CDS 101: Lecture 2.1
System Modeling

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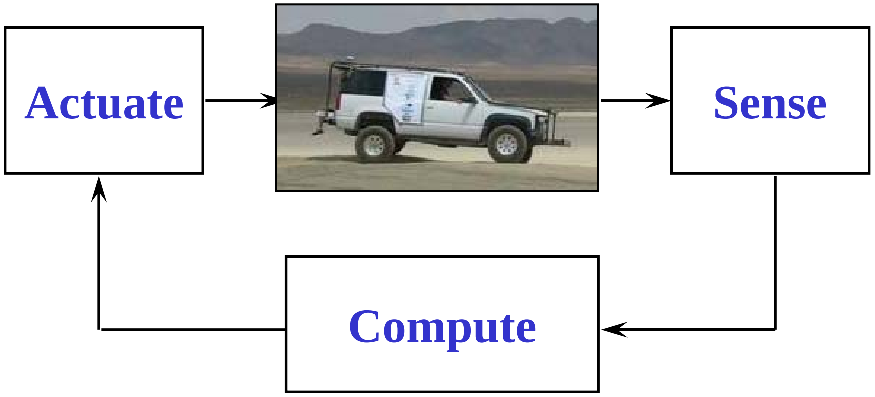
Goals:

- Define what a model is and its use in answering questions about a system
- Introduce the concepts of state, dynamics, inputs and outputs
- Provide examples of common modeling techniques: differential equations, difference equations, finite state automata

Reading:

- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 2
- Advanced: Lewis, *A Mathematical Approach to Classical Control*, Ch 1

Review from last week



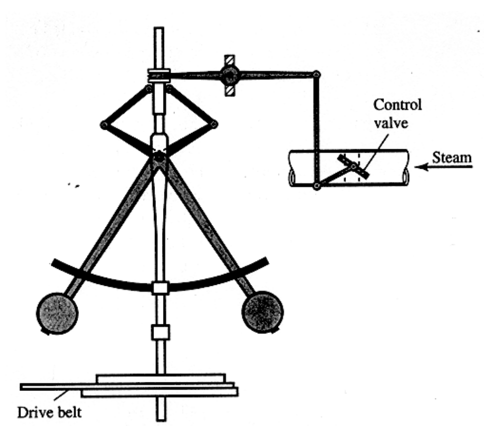
Control =

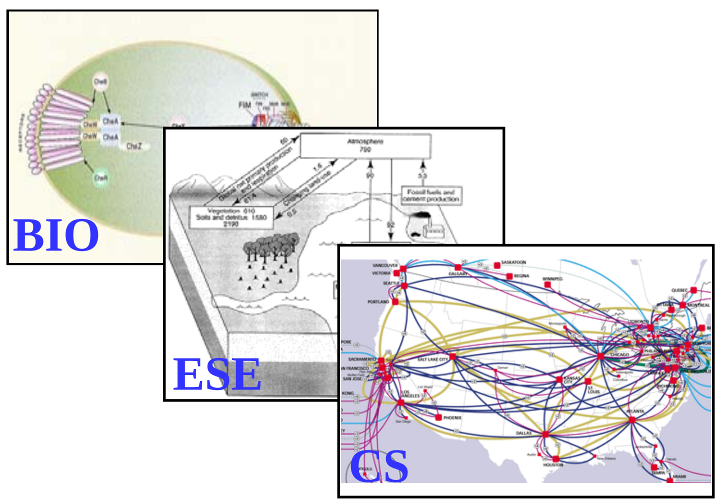
Sensing + Computation + Actuation

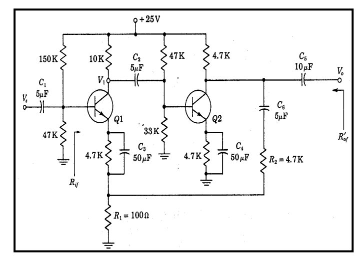
Feedback Principles

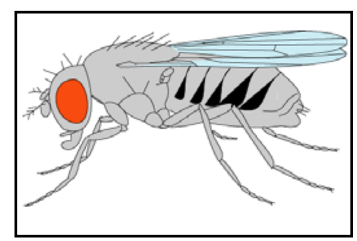
- Robustness to Uncertainty
- Design of Dynamics

