CDS 101: Lecture 5-1
Reachability and State Space Feedback

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23 October 2006

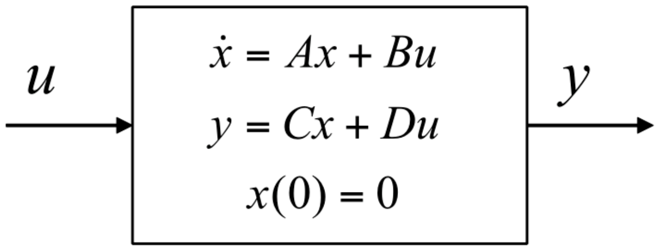
Goals:

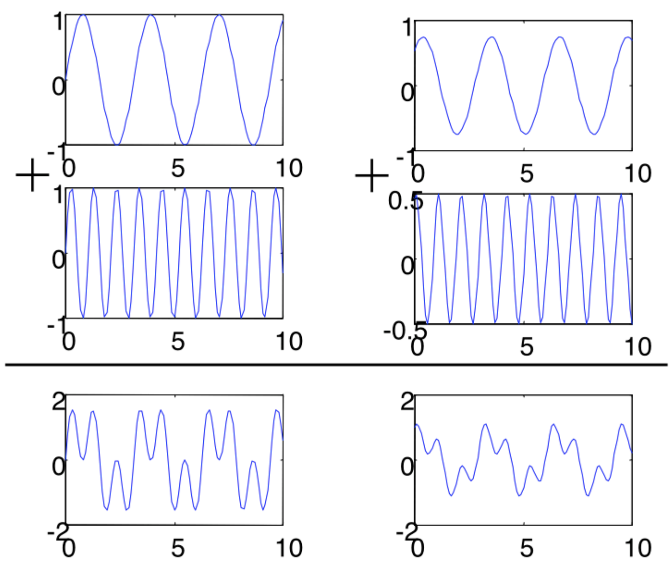
- Define reachability of a control system
- Give tests for reachability of linear systems and apply to examples
- Describe the design of state feedback controllers for linear systems

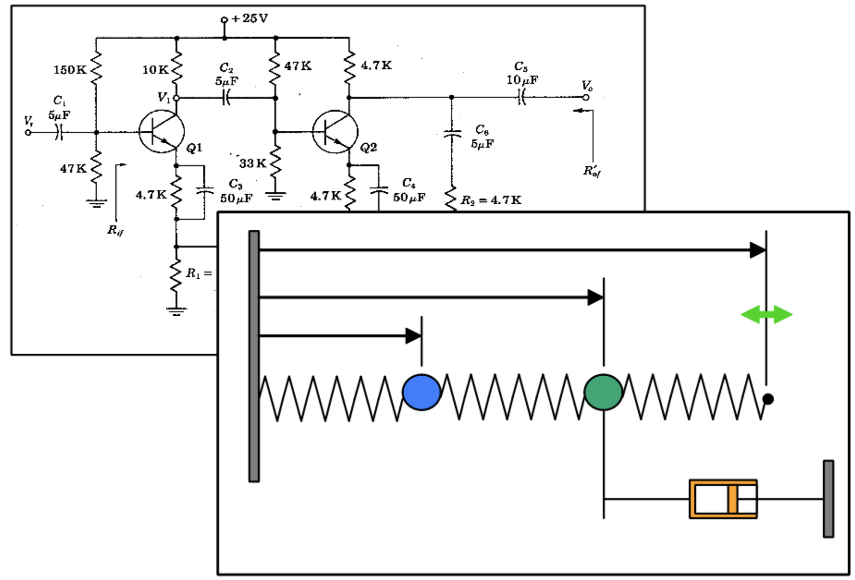
Reading:

- Åström and Murray, *Feedback Systems,* Ch 6

Review from Last Week







$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Properties of linear systems

- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems

Control Design Concepts

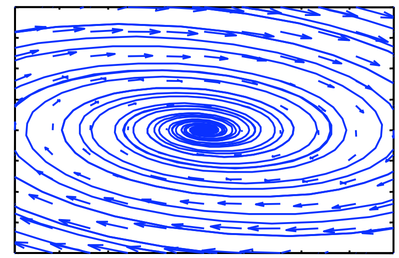
System description: single input, single output system (MIMO also OK)

$$\dot{x} = f(x,u) \qquad x \in \mathrm{R}^n, x(0) \text{ given}$$
$$y = h(x,u) \qquad u \in \mathrm{R}, y \in \mathrm{R}$$

Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" $u=\alpha(x)$ such that

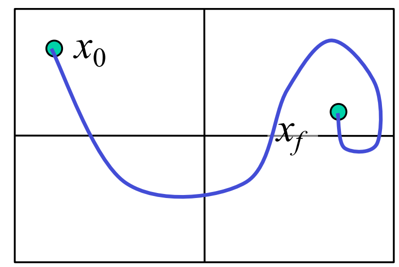
$$\lim_{t\to\infty} x(t) = x_e \text{ for all } x(0) \in \mathrm{R}^n$$



Reachability: steer the system between two points

- Given $x_o, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that

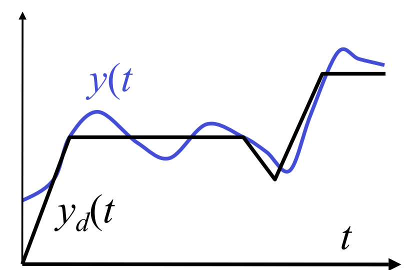
$$\dot{x} = f(x, u(t)) \text{ takes } x(t_0) = x_0 \rightarrow x(T) = x_f$$



Tracking: track a given output trajectory

- Given $y_d(t)$, find $u=\alpha(x,t)$ such that

$$\lim_{t\to\infty} \big(y(t) - y_d(t)\big) = 0 \text{ for all } x(0) \in \mathrm{R}^n$$



Reachability of Input/Output Systems

$$\dot{x} = f(x,u) \quad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = h(x,u) \quad u \in \mathbb{R},\ y \in \mathbb{R}$$

Defn An input/output system is *reachable* if for any $x_o, x_f \in \mathbb{R}^n$ and any time $T > 0$ there exists an input $u_{[0,T]} \in \mathbb{R}$ such that the solution of the dynamics starting from $x(0)=x_0$ and applying input $u(t)$ gives $x(T)=x_f$.

Remarks

- In the definition, x_0 and x_f do not have to be equilibrium points ⇒ we don't necessarily stay at x_f after time T.
- Reachability is defined in terms of states ⇒ doesn't depend on output
- For *linear systems,* can characterize reachability by looking at the general solution:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

- If integral is "surjective" (as a linear operator), then we can find an input to achieve any desired final state.

Tests for Reachability

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx + Du \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Thm A linear system is reachable if and only if the $n \times n$ *reachability matrix*

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

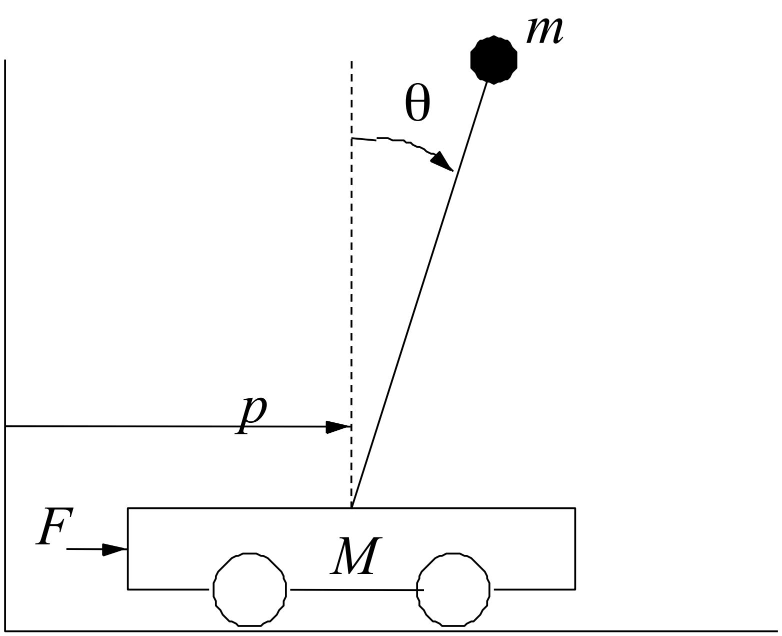
is full rank.

