CDS 101: Lecture 3.1
Stability and Performance

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11 October 2004

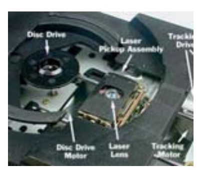
Goals:
- Describe different types of stability for an equilibrium point
- Explain the difference between local/global stability, and related concepts
- Describe performance measures for (controlled) systems, including transients and steady state response

Reading:
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 3

Review from Last Week

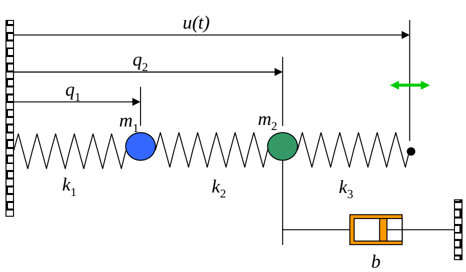
Model = state, inputs, outputs, dynamics

$$\frac{dx}{dt} = f(x,u) \qquad y = h(x)$$



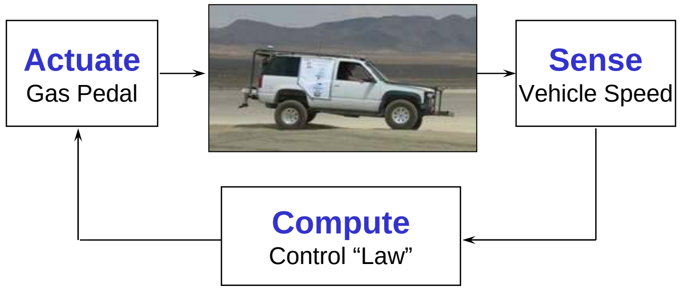
$$x_{k+1} = f(x_k, u_k) \qquad y_{k+1} = h(x_{k+1})$$

***Principle*: Choice of model depends on the questions you want to answer**




```
function dydt = f(t,y, k1, k2,
k3,   m1, m2, b, omega)
u = 0.00315*cos(omega*t);
dydt = [
  y(3);
  y(4);
  -(k1+k2)/m1*y(1) +
      k2/m1*y(2);
  k2/m2*y(1) - (k2+k3)/m2*y(2)
      - b/m2*y(4) + k3/m2*u ];
```

Today: Stability and Performance



Goal #1: Stability

- Check if *closed loop* response is stable

$\dot{x} = f(x,u) \qquad u = k(x)$

control law
system input

Goal #2: Performance

- Look at ability to track changes in reference and reject disturbances

$\dot{x} = f(x, k(x,r), d)$

$y = h(x)$

disturbance
reference
measured output

Goal #3: Robustness (later)

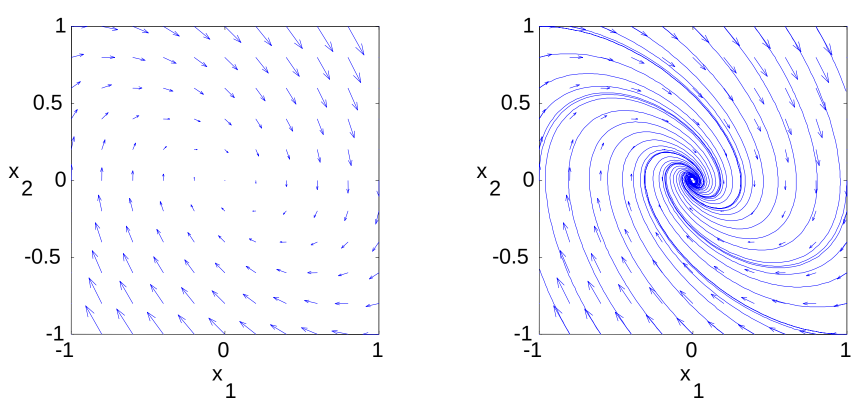
Phase Portraits (2D systems only)

