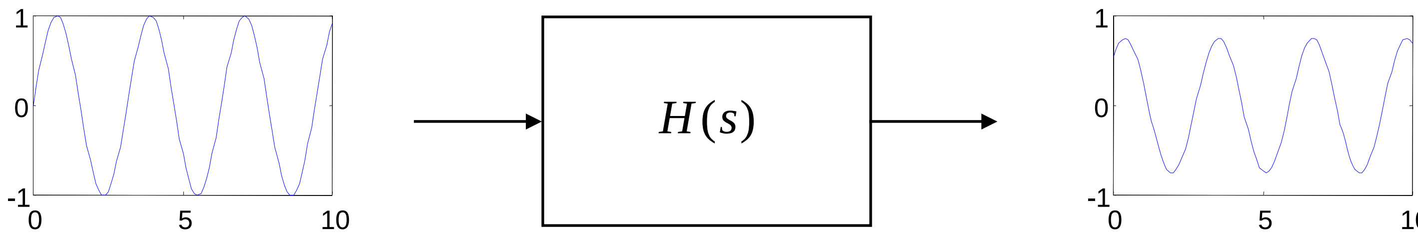


Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity $\Rightarrow$ can construct response to any input (via Fourier decomposition)

Transfer Functions

"Defn." The *transfer function* for a linear system $\Sigma = (A, B, C, D)$ is a function $H(s)$, $s \in \mathcal{C}$ such that $H(j\omega)$ gives the gain and phase of the response to a sinusoid at frequency ω:



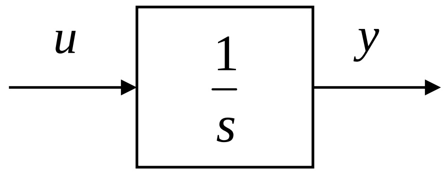
$u = A\sin(\omega t)$ $\qquad$ $y = |H(j\omega)| A \sin(\omega t + \measuredangle H(j\omega))$

$$H(j\omega) = \alpha + j\beta \qquad |H(j\omega)| = \sqrt{\alpha^2 + \beta^2} \qquad \measuredangle H(j\omega) = \tan^{-1}(\beta/\alpha)$$

Example: single "integrator"

$$\dot{x} = u$$
$$y = x$$

$$u = A\sin(\omega t)$$
$$y = (A/\omega)\sin(\omega t - \tfrac{\pi}{2})$$



$$\text{"}y = H(s)u\text{"}$$

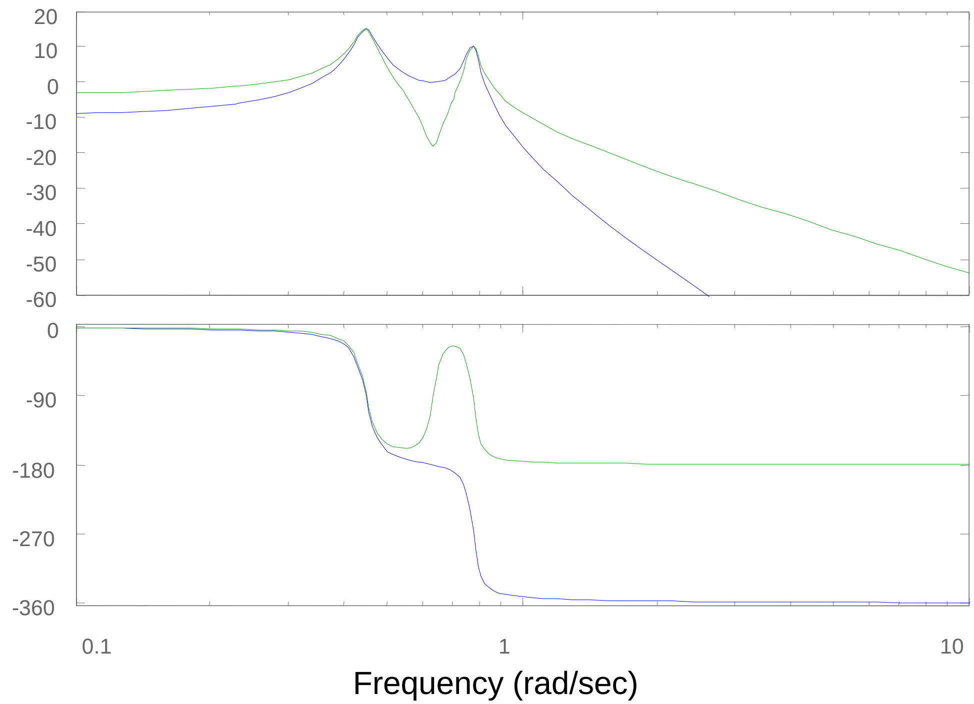
$$|H(j\omega)| = 1/\omega$$
$$\measuredangle H(j\omega) = -\pi/2$$

