CDS 110: Lecture 2-2
Modeling Using Differential Equations

Richard M. Murray and Hideo Mabuchi

4 October 2006

Goals:

- Provide a more detailed description of the use of ODEs for modeling
- Provide examples of the type of analysis that can be done using ODEs

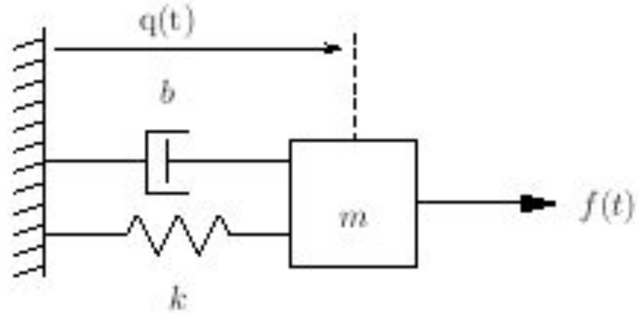
Reading:

- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 2
- Advanced: Lewis, *A Mathematical Approach to Classical Control*, Ch 1

Review: Second Order Differential Equations (Ma 1)

Damped oscillator dynamics

$$m\ddot{q} + c\dot{q} + kq = f(t)$$



Homogeneous solution: f(t) = 0

- Guess form of the solution: $q(t) = e^{\alpha t}(A\cos\omega t + B\sin\omega t)$
- Substitute into ODE and solve for the constants

$$0 = e^{\alpha t}\Big(\big(B(c + 2\alpha m)\omega + A\left(m\alpha^2 + c\alpha - m\omega^2 + k\right)\big)\cos(\omega t)$$
$$+ \left(Bm\alpha^2 + Bc\alpha - 2Am\omega\alpha - Bm\omega^2 + Bk - Ac\omega\right)\sin(\omega t)\Big)$$

$$q_0 = A$$
$$v_0 = A\alpha + B\omega$$

Solve for *A* & *B*

Coefficents of sin/cos must be zero
Use to solve for α, ω

- Simplify the solution by pulling out common terms

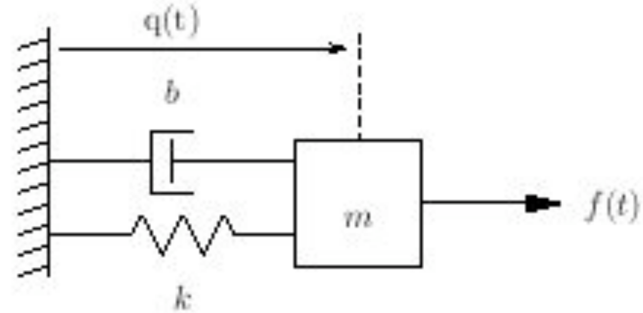
$$q(t) = e^{-\zeta\omega_0 t}\left(q_0\cos\omega_d t + \left(\frac{\zeta\omega_0}{\omega_d}q_0 + \frac{1}{\omega_d}v_0\right)\sin\omega_d t\right)$$

$$\omega_0 = \sqrt{k/m}$$
$$\zeta = \frac{1}{2}\sqrt{c^2/km}$$
$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

- Note: this solution holds when $\zeta < 1$

Second Order Differential Equations, ctd

$$m\ddot{q} + c\dot{q} + kq = f(t)$$



Particular response: zero initial conditions

- $q(0) = 0, \; \dot{q}(0) = 0$
- Response to constant (step) input, *f*(*t*) = *F*

$$q(t) = \frac{F}{m\omega_0^2}\left(1 - e^{-\zeta\omega_0 t}\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t}\sin\omega_d t\right)$$

- Response to sinusoidal input, *f*(*t*) = *A* sin ω *t*

$$q(t) = MA\sin(\omega t + \theta) \qquad Me^{j\theta} = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + 2j\zeta\omega_0\omega}.$$

- Form of the solution: sinusoid at same frequency, with shift in mag & phase
- Solving by hand is a mess; we will learn much better ways later

Complete solution: homogeneous + particular

More General Forms of Differential Equations

State space form

$$\frac{dx}{dt} = f(x,u) \qquad y = h(x,u)$$

General form

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

Linear system

$$x \in \mathbb{R}^n,\ u \in \mathbb{R}^p \qquad y \in \mathbb{R}^q$$

- *x* = state; *n*th order
- *u* = input; will usually set p = 1
- *y* = output; will usually set q = 1

Higher order, linear ODE

$$\frac{d^n q}{dt^n} + a_1 \frac{d^{n-1} q}{dt^{n-1}} + \cdots + a_n q = u$$

$$y = b_1 \frac{d^{n-1} q}{dt^{n-1}} + \cdots + b_{n-1}\dot{q} + b_n q$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d^{n-1}q/dt^{n-1} \\ \vdots \\ dq/dt \\ q \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \dots & & -a_n \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & & & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} x + du.$$

Analytical Solutions of ODEs

Scalar systems

$$\frac{dx}{dt} = ax + u \qquad x_h(t) = e^{at}x_0 \qquad u = A \sin \omega_1 t$$

$$y = x \qquad\qquad y = -A\frac{-\omega_1 e^{at} + \omega_1 \cos \omega_1 t + a \sin \omega_1 t}{a^2 + \omega_1^2}$$