Remarks

- *Very* simple test to apply. In MATLAB, use ctrb(A,B) and check rank w/ det()
- If this test is satisfied, we say "the pair (A,B) is reachable"
- Some insight into the proof can be seen by expanding the matrix exponential

$$\begin{aligned} e^{A(T-\tau)}B &= \left(I + A(T-\tau) + \frac{1}{2}A^2(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}(T-\tau)^{n-1} + \cdots\right)B \\ &= B + AB(T-\tau) + \frac{1}{2}A^2B(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}B(T-\tau)^{n-1} + \cdots \end{aligned}$$

Example #1: Linearized pendulum on a cart



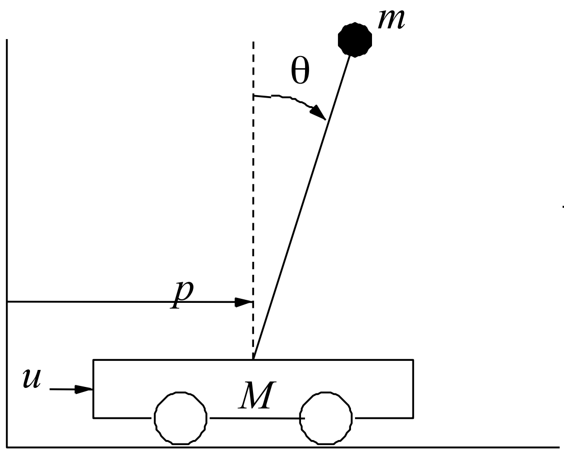
Question: can we locally control the position of the cart by proper choice of input?

Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{M_t J_t - m^2 l^2} & \frac{-c J_t}{M_t J_t - m^2 l^2} & \frac{-\gamma J_t l m}{M_t J_t - m^2 l^2} \\ 0 & \frac{M_t m g l}{M_t J_t - m^2 l^2} & \frac{-c l m}{M_t J_t - m^2 l^2} & \frac{-\gamma M_t}{M_t J_t - m^2 l^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_t J_t - m^2 l^2} \\ \frac{l m}{M_t J_t - m^2 l^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x,$$

Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot p \\ \dot\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & \cancelto{0}{\frac{-cJ_t}{\mu}} & \cancelto{0}{\frac{-\gamma J_t l m}{\mu}} \\ 0 & \frac{M_t m g l}{\mu} & \cancelto{0}{\frac{-clm}{\mu}} & \cancelto{0}{\frac{-\gamma M_t}{\mu}} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot p \\ \dot\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{lm}{\mu} \end{bmatrix} u$$

$$\mu = M_t J_t - m^2 l^2$$

- Simplify by setting $c, \gamma = 0$

Reachability matrix

$$W_r = \begin{bmatrix} 0 & \frac{J_t}{\mu} & 0 & \frac{gl^3m^3}{\mu^2} \\ 0 & \frac{lm}{\mu} & 0 & \frac{gl^2m^2(m+M)}{\mu^2} \\ \frac{J_t}{\mu} & 0 & \frac{gl^3m^3}{\mu^2} & 0 \\ \frac{lm}{\mu} & 0 & \frac{gl^2m^2(m+M)}{\mu^2} & 0 \end{bmatrix}.$$

B AB A^2B A^3B

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- ⇒ reachable as long as

$$\det(W_r) = \frac{g^2 l^4 m^4}{\mu^4} \neq 0$$

- ⇒ can "steer" linearization between points by proper choice of input

Control Design Concepts

System description: single input, single output system (MIMO also OK)