Phase plane plots show 2D dynamics as *vector fields* & *stream functions*

- Plot *f*(*x*) as a vector on the plane; stream lines follow the flow of the arrows

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$

```
phaseplot('dosc', ...
  [-1 1 10], [-1 1 10], ...
  boxgrid([-1 1 10], [-1 1 10]));
```

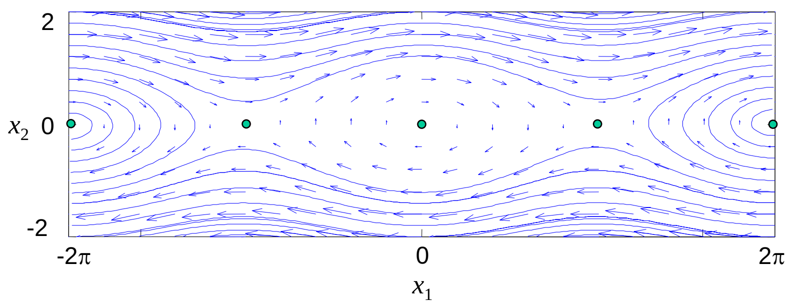


Equilibrium Points

Equilibrium points represent stationary conditions for the dynamics

The *equilibria* of the system $\dot{x} = f(x)$ are the points x_e such that $f(x_e) = 0$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) \end{bmatrix} \quad \Rightarrow \quad x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$



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Stability of Equilibrium Points

An equilibrium point is:

Asymptotically stable if all nearby initial conditions converge to the equilibrium point

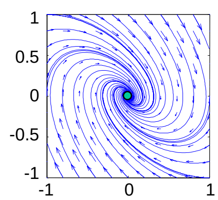
- Equilibrium point is an *attractor* or *sink*

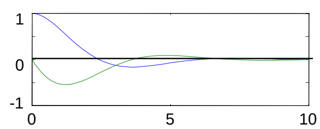
Unstable if some initial conditions diverge from the equilibrium point

- Equilibrium point is a *source* (or *saddle*)

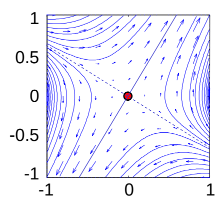
Stable if initial conditions that start near the equilibrium point, stay near

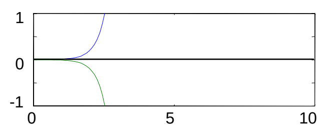
- Equilibrium point is a *center*



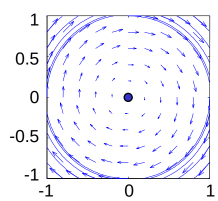


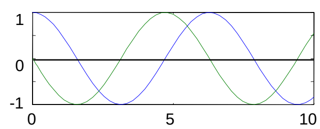
$$\lim_{t\to\infty} x(t) = x_e \;\; \forall \;\; \|x(0) - x_e\| < \varepsilon$$





$$\lim_{t\to\infty} \|x(t)\| = \infty \text{ for some } x(0)$$





$$\|x(t) - x_e\| < \varepsilon \;\; \forall t, \|x(0) - x_e\| < \delta_\varepsilon$$

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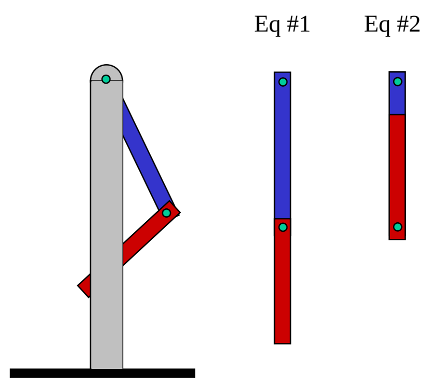
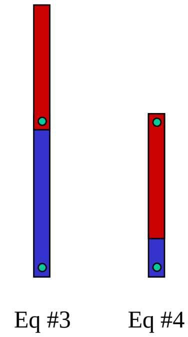
Example #1: Double Inverted Pendulum