Decoupled systems

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u$$

$$y = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix} x + Du.$$

$$\dot{x}_i = \lambda_i x_i + \beta_i u$$

$$x_i(t) = e^{\lambda_i t} x(0) + \int_0^t e^{\lambda_i (t-\tau)} \beta_i u(\tau)\, d\tau.$$

- Effect of input modeled by "convolution integral"

General solutions

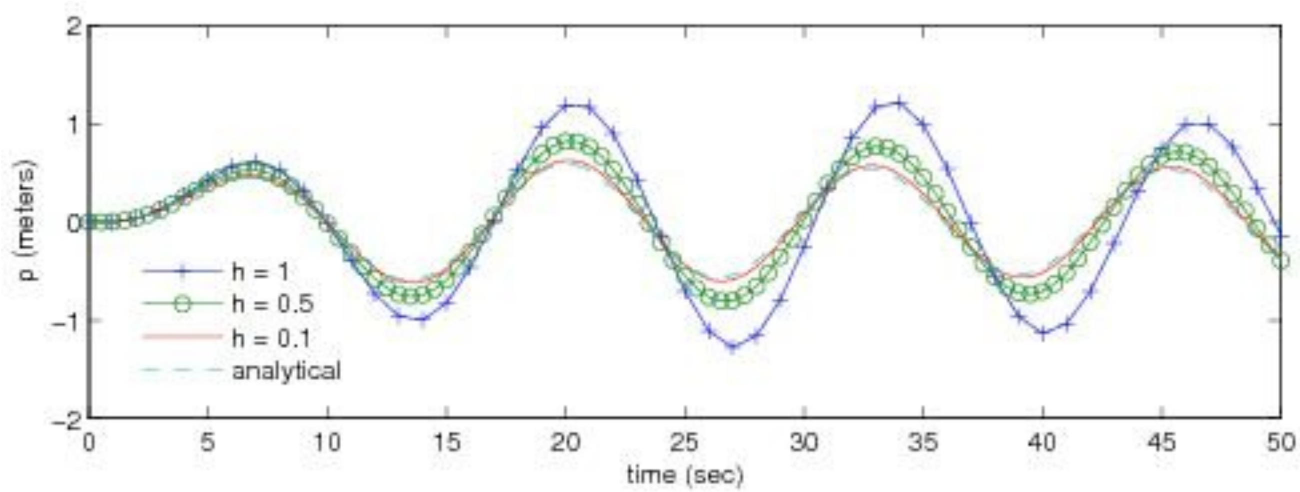
- Linear systems: use Jordan canonical form and "matrix exponential" (more later)
- Nonlinear system: generally no closed form solutions, expect in special cases

Numerical Solution of ODEs

Numerical simulation: Euler integration

$$\frac{dx}{dt} = \lim_{\epsilon \to 0} \frac{x(t+\epsilon) - x(t)}{\epsilon} \qquad \Longrightarrow \qquad x(t+\epsilon) \approx x(t) + \epsilon f(x(t), u(t)).$$

- If ε chosen sufficently small, get good approximation analytical solution
- Solution is in the form of a difference equation (with step size ε)



- More accurate algorithms: build better approximation to the derivative
- Faster algorithms: choose the step size based on how quickly solution is changing
- Example: Runga Kutta (ode45)

Analyzing Models using ODEs: Frequency Response

How does linear system respond to sinusoidal inputs?

$$m\ddot{q} + c\dot{q} + kq = f(t).$$

$$f(t) = A \sin \omega t.$$

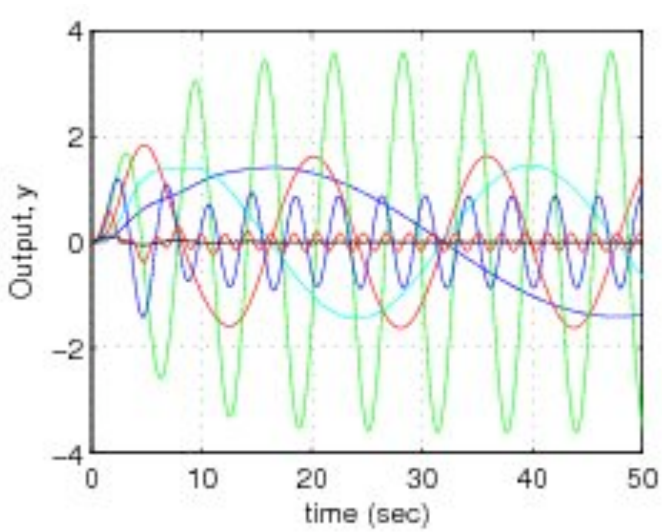
$$q(t) = g(\omega) \sin(\omega t + \phi(\omega)),$$