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Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in R^n$, find control "law" $u=\alpha(x)$

  such that

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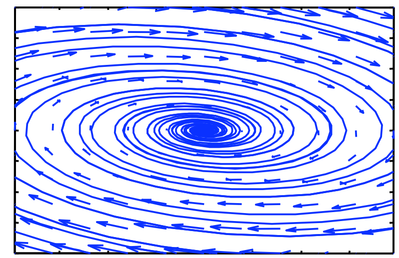

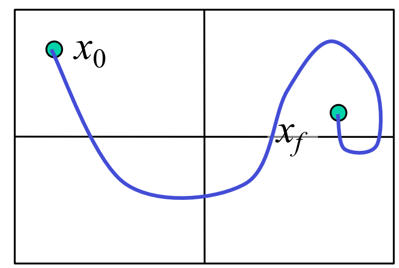

✓ **Reachability: steer the system between two points**

- Given $x_0, x_f \in R^n$, find an input $u(t)$ such that

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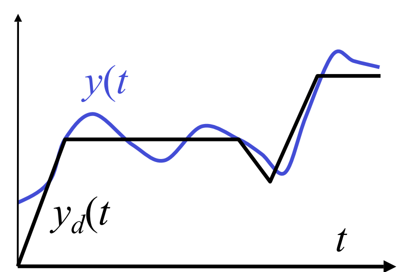


Tracking: track a given output trajectory

- Given $y_d(t)$, find $u=\alpha(x,t)$ such that

$$\lim_{t\to\infty} \left(y(t) - y_d(t) \right) = 0 \text{ for all } x(0) \in R^n$$

State space controller design for linear systems

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx + Du \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT} x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)} Bu(\tau)\, d\tau$$

Goal: find a linear control law $u = -Kx$ such that the closed loop system

$$\dot{x} = Ax - BKx = (A - BK)x$$

is stable at $x_e = 0$.

Remarks

- Stability based on eigenvalues ⇒ use K to make eigenvalues of $(A+BK)$ stable
- Can also link eigenvalues to *performance* (eg, initial condition response)
- Question: when can we place the eigenvalues anyplace that we want?

Theorem The eigenvalues of $(A - BK)$ can be set to arbitrary values if and only if the pair (A, B) is reachable.

MATLAB: K = place(A, B, eigs)

Example #2: Predator prey

Natural dynamics

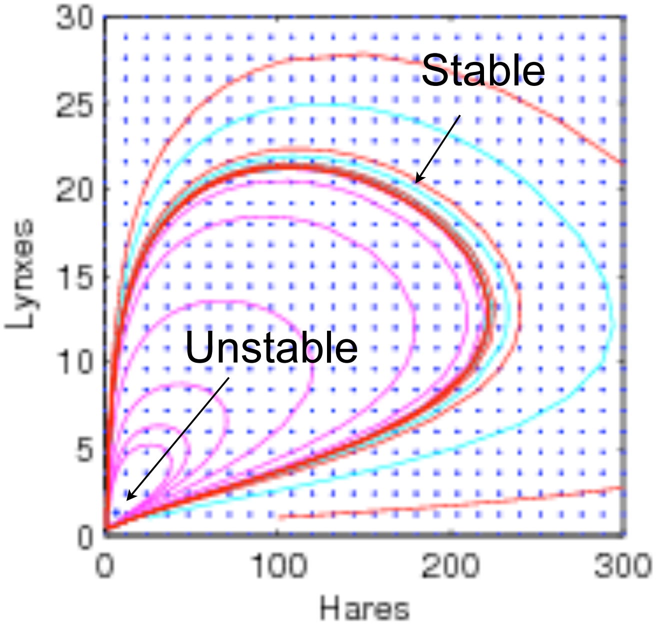
$$\frac{dH}{dt} = r_h H \left(1 - \frac{H}{K}\right) - \frac{aHL}{1 + aHT_h} \qquad H \geq 0$$

$$\frac{dL}{dt} = r_l L \left(1 - \frac{L}{kH}\right) \qquad L \geq 0$$

Controlled dynamics: modulate food supply

$$\frac{dH}{dt} = (r_h + u) H \left(1 - \frac{H}{K}\right) - \frac{aHL}{1 + aHT_h}$$

$$\frac{dL}{dt} = r_l L \left(1 - \frac{L}{kH}\right)$$



Q1: can we move from some initial population of foxes and rabbits to a specified one in time *T* by modulation of the food supply?