Transfer functions and frequency response

H(jω) is like a complex function representation of the Bode plot…

$|H(j\omega)|$

$\measuredangle H(j\omega)$



One way to determine the transfer function of a given system is to fit the frequency response by a (rational) complex function. This works well in practice for so-called "minimum phase" systems, but otherwise can be tricky…

Transfer functions from state-space models

Thm. The *transfer function* for a linear system $\Sigma=(A,B,C,D)$ is given by

$$H(s) = C(sI - A)^{-1}B + D \quad s \in \mathrm{C}$$

Thm. The transfer function $H(s)$ corresponding to $\Sigma=(A,B,C,D)$ has the following properties:

- $H(s)$ is a ratio of polynomials $n(s)/d(s)$ where $d(s)$ is the *characteristic equation* for the matrix A and $n(s)$ has order less than or equal to $d(s)$.
- The *zero initial state* frequency response of Σ has gain $|H(j\omega)|$ and phase $\mathrm{P}H(j\omega)$:

$$u = A\sin(\omega t)$$
$$y = \left|H(j\omega)\right| A\sin\left(\omega t + \mathrm{R}H(j\omega)\right)$$

Remarks

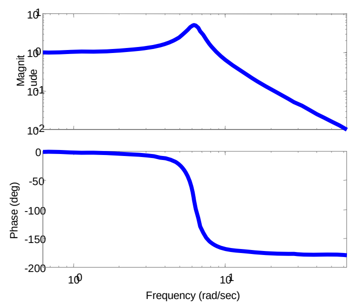
- Formally, can show that $H(s)$ is the *Laplace transform* of the impulse response of Σ
- "$y=H(s)u$" is formally $Y(s)=H(s)U(s)$ where $Y(s)$ and $U(s)$ are the Laplace transforms of $y(t)$ and $u(t)$. (Multiplication in the Laplace domain corresponds to convolution.)

Series Interconnections

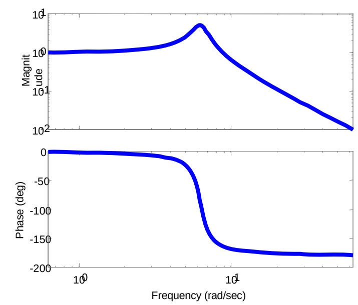
Q: what happens when we connect two systems together *in series*?

$u_1 \rightarrow$
$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$y_1 = C_1 x_1 + D_1 u_1$$
$\rightarrow y_1 \quad u_2 \rightarrow$
$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y_2 = C_2 x_2 + D_2 u_2$$
$\rightarrow$