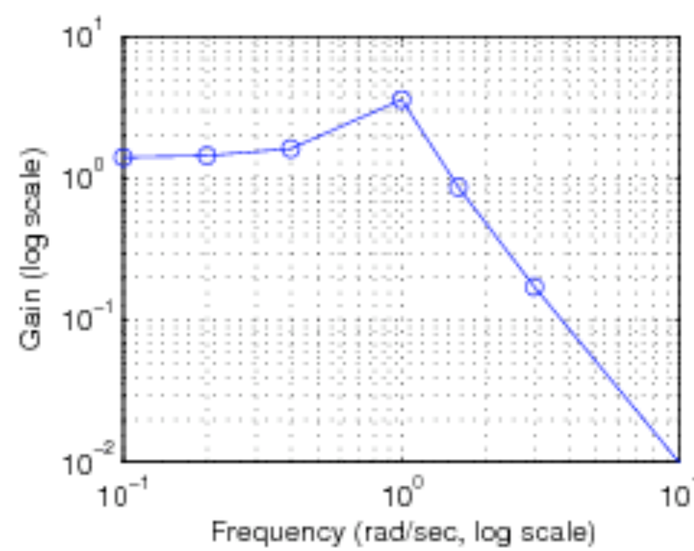
magnitude → $g(\omega)$

phase → $\phi(\omega)$

General properties

- Linear systems: sinusoidal input at frequency $\omega \Rightarrow$ sinusoidal output at frequency ω
- Gain = $\dfrac{\text{output magnitude}}{\text{input magnitude}} = \dfrac{g(\omega)}{A}$
- Phase: shift in input sinusoid versus output sinusoid



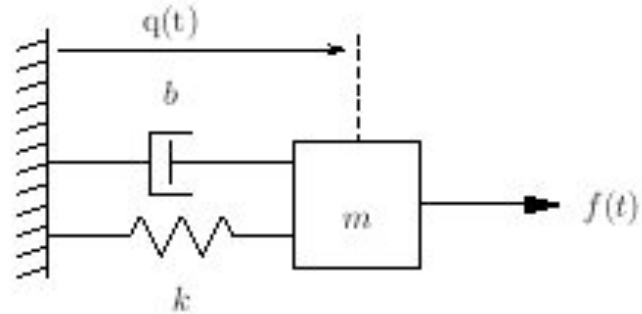


Analyzing Models Using ODEs: Stability

ODEs can also be used to prove stability of a systems

- Try to reason about the long term behavior of *all* solutions
- Stability ≈ all solutions return to equilibrium point (more precise defn later)

$$m\ddot{q} + c\dot{q} + kq = 0$$



Example: spring mass system

- Can we show that all solutions return to rest w/out explicitly solving ODE?
- Idea: look at how energy evolves in time

- Start by writing equations in state space form

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{bmatrix} \qquad \begin{aligned} x_1 &= q \\ x_2 &= \dot{q} \end{aligned}$$

- Compute energy and its derivative

$$V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2. \qquad \frac{dV}{dt} = kx_1\dot{x}_1 + mx_2\dot{x}_2$$

$$= kx_1x_2 + mx_2(-\frac{b}{m}x_2 - \frac{k}{m}x_1) = -bx_2^2,$$

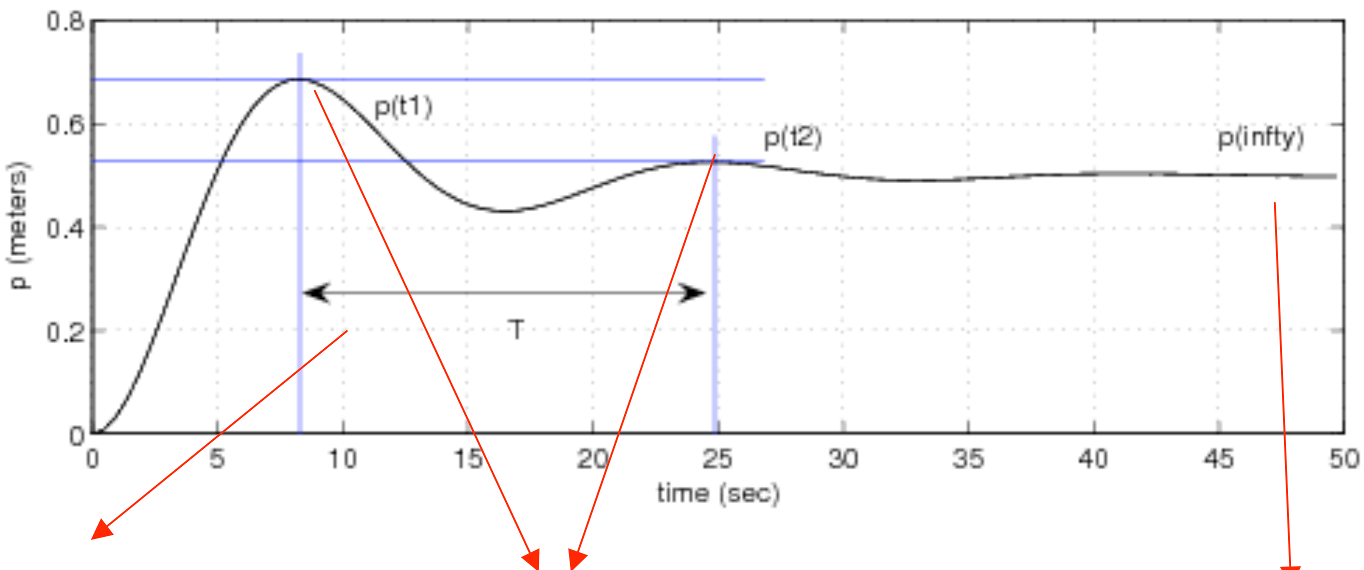
- Energy is positive ⇒ x_2 must eventually go to zero
- If x_2 goes to zero, can show that x_1 must also approach zero (Lasalle, W3)