Q2: can we *stabilize* the population around the desired equilibrium point

Approach: try to answer this question *locally*, around the natural equilibrium point

Example #2: Problem setup

Equilibrium point calculation

$$\frac{dH}{dt} = (r_h + u)H\left(1 - \frac{H}{K}\right) - \frac{aHL}{1 + aHT_h}$$
$$\frac{dL}{dt} = r_l L\left(1 - \frac{L}{kH}\right)$$

- x_e = (6.5, 1.3), u_e = 0, y_e = 6.5

```
% Compute the equil point
% predprey.m contains dynamics
f = inline('predprey(0,x)');
xeq = fsolve(f, [10, 2]);

% Compute linearization
A = [
  rH - (2*H0*rH)/K - (a*L0)...
  ..., rL - (2*L0*rL)/(H0*k)
];
B = [H0*(1 - H0/K); 0];
```

Linearization

- Compute linearization around equil. point, x_e:

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

- Redefine local variables: $z = x - x_e$, $v = u - u_e$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{aL_0}{(aH_0T_h+1)^2} - \frac{2H_0r_h}{K} + r_h & -\frac{aH_0}{aH_0T_h+1} \\ \frac{L_0^2 r_l}{H_0^2 k} & r_l - \frac{2L_0 r_l}{H_0 k} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0\left(1 - \frac{H_0}{K}\right) \\ 0 \end{bmatrix} v$$

Example #2: Stabilization via eigenvalue assignment

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{aL_0}{(aH_0T_h+1)^2} - \frac{2H_0r_h}{K} + r_h & -\frac{aH_0}{aH_0T_h+1} \\ \frac{L_0^2 r_l}{H_0^2 k} & r_l - \frac{2L_0 r_l}{H_0 k} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0\left(1 - \frac{H_0}{K}\right) \\ 0 \end{bmatrix} v$$

Control design:

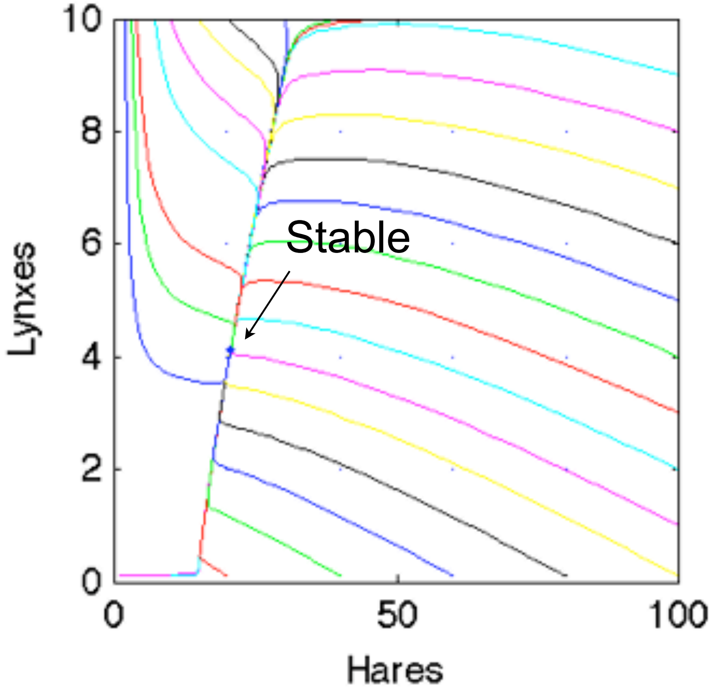
$$v = -Kz + k_r r$$
$$u = u_e + K(x - x_e) + k_r(r - y_e)$$

Place poles at stable values

- Choose λ=-1, -2
- K = place(A, B, [-1; -2]);