Many examples of feedback and control in natural & engineered systems:









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Model-Based Analysis of Feedback Systems

Analysis and design based on *models*

- A model provides a *prediction* of how the system will behave
- Feedback can give counter-intuitive behavior; models help sort out what is going on
- For control design, models don't have to be exact: *feedback* provides robustness

Control-oriented models: *inputs* and *outputs*

The model you use depends on the questions you want to answer

- A single system may have many models
- Time and spatial scale must be chosen to suit the questions you want to answer
- Formulate questions *before* building a model

Weather Forecasting

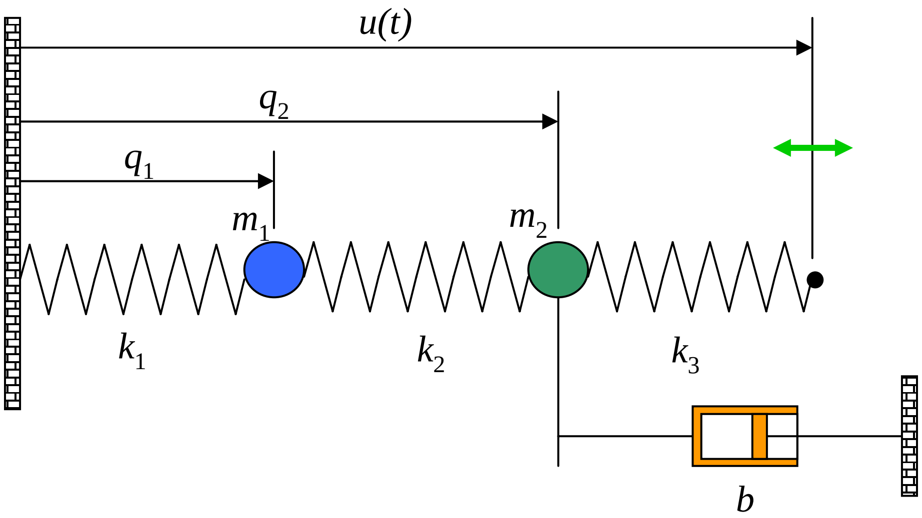
- **Question 1: how much will it rain tomorrow?**
- **Question 2: will it rain in the next 5-10 days?**
- **Question 3: will we have a drought next summer?**

Different questions ⇒ different models

RMM16 Use a different model, more related to feedback and control

Richard Murray, 10/4/2003

Example #1: Spring Mass System



Applications

- Flexible structures (many apps)
- Suspension systems (eg, “Bob”)
- Molecular and quantum dynamics