Modeling from Experiments

Example: spring mass system

- Measure response of system to a step input

$$q(t) = \frac{F_0}{k}\left(1 - e^{-\frac{bt}{2m}}\left[\cos(\frac{\sqrt{4km-b^2}}{2m}t) - \frac{1}{\sqrt{4km-b^2}}\sin(\frac{\sqrt{4km-b^2}}{2m}t)\right]\right)$$



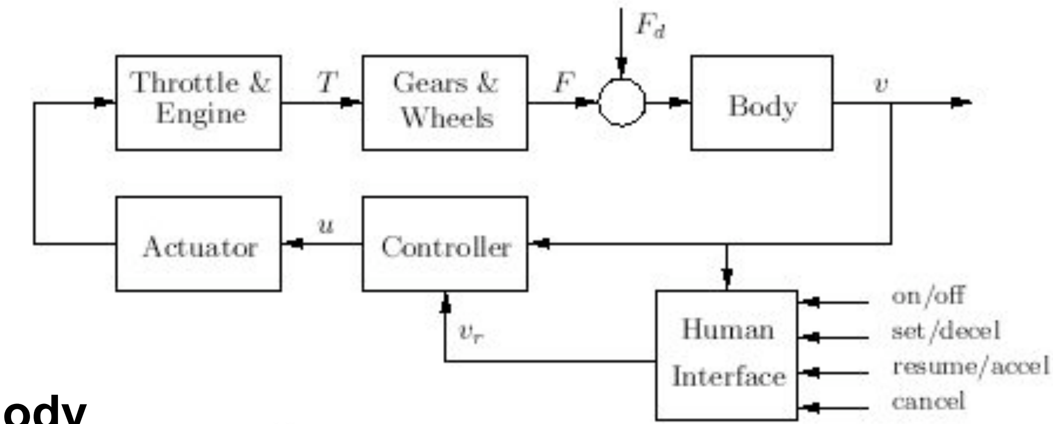
$$\frac{2\pi}{T} = \frac{\sqrt{4km-b^2}}{2m}$$

$$\log(q(t_1) - F_0/k) - \log(q(t_2) - F_0/k) = \frac{b}{2m}(t_2 - t_1)$$

$$q(\infty) = \frac{F_0}{k}$$

Block Diagrams

Block diagrams separate components of a system into manageable units



Example: cruise control

- Each block corresponds to a portion of the overall dynamics
- Write out the individual blocks as input/output systems

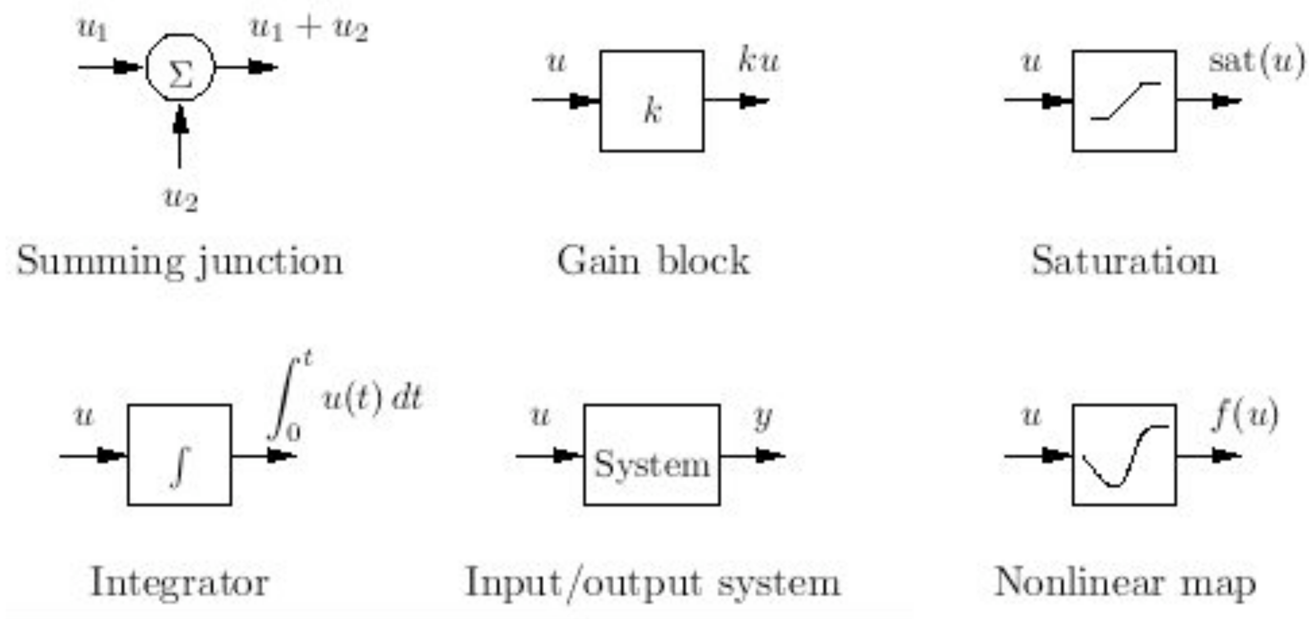
Body

- Dynamics: $m\frac{dv}{dt} = F - F_d.$
- State: v - velocity of vehicle
- Inputs: F, F_d - force from wheels, external disturbances (wind, hills, etc)
- Output: v - velocity of vehicle