Two series coupled pendula

- States: pendulum angles (2), velocities (2)
- Dynamics: $F = ma$ (balance of forces)
- Dynamics are very nonlinear





Stability of equilibria

- Eq #1 is stable
- Eq #3 is unstable
- Eq #2 and #4 are unstable, but with some stable “modes”

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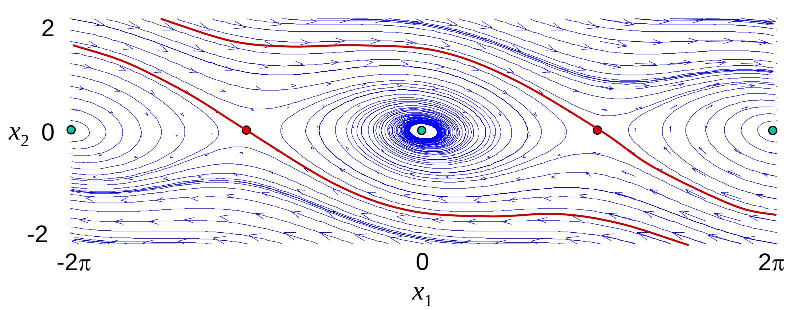
Local versus Global Behavior

Stability is a *local* concept

- Equilibrium points define the local behavior of the dynamical system
- Single dynamical system can have stable *and* unstable equilibrium points

Region of attraction

- Set of initial conditions that converge to a given equilibrium point



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Example #2: Predator Prey (ODE version)

Continuous time (ODE) version of predator prey dynamics:

$$\dot{R} = r_R R\left(1 - \frac{R}{K}\right) - \frac{aRF}{1 + aRT_h}$$
$$\dot{F} = r_F F\left(1 - \frac{F}{kR}\right)$$

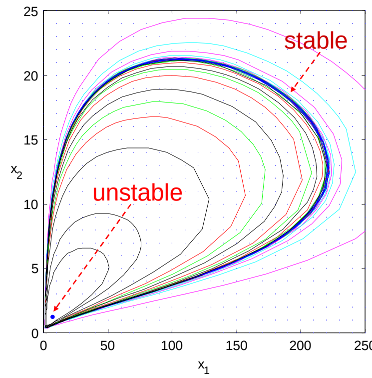
- Continuous time (ODE) model
- MATLAB: predprey.m (from web page)

Equilibrium points

- ~(6.5, 1.3): unstable ⇒ no steady state population

Invariant curves (3)

- Start on curve, stay on curve
- “Limit cycle” ⇒ population of each species oscillates over time
- This is a *global* feature of the dynamics (not local to an equilibrium point)



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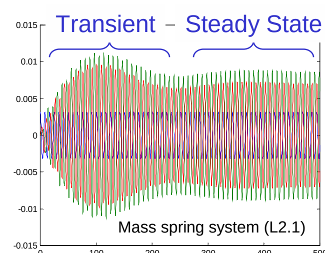
Input/Output Performance

Return to system with inputs

- How does system response to changes in input values?

Transient response:

- What happens right after a new input is applied



Steady state response:

- What happens a long time after the input is applied

Stability vs input/output performance

- Systems that are close to instability typically exhibit poor input/output performance
- Nearly unstable systems (slow convergence) often exhibit “ringing” (highly oscillatory response to [non-periodic] inputs)