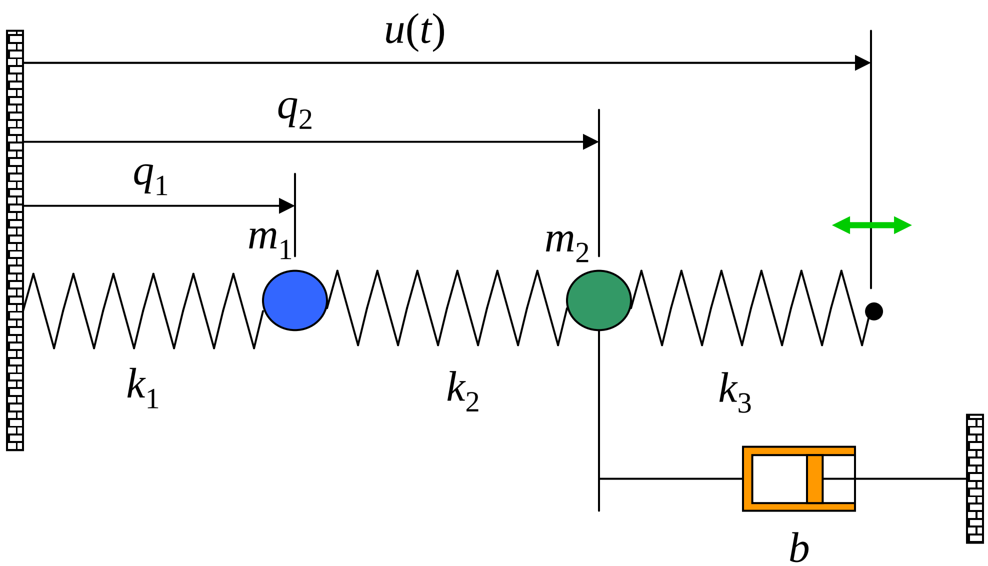
Questions we want to answer

- How much do masses move as a function of the forcing frequency?
- What happens if I change the values of the masses?
- Will Bob fly into the air if I take that hill at 25 mph?

Modeling assumptions

- Mass, spring, and damper constants are fixed and known
- Springs satisfy Hooke’s law
- Damper is (linear) viscous force, proportional to velocity

Modeling a Spring Mass System



Model: rigid body physics (Ph 1)

- Sum of forces = mass $*$ acceleration
- Hooke's law: $F = k(x - x_{rest})$
- Viscous friction: $F = b\,v$

$$
\begin{aligned}
m_1 \ddot{q}_1 &= k_2(q_2 - q_1) - k_1 q_1 \\
m_2 \ddot{q}_2 &= k_3(u - q_2) - k_2(q_2 - q_1) - b\dot{q}_2
\end{aligned}
$$

Converting models to state space form

- Construct a *vector* of the variables that are required to specify the evolution of the system
- Write dynamics as a *system* of first order differential equations:

$$
\frac{dx}{dt} = f(x,u) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^p
$$

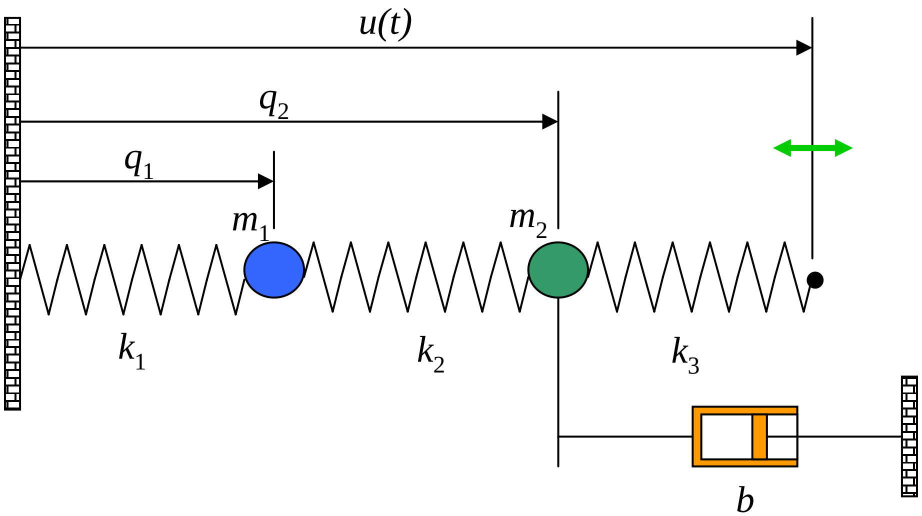
$$
y = h(x) \qquad y \in \mathbb{R}^q
$$