Dynamic versus state blocks

- Some blocks represent static relationships (no states); eg, gears and wheels

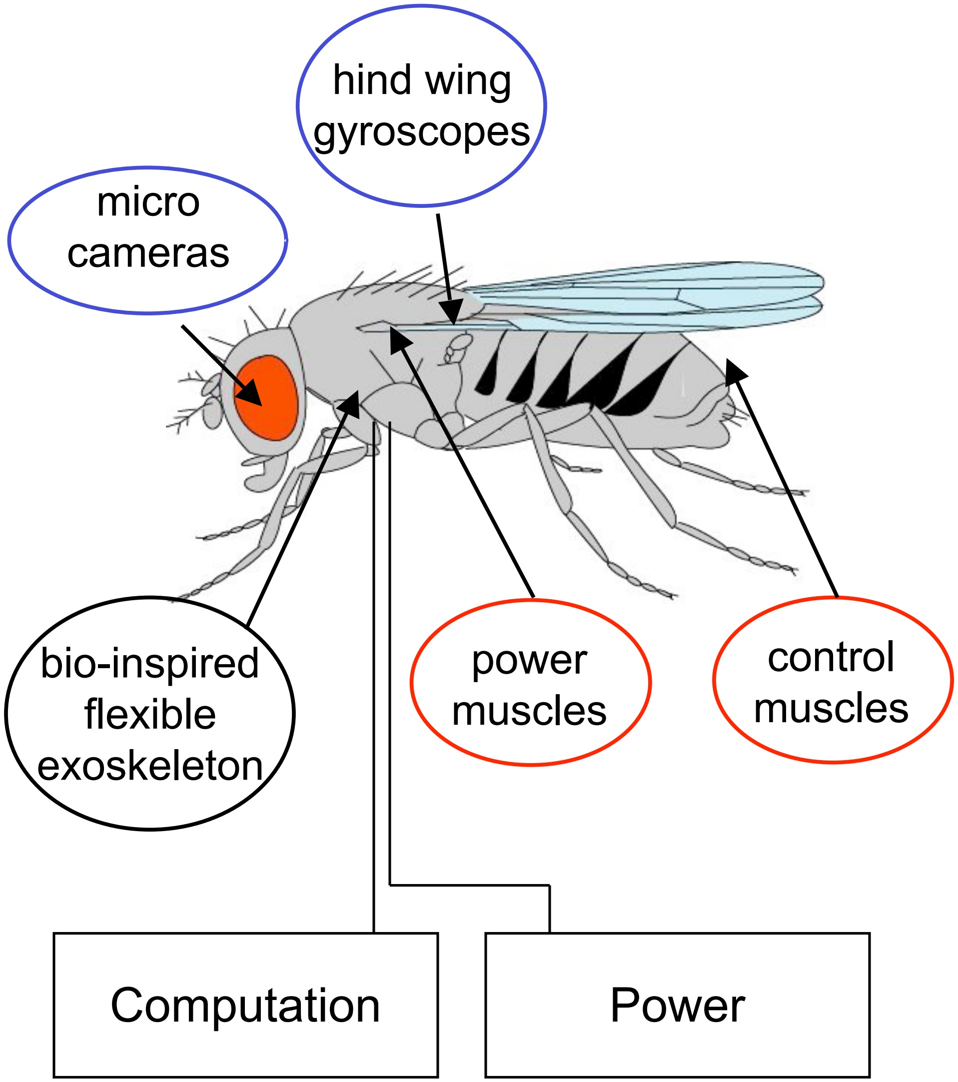
Standard Block Diagram Notation



Remarks

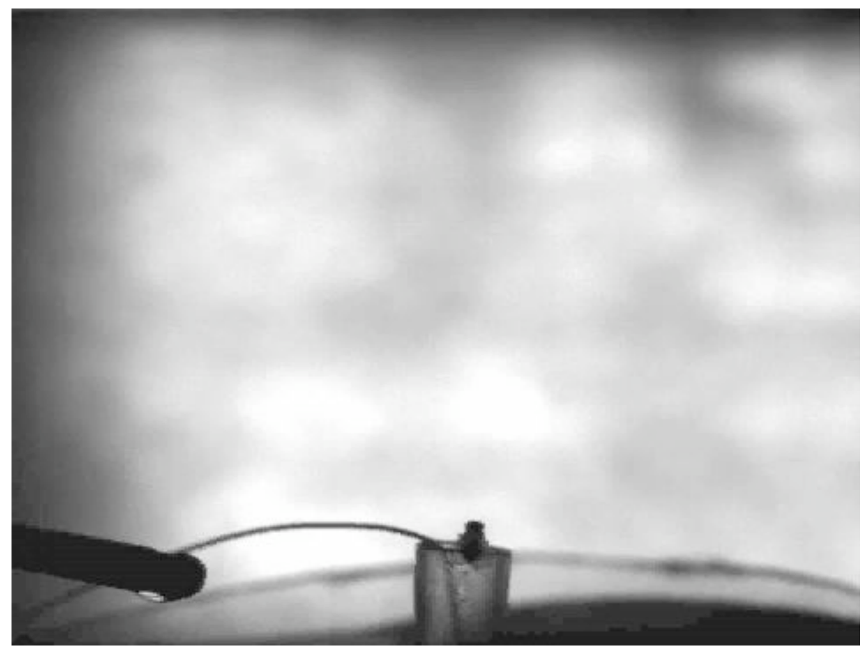
- SIMULINK uses slightly different symbols in a few places (eg, gain block)