$u = A\sin(\omega t) \rightarrow$



$\rightarrow y = g_1 A\sin(\omega t + \phi_1) \rightarrow$



$\rightarrow y_2 = g_1 g_2 A \cdot \sin(\omega t + \phi_1 + \phi_2)$

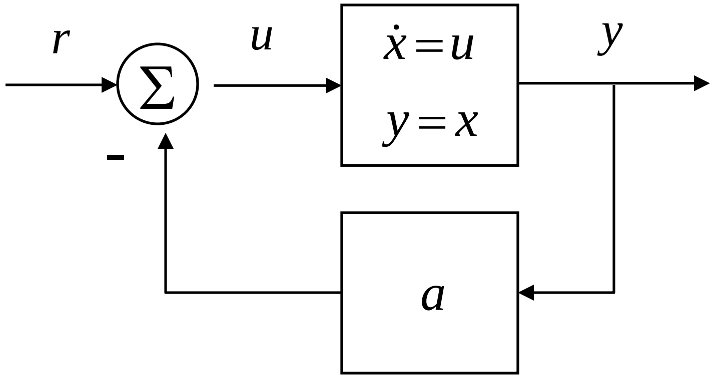
A: Transfer functions *multiply*

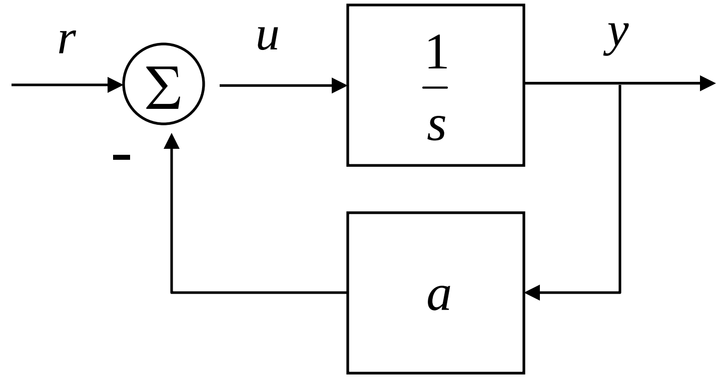
- Gains multiply
- Phases add
- Generally: transfer functions well formulated for frequency domain interconnections

$u_1 \rightarrow H_1(s) \rightarrow u_2 = y_1 \rightarrow H_2(s) \rightarrow y_2$

$u_1 \rightarrow H_1(s)H_2(s) \rightarrow y_2 = H_1(s)H_2(s)u_1$

Feedback Interconnection





State space derivation

$$\dot{x} = u = r - ay = -ax + r$$

$$y = x$$

Frequency response $\quad r = A\sin(\omega t)$

$$y = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right| \sin\left(\omega t - \tan^{-1}\left(\tfrac{\omega}{a}\right)\right)$$