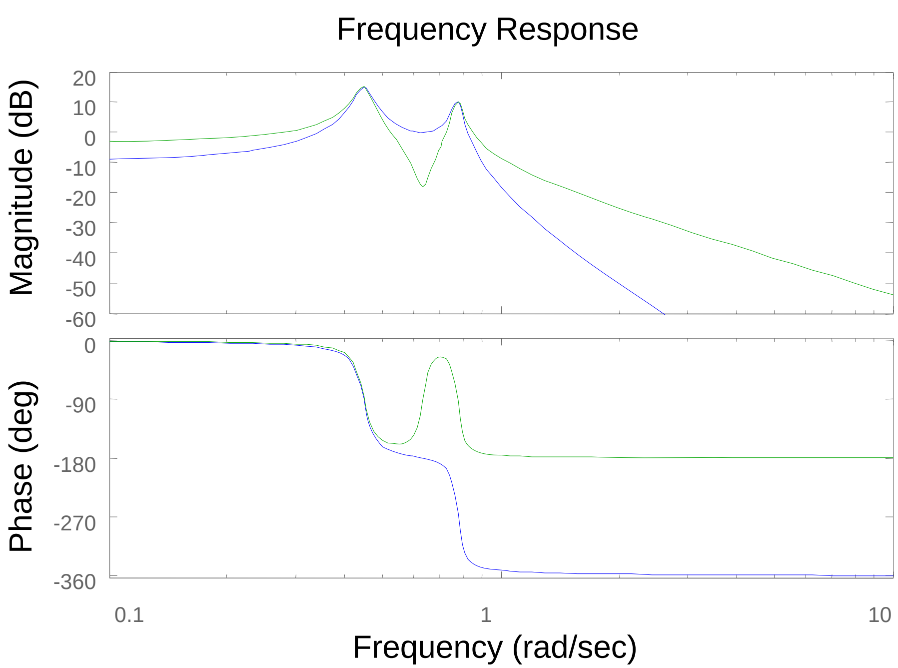
$$
\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m} q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{b}{m}\dot{q}_2 \end{bmatrix}
$$

$$
y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
$$

"State space form"

Frequency Response for a Mass Spring System





Steady state frequency response

- Force the system with a sinusoid
- Plot the "steady state" response, after transients have died out
- Plot relative magnitude and phase of output versus input (more later)

Matlab simulation (see handout)

```
function dydt = f(t, y, ...)
u = 0.00315*cos(omega*t);
dydt = [
  y(3);
  y(4);
  -(k1+k2)/m1*y(1) + k2/m1*y(2);
  k2/m2*y(1) - (k2+k3)/m2*y(2)
      - b/m2*y(4) + k3/m2*u ];

t,y] = ode45(dydt,tspan,y0,[], k1,
k2, k3, m1, m2, b, omega);
```

RMM14

Modeling Terminology

***State* captures effects of the past**

- independent physical quantities that determines future evolution (absent external excitation)

***Inputs* describe external excitation**

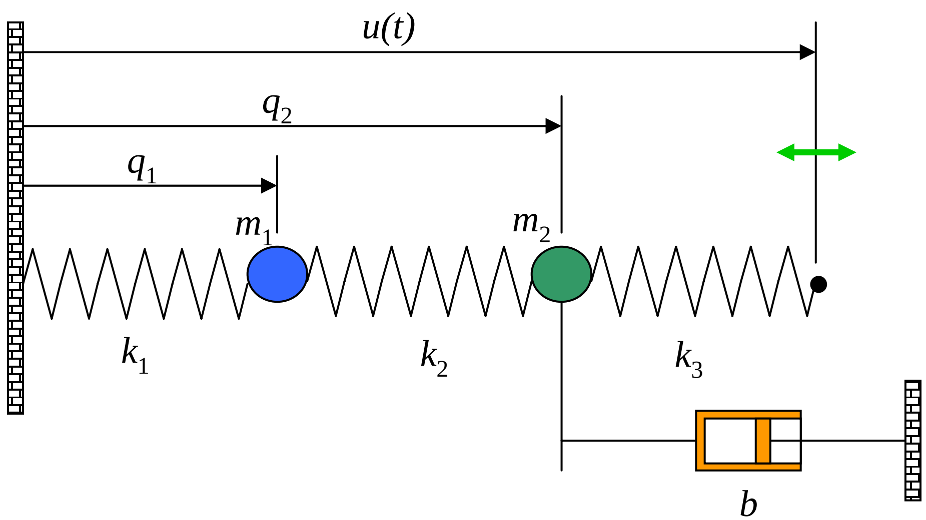
- Inputs are *extrinsic* to the system dynamics (externally specified)