Example: Hovering Mesoscale Robot (HOMER)



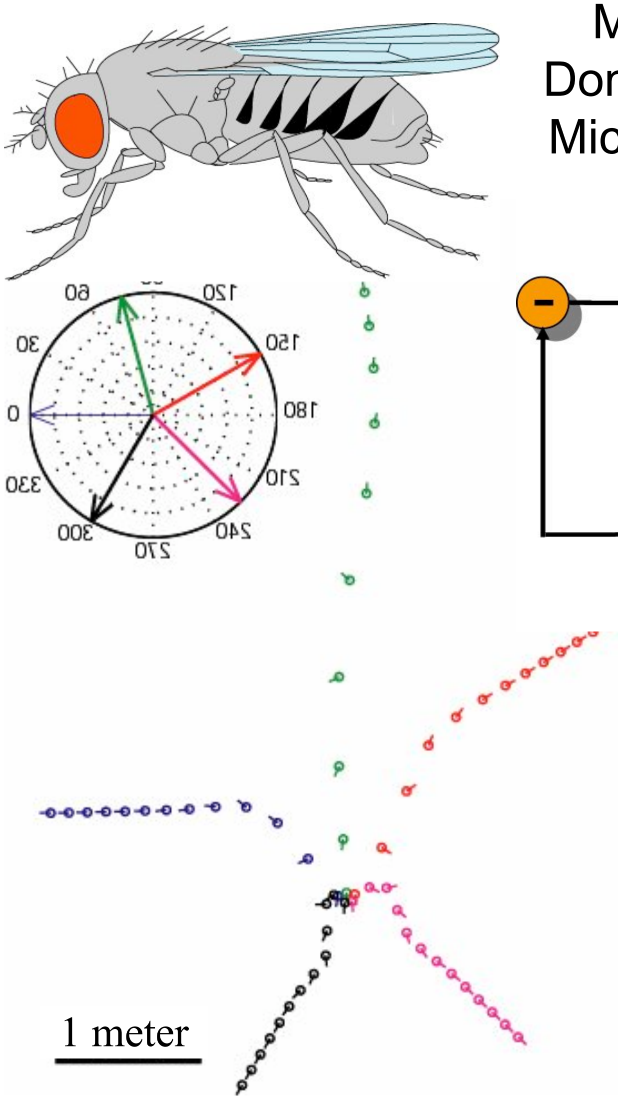
Project Goals

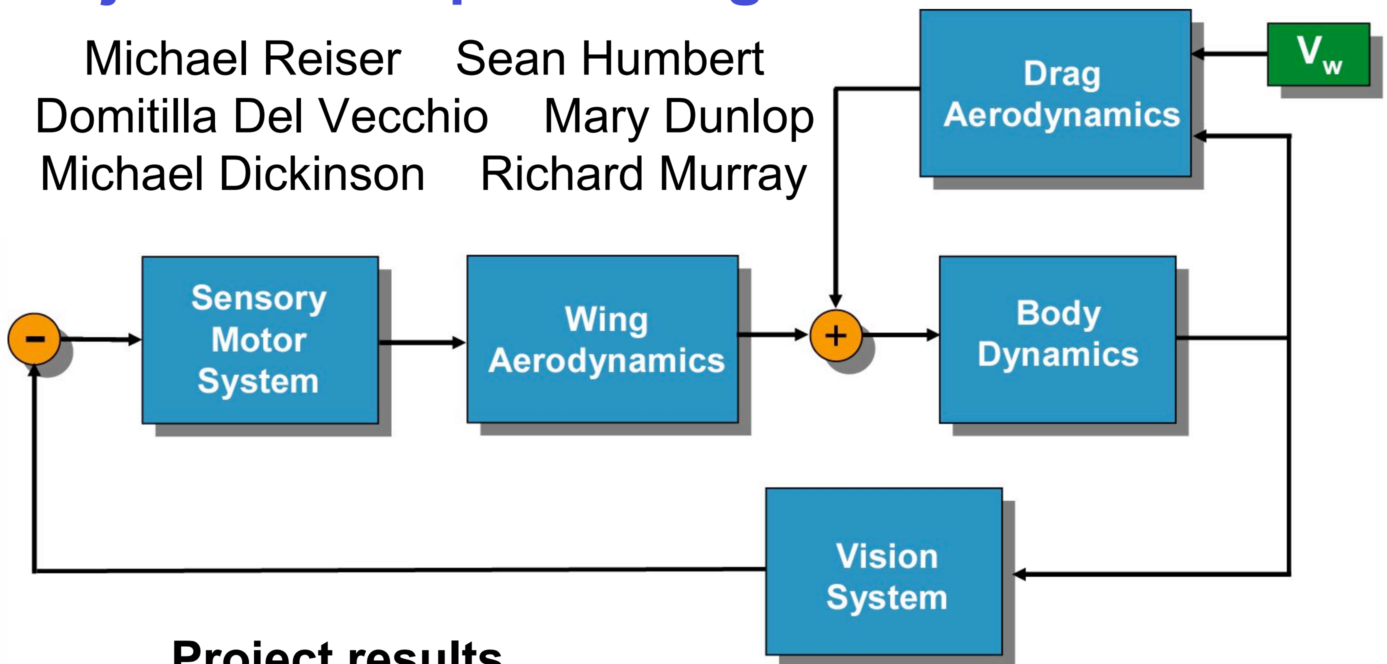
- Characterize and reverse engineer the sensory-motor control system of the fly
- Apply salient features to the design of micro air vehicles and other autonomous systems
- Experimentation and modeling key components of flight control system: (1) take-off, (2) robustness to wing gust, (3) chemical tracking, and (4) sensory fusion (visual, gyro)