Transfer function derivation

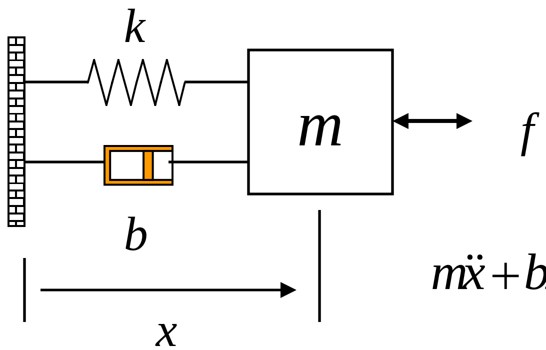
$$y = \frac{u}{s} = \frac{r - ay}{s}$$

$$y = \frac{r}{s+a} = H(s)r$$

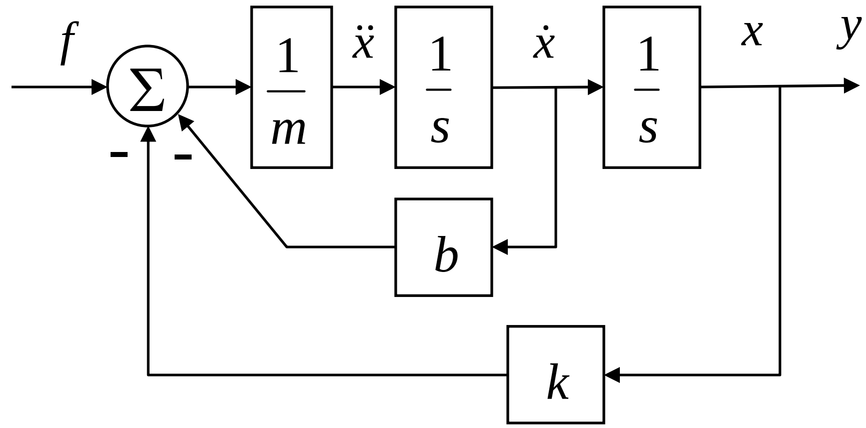
Frequency response

$$y = \left|H(j\omega)\right| \sin\left(\omega t + \angle H(j\omega)\right)$$

Example: mass spring system



$$m\ddot{x} + b\dot{x} + kx = f$$



Rewrite in terms of "block diagram"

- Represent integration using 1/*s*
- Include spring and damping through feedback terms
- Determine the transfer function through algebraic manipulation
- Claim: resulting transfer function captures the frequency response

$$y = \frac{1}{m}\cdot\frac{1}{s}\cdot\frac{1}{s}\left(f - b\dot{x} - kx\right) = \frac{1}{ms^2}f - \frac{b}{ms}y - \frac{k}{ms^2}y$$

$$\left(1 + \frac{b}{ms} + \frac{k}{ms^2}\right)y = \frac{1}{ms^2}f$$

$$y = \frac{1}{ms^2 + bs + k}f$$

$$\boxed{H(s) = \frac{1}{ms^2 + bs + k}}$$

Poles and Zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$H(s) = \frac{n(s)}{d(s)}$$
$$d(s) = \det(sI - A)$$

- Roots of $d(s)$ are called *poles* of $H(s)$
- Roots of $n(s)$ are called *zeros* of $H(s)$

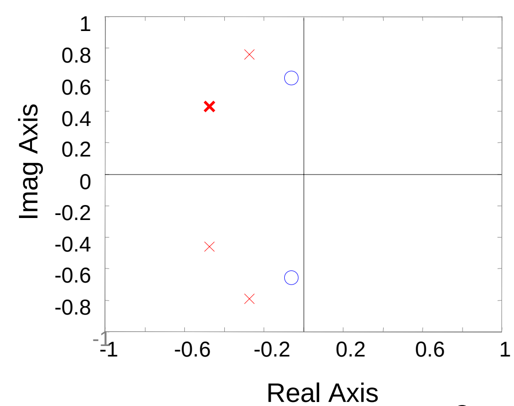
Poles of $H(s)$ determine the stability of the (closed loop) system

- Denominator of transfer function = characteristic polynomial of state space system
- Provides easy method for computing stability of systems
- Right half plane (RHP) poles (Re > 0) correspond to unstable systems

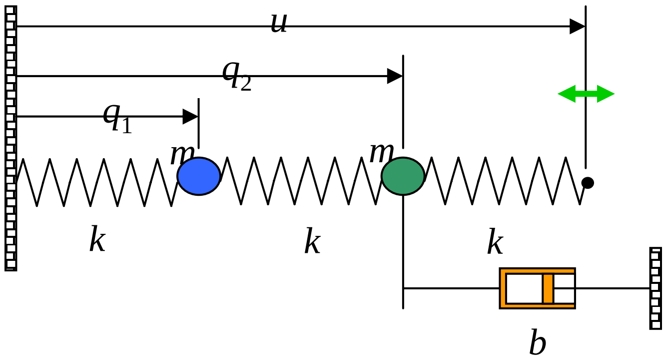
Zeros of $H(s)$ related to frequency ranges with limited transmission

- A pure imaginary zero at $s=j\omega_z$ blocks any output at that frequency ($H(j\omega_z) = 0$)
- Zeros provide limits on performance, especially RHP zeros (more on this later)

$$H(s) = k\frac{s^2 + b_1 s + b_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \xrightarrow{\text{pzmap}}$$