***Dynamics* describes state evolution**

- update rule for system state
- function of current state and any external inputs

***Outputs* describe measured quantities**

- Outputs are function of state and inputs $\Rightarrow$ not independent variables
- Outputs are often *subset* of state



Example: spring mass system

- State: position and velocities of each mass: $q_1, q_2, \dot{q}_1, \dot{q}_2$
- Input: position of spring at right end of chain: $u(t)$
- Dynamics: basic mechanics
- Output: measured positions of the masses: q_1, q_2

Slide 8

RMM14 Put an example back in (cut out of previous)
Richard Murray, 10/4/2003

Modeling Properties

Choice of state is not unique

- There may be *many* choices of variables that can act as the state
- Trivial example: different choices of units (scaling factor)
- Less trivial example: sums and differences of the mass positions (HW 2.4)

Choice of inputs and outputs depends on point of view

- Inputs: what factors are *external* to the model that you are building
  - Inputs in one model might be outputs of another model (eg, the output of a cruise controller provides the input to the vehicle model)
- Outputs: what physical variables (often states) can you *measure*
  - Choice of outputs depends on what you can sense and what parts of the component model interact with other component models

Can also have different *types* of models

- Ordinary differential equations for rigid body mechanics
- Finite state machines for manufacturing, Internet, information flow
- Partial differential equations for fluid flow, solid mechanics, etc

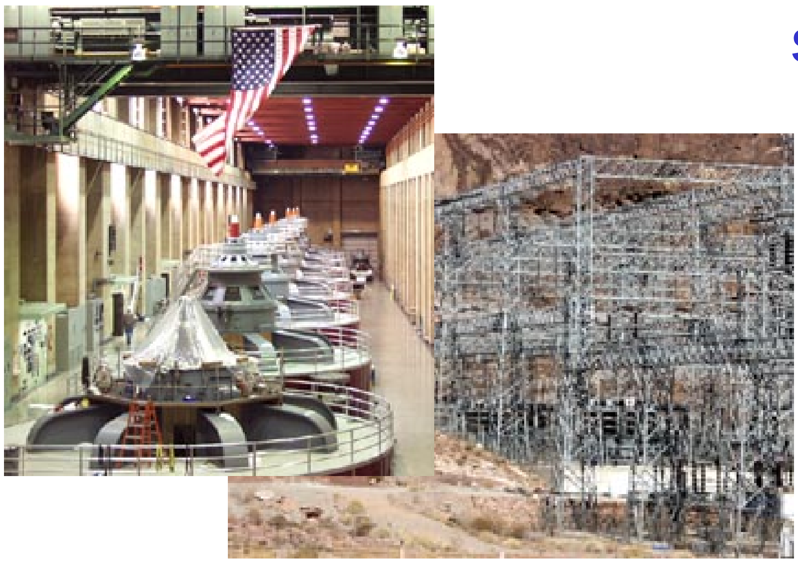
Differential Equations

Differential equations model continuous evolution of state variables

- Describe the rate of change of the state variables
- Both state and time are continuous variables

$$\frac{dx}{dt} = f(x,u)$$
$$y = h(x)$$

Example: electrical power grid



Swing equations

$$\ddot{\delta}_1 + D_1\dot{\delta}_1 = \omega_0\left(P_1 - B\sin(\delta_1 - \delta_2) + G\cos(\delta_1 - \delta_2)\right)$$
$$\ddot{\delta}_2 + D_1\dot{\delta}_2 = \omega_0\left(P_2 - B\sin(\delta_1 - \delta_2) + G\cos(\delta_1 - \delta_2)\right)$$