Vision as a Compensatory Mechanism for Disturbance Rejection in Upwind Flight

Michael Reiser   Sean Humbert
Domitilla Del Vecchio   Mary Dunlop
Michael Dickinson   Richard Murray





Project results

- Interconnected simplified models that provide bio-realistic behavior for upwind flight

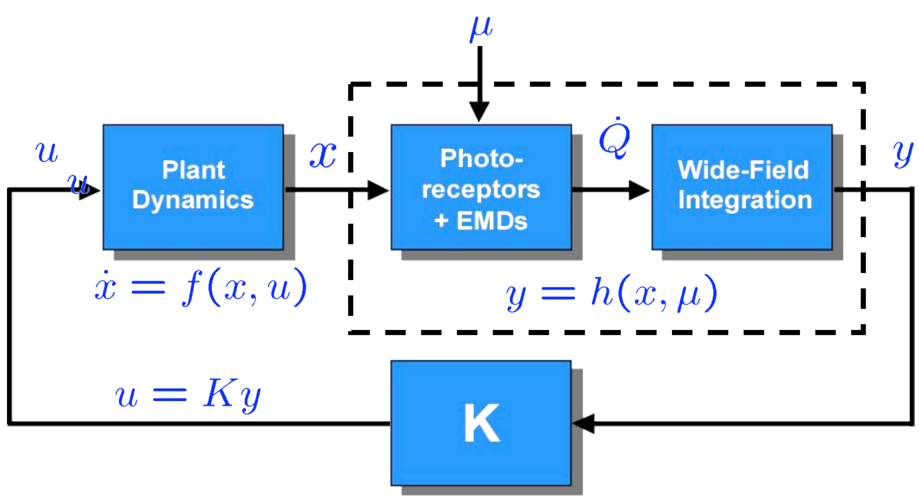
Insights

- Low level (fast!) vision and sensory motor processing capable of generated complex behaviors that achieve desired response

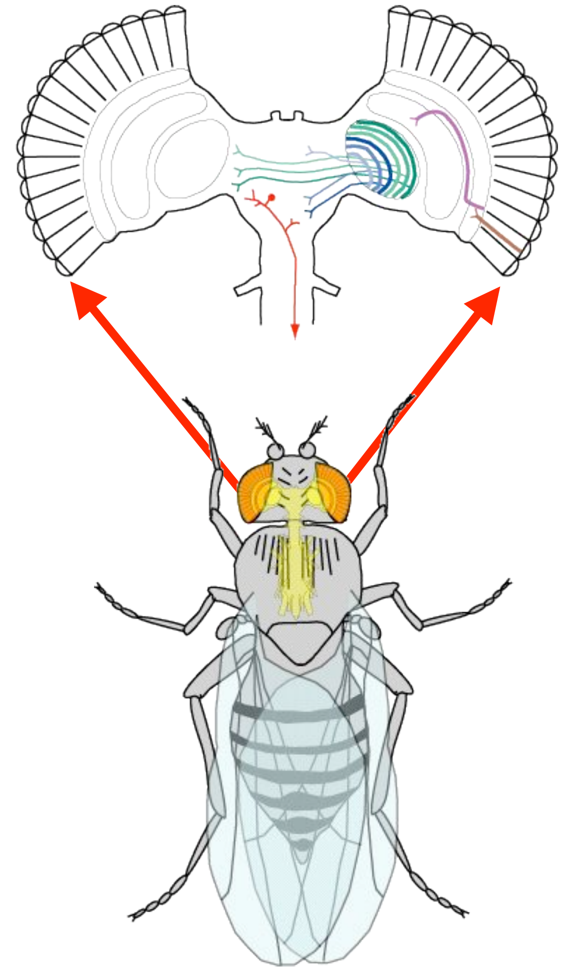
Vision-Based Navigation Using Wide-Field Integration

Sean Humbert (U. Maryland)