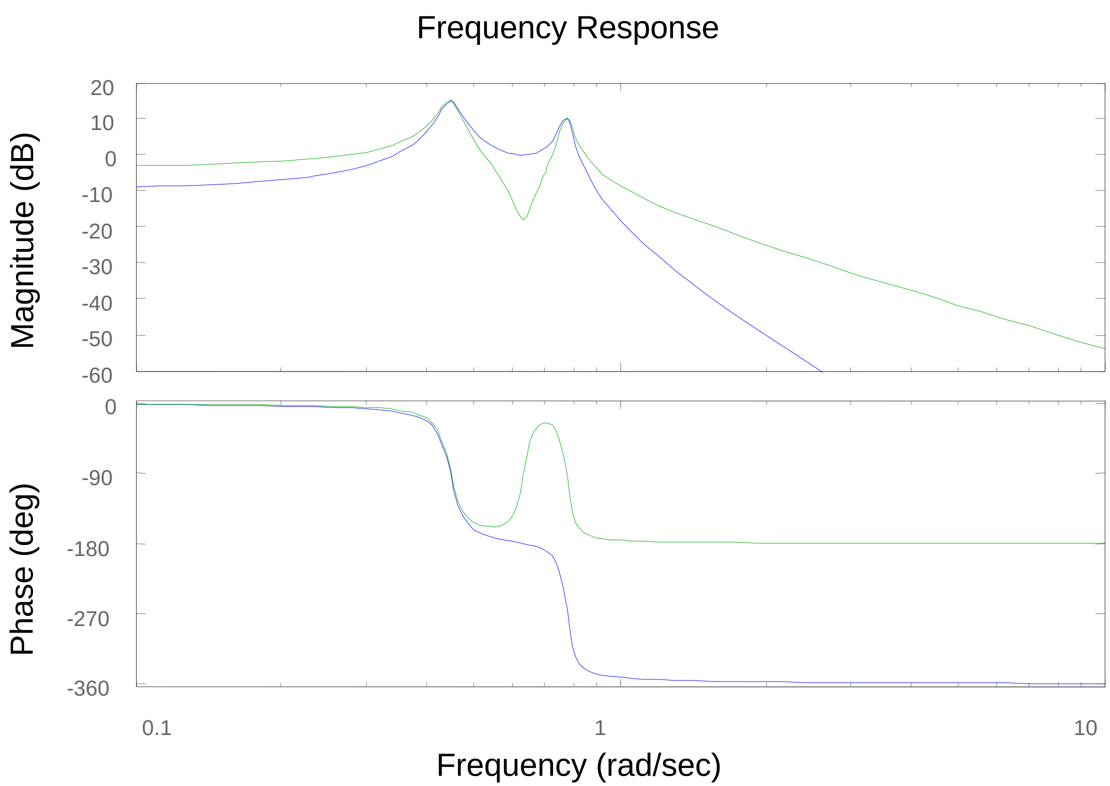


Example: Coupled Masses



$$H_{q_1 f} = \frac{0.04}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

$$H_{q_2 f} = \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$



Poles (H_{q1f} and H_{q2f})

- $-0.0200 \pm 0.7743j$
- $-0.0200 \pm 0.4468j$

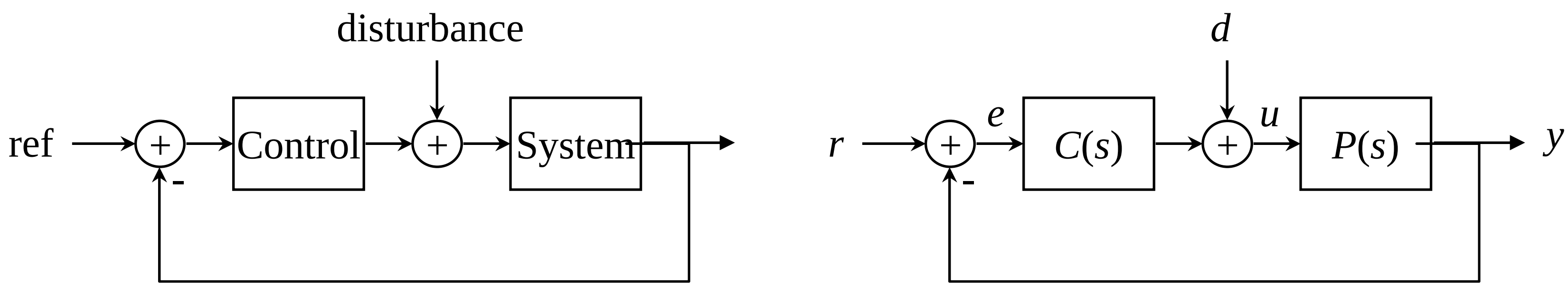
Zeros (H_{q2f})