- Describe how generator rotor angles (δ_i) interact through the transmission line (G, B)
- Stability of these equations determines how loads on the grid are accommodated

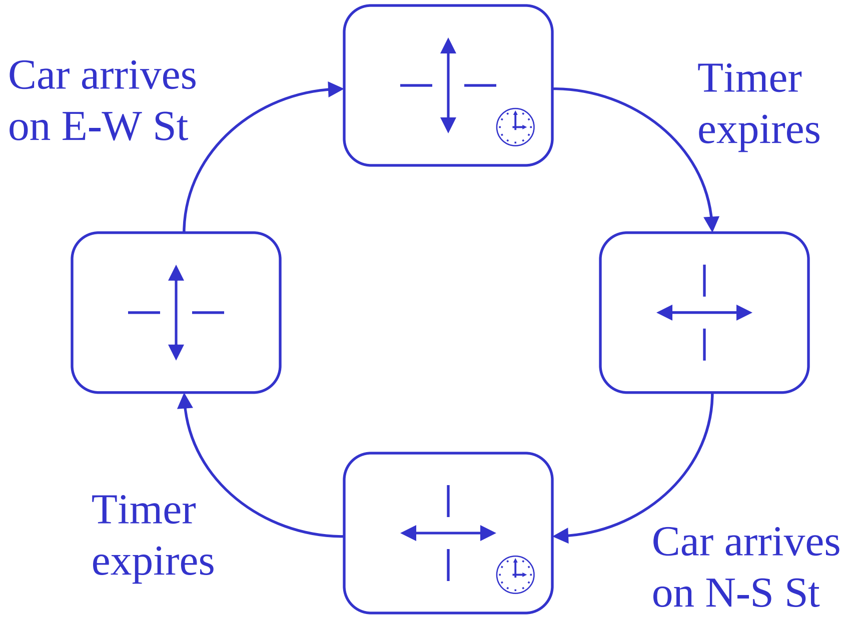
State:	rotor angles, velocities ($\delta_i, \dot{\delta}_i$)
Inputs:	power loading on the grid (P_i)
Outputs:	voltage levels and frequency (based on rotor speed)

Finite State Machines

Finite state machines model discrete transitions between finite # of states

- Represent each configuration of system as a state
- Model transition between states using a graph
- Inputs force transition between states

Example: Traffic light logic



State:	current pattern of lights that are on + internal timers
Inputs:	presence of car at intersections
Outputs:	current pattern of lights that are on

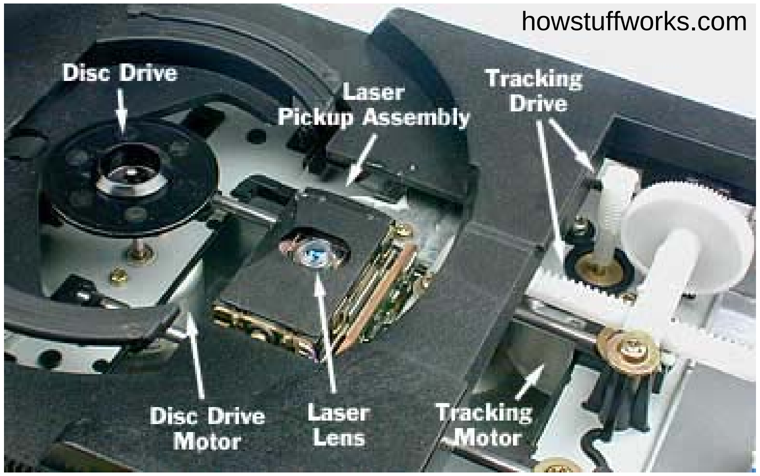
Difference Equations

Difference eqs model discrete transitions between continuous variables

- "Discrete time" description (clocked transitions)
- New state is function of current state + inputs
- State is represented as a *continuous* variable

$$x_{k+1} = f(x_k, u_k)$$
$$y_{k+1} = h(x_{k+1})$$

Example: CD read/write head *controller* (implemented on DSP)