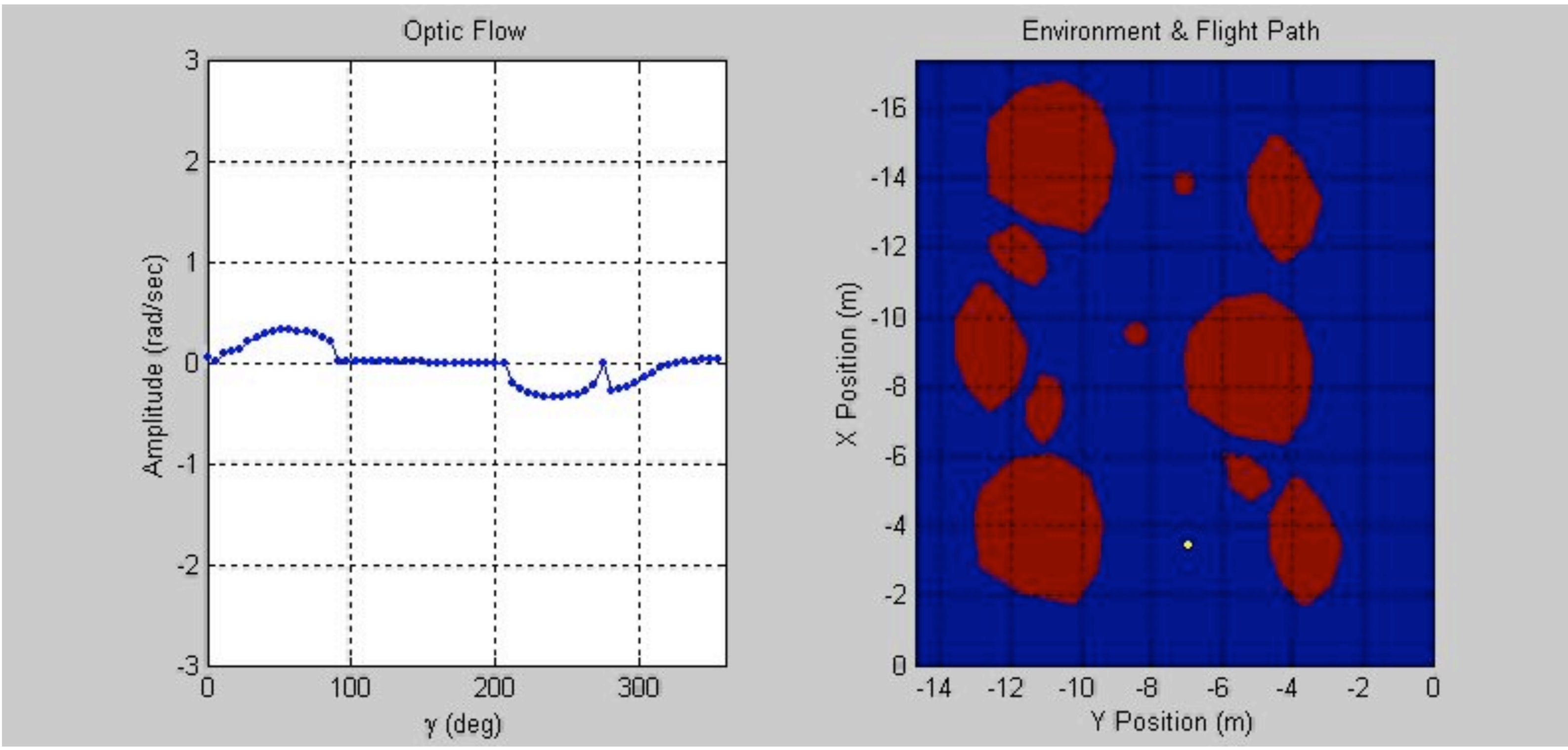
- **Approach**
  - **Understand & characterize wide field integration processing in Drosophila**
  - **Near 360° optical flow processing**
  - **Very fast coupling to flight actuation**



Flight Stabilization and Obstacle Avoidance



Engineering Applications in Vision-Based Navigation



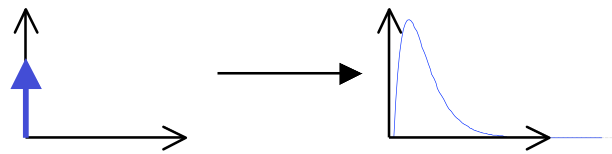
Preview: Linear Control Systems and Convolution

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\longrightarrow$ $y(t) = \underbrace{Ce^{At}x(0)}_{\text{Homogeneous via } \textit{matrix exponential}} +$ particular

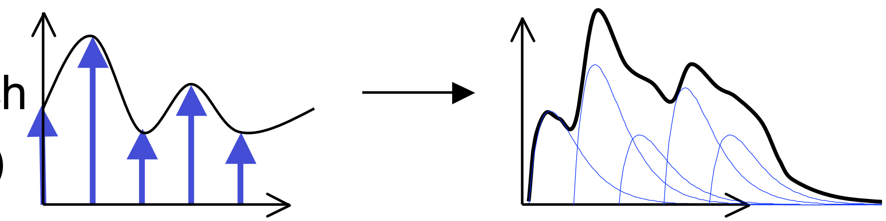
Impulse response, $h(t) = Ce^{At}B$

- Response to input "impulse"
- Equivalent to "Green's function"



Linearity ⇒ compose response to arbitrary *u*(*t*) using *convolution*

- Decompose input into "sum" of shifted impulse functions
- Compute impulse response for each
- "Sum" impulse response to find $y(t)$



Complete solution: use integral instead of "sum"

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input ⇒ 2X output when $x(0) = 0$