- $-0.0200 \pm 0.6321j$

Interpretation

- Zeros in H_{q2f} give low response at $\omega \approx 0.6321$

Control Analysis and Design Using Transfer Functions



Transfer functions provide a method for “block diagram algebra”

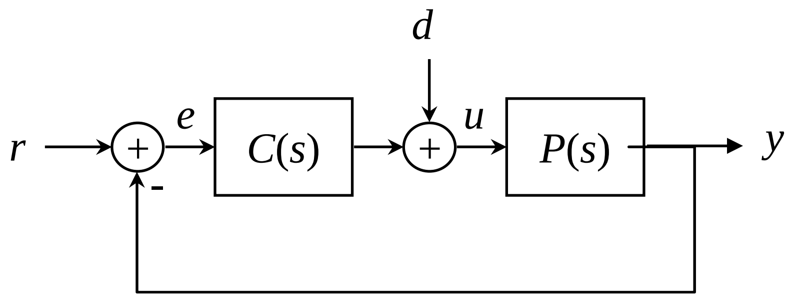
- Easy to compute transfer functions between various inputs and outputs
  - $H_{er}(s)$ is the transfer function between the reference and the error
  - $H_{ed}(s)$ is the transfer function between the disturbance and the error

Transfer functions provide a method for performance specification

- Since transfer functions provide frequency response directly, it is convenient to work in the “frequency domain”
  - $H_{er}(s)$ should be small in the frequency range 0 to 10 Hz (good tracking)

Block Diagram Algebra

Basic idea: treat transfer functions as multiplication, write down equations



$$y = P(s)u$$
$$u = d + C(s)e$$
$$e = r - y$$

Manipulate equations to compute desired signals

$$\begin{aligned} e &= r - y \\ &= r - P(s)u \\ &= r - P(s)(d + C(s)e) \end{aligned}$$

$$(1 + P(s)C(s))e = r - P(s)d$$

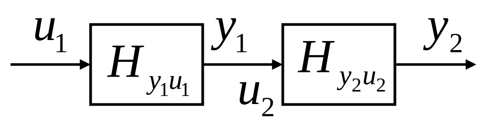
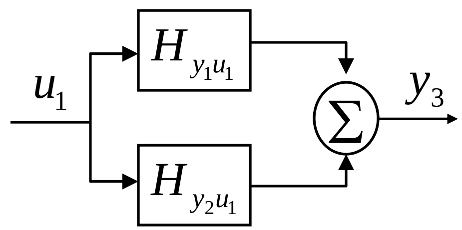
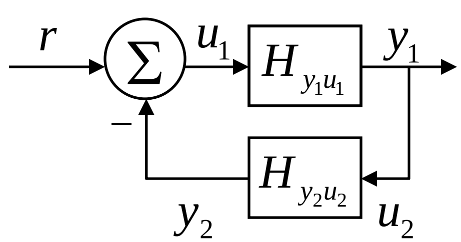
$$e = \underbrace{\frac{1}{1 + P(s)C(s)}}_{H_{er}} r - \underbrace{\frac{P(s)}{1 + P(s)C(s)}}_{H_{ed}} d$$

Note: linearity gives superposition of terms

Algebra works because we are working in frequency domain

- Time domain (ODE) representations are not as easy to work with
- Formally, all of this works because of Laplace transforms (ACM 95/100)