State: estimated center, wobble

Inputs: read head signal

Outputs: commanded motion

Controller operation (every 1/44,100 sec)

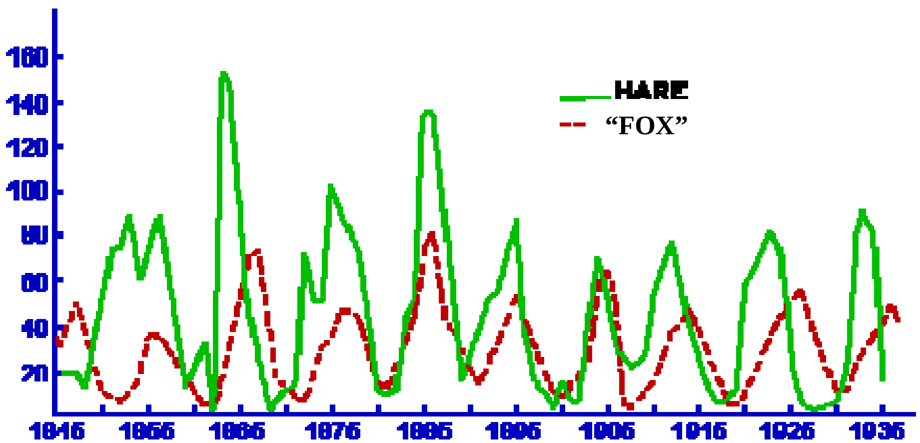
- Get analog signal from read head
- Determine the data (1/0) plus estimate the location of the track center
- Update estimate of "wobble"
- Compute where to position disk head for next read (limited by motor torque)

Performance specification

- Keep disk head on track center
- Reject disturbances due to disk shape, shaking and bumps, etc

RMM7

Example #2: Predator Prey



http://www.math.duke.edu/education/ccp/materials/diffeq/predprey/contents.html

Questions we want to answer

- Given the current population of rabbits and foxes, what will it be next year?
- If we hunt down lots of foxes in a given year, what will the effect on the rabbit and fox population be?
- How do long term changes in the amount of rabbit food available affect the populations?

Modeling assumptions

- The predator species is totally dependent on the prey species as its only food supply.
- The prey species has an external food supply and no threat to its growth other than the specific predator.

Slide 13

RMM7 Tie to Hideo's Friday lecture, if possible
Richard Murray, 9/4/2002

Example #2: Predator Prey (2/2)

Discrete Lotka-Volterra model

- State
  - R_k # of rabbits in period k
  - F_k # of foxes in period k
- Inputs (optional)
  - u_k amount of rabbit food
- Outputs: # of rabbits and foxes
- Dynamics: Lotka-Volterra eqs

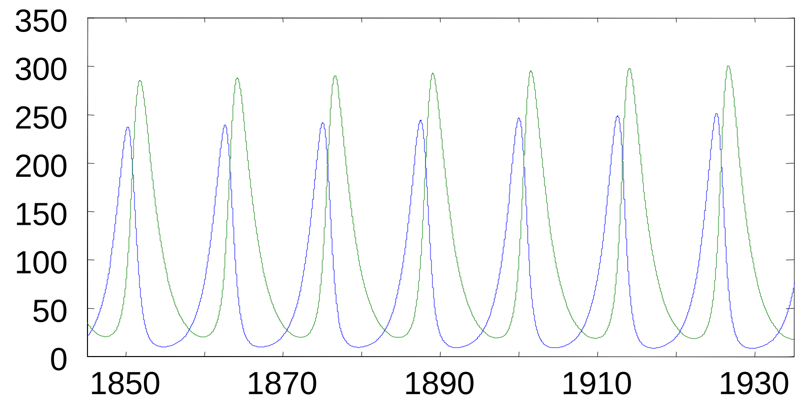
$$R_{k+1} = R_k + b_r(u)R_k - aF_k R_k$$

