Goals:
• Describe limits of performance on feedback systems
• Introduce Bode’s integral formula and the “waterbed” effect
• Show some of the limitations of feedback due to RHP poles and zeros

Reading:
• Åström and Murray, Feedback Systems, Ch 11