Modify NL dynamics to include control

$$\frac{dH}{dt} = (r_h + u)H\left(1 - \frac{H}{K}\right) - \frac{aHL}{1 + aHT_h}$$
$$\frac{dL}{dt} = r_l L\left(1 - \frac{L}{kH}\right)$$



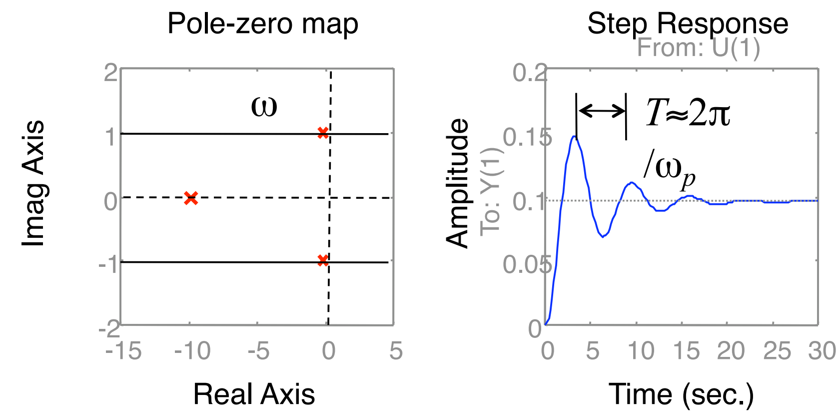
Implementation Details

Eigenvalues determine performance

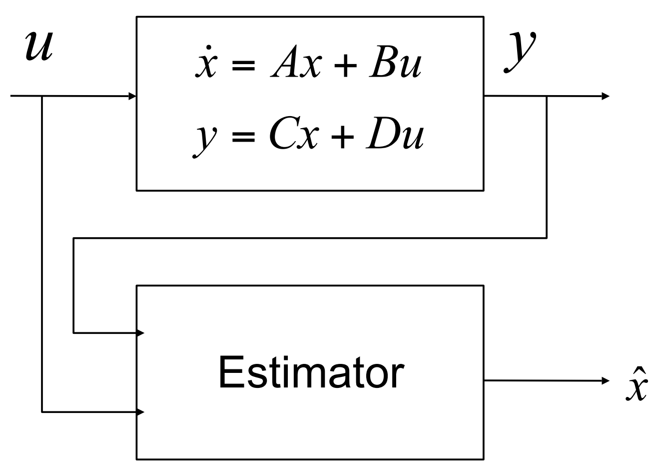
- For each eigenvalue $\lambda_i = \sigma_i + j\omega_i$, get contribution of the form

$$y_i(t) = e^{-\sigma t}\left(a \sin(\omega t) + b \cos(\omega t)\right)$$

- Repeated eigenvalues can give additional terms of the form $t^k e^{\sigma + j\omega}$



Use *estimator* to determine the current state if you can't measure it



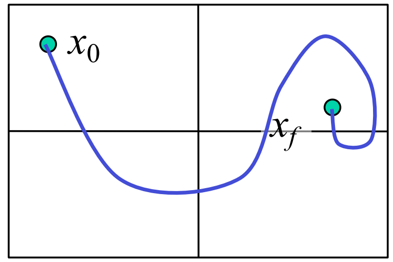
- Estimator looks at inputs and outputs of plant and estimates the current state
- Can show that if a system is *observable* then you can construct and estimator
- Use the *estimated* state as the feedback

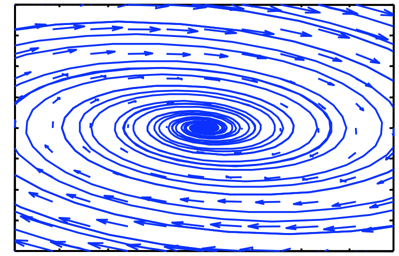
$$u = K\hat{x}$$

- *Kalman* filter is an example of an estimator

Summary: Reachability and State Space Feedback

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$





$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$u = -Kx + k_r r$$

Key concepts

- Reachability: find u s.t. $x_0 \rightarrow x_f$
- Reachability rank test for linear systems
- State feedback to assign eigen-values