$$F_{k+1} = F_k - d_f F_k + aF_k R_k$$

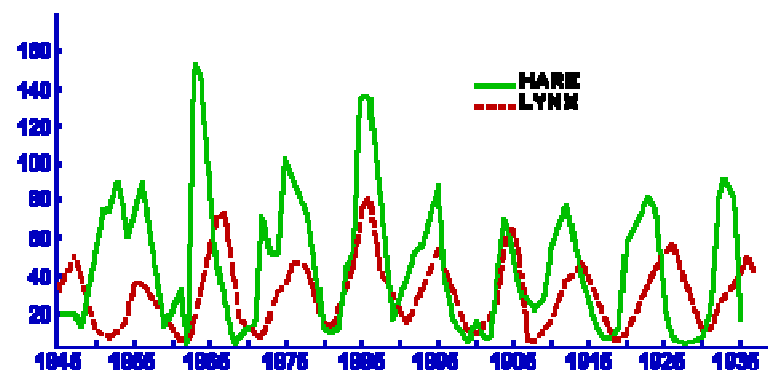
- Parameters/functions
  - $b_r(u)$ rabbit birth rate (per year) (depends on food supply)
  - d_f fox death rate (per year)
  - a interaction term

Matlab simulation (see handout)

- Discrete time model, "simulated" through repeated addition

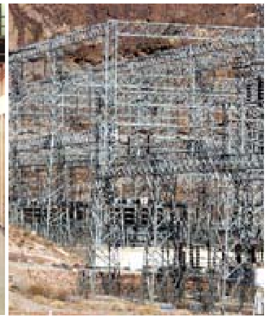


Comparison with data



Summary: System Modeling

Model = state, inputs, outputs, dynamics



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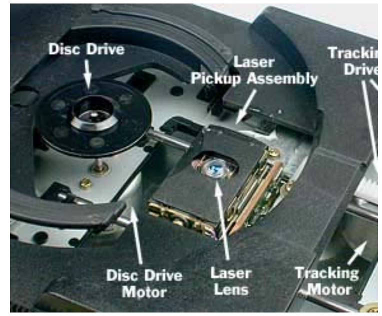
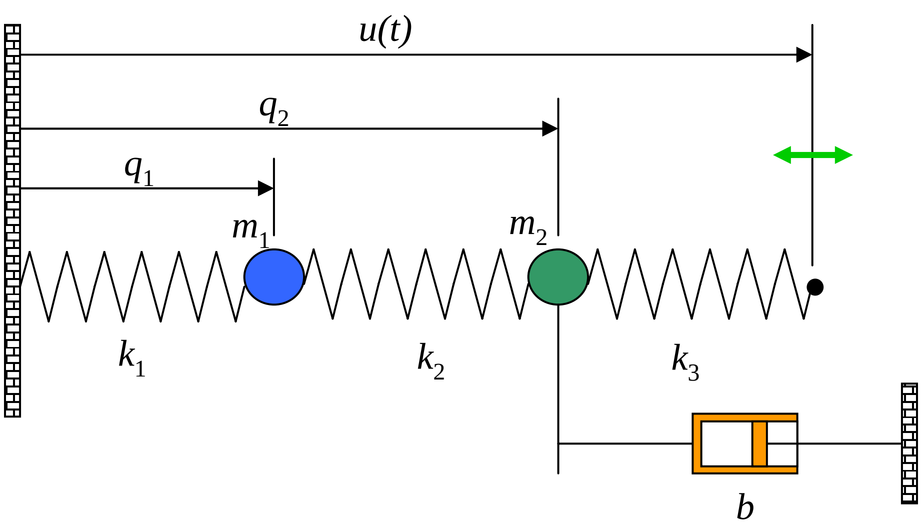


$$x_{k+1} = f(x_k, u_k)$$
$$y_{k+1} = h(x_{k+1})$$

***Principle*: Choice of model depends on the questions you want to answer**

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```