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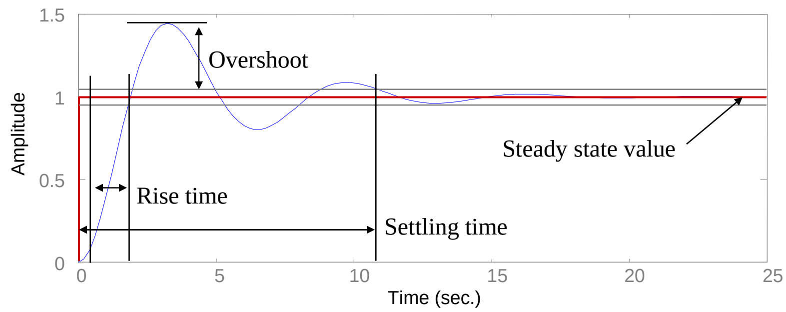
Step Response

Output characteristics in response to a "step" input

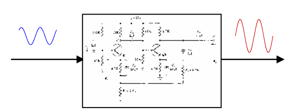
- Rise time: time required to move from 5% to 95% of final value
- Overshoot: ratio between amplitude of first peak and steady state value
- Settling time: time required to remain w/in *p*% (usually 2%) of final value
- Steady state value: final value at t = ∞



Frequency Response

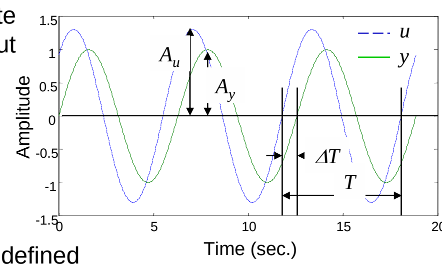
Measure the *steady state* response of the system to sinusoidal input

- Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
- Individual sinusoids are good *test signals* for measuring performance in many systems (eg, seasonal cycles in temperature)



Approach: plot input and output, measure *relative* amplitude and phase

- Use MATLAB or SIMULINK to generate response of system to sinusoidal output
- Gain = A_y/A_u
- Phase = $2\pi \cdot \Delta T/T$



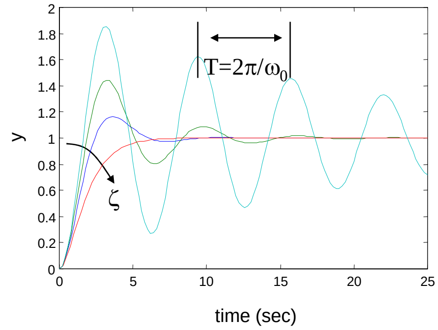
May not work for *nonlinear* systems

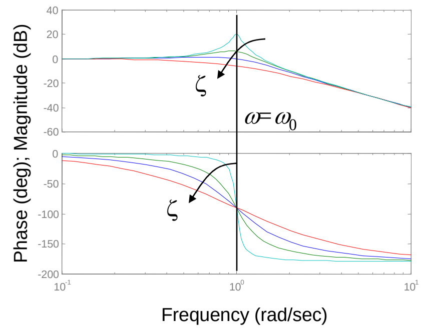
- System nonlinearities can cause *harmonics* to appear in the output
- Amplitude and phase may not be well-defined
- For *linear* systems, frequency response is always well defined (week 6)

Second Order Systems

Important class of systems in many applications areas

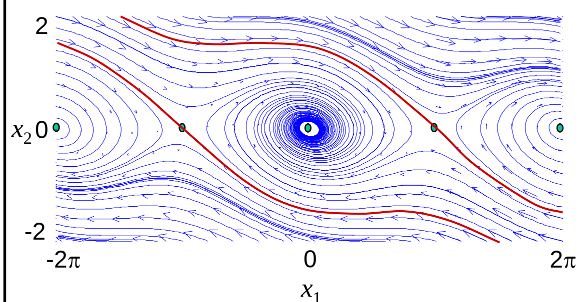
$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \quad \longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_0^2 & 2\zeta\omega_0\end{bmatrix}x + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u$$





- Analytical formulas exist for overshoot, rise time, settling time, etc
- Frequency response can also be analytically derived

Summary: Stability and Performance



Key topics for this lecture

- Stability of equilibrium points
- Local versus global behavior
- Performance specification via step and frequency response