Block Diagram Algebra

<table>
<tr><th>Type</th><th>Diagram</th><th>Transfer function</th></tr>
<tr><td>Series</td><td>
</td><td>$$H_{y_2u_1} = H_{y_2u_2}H_{y_1u_1} = \frac{n_1n_2}{d_1d_2}$$</td></tr>
<tr><td>Parallel</td><td>
</td><td>$$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \frac{n_1d_2 + n_2d_1}{d_1d_2}$$</td></tr>
<tr><td>Feedback</td><td>
</td><td>$$H_{y_1r} = \frac{H_{y_1u_1}}{1 + H_{y_1u_1}H_{y_2u_2}} = \frac{n_1d_2}{n_1n_2 + d_1d_2}$$</td></tr>
</table>

- These are the basic manipulations needed; some others are possible
- Formally, could work all of this out using the original ODEs ($\Rightarrow$ nothing *really* new)

MATLAB manipulation of transfer functions

Creating transfer functions

- [num, den] = ss2tf(A, B, C, D)
- sys = tf(num, den)
- num, den = [1 a b] $\rightarrow s^2 + as + b$

Interconnecting blocks

- sys= series(sys1, sys2), parallel, feedback

Computing poles and zeros

- pole(sys), zero(sys)
- pzmap(sys)

I/O response

- step(sys), bode(sys)

```
» tf(sys)
Transfer function:
      1
---------------
s^2 + 0.2 s + 1
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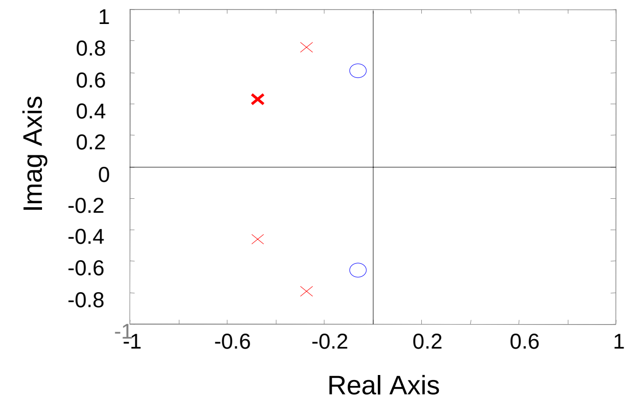
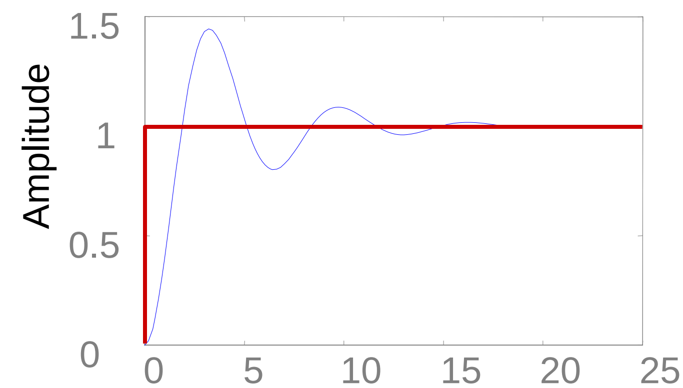
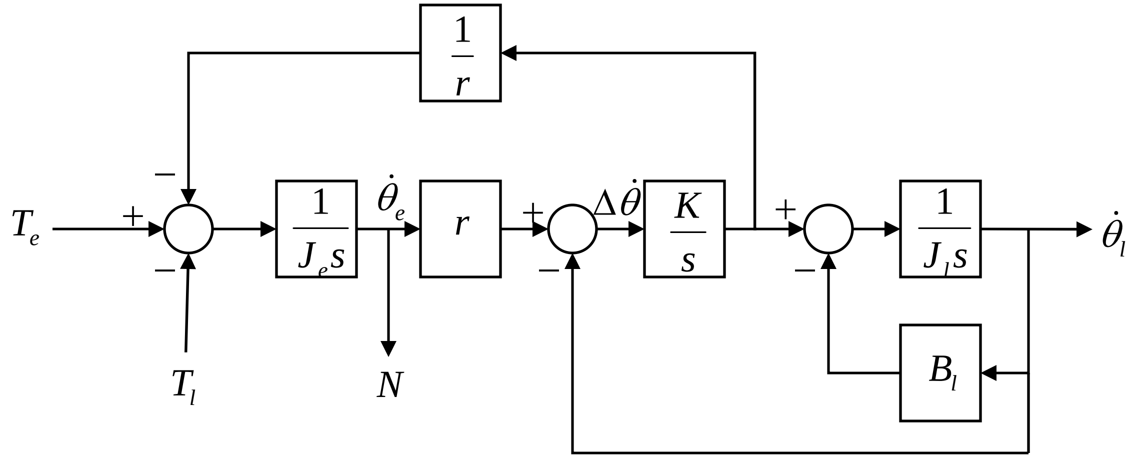
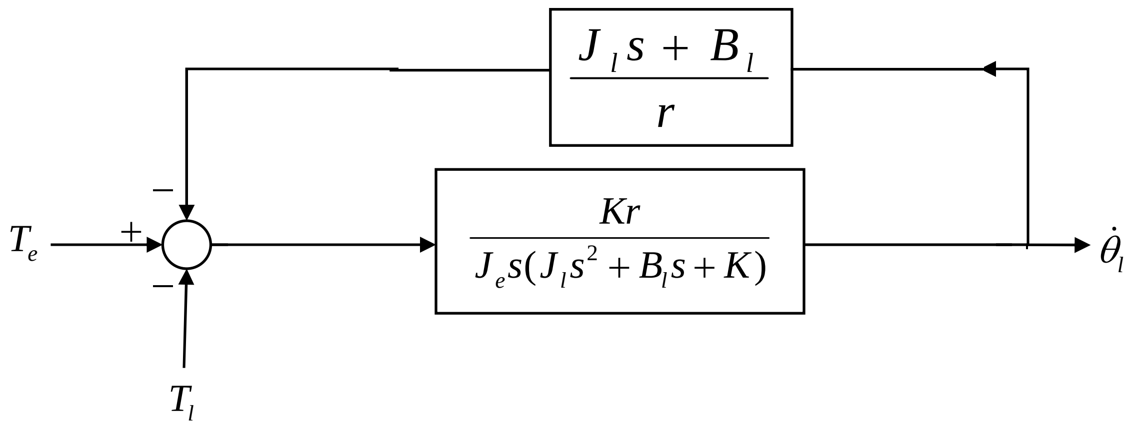




Example: Engine Control of a GM Astro





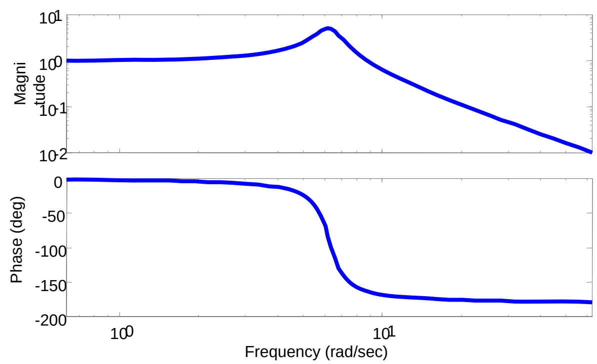
$$H_{\theta_l T_e}(s) = \frac{Kr}{J_e J_l s^3 + J_e B_l s^2 + (J_e K + K J_l)s + K B_l}$$

Summary: Frequency Response & Transfer Functions

$$u = A\sin(\omega t)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$

$$y_{ss} = |H(j\omega)| A \cdot \sin\left(\omega t + \angle H(j\omega)\right)$$



$$H(s) = C(sI - A)^{-1}B + D$$

$$H_{y_2u_1} = H_{y_2u_2} H_{y_1u_1} = \frac{n_1 n_2}{d_1 d_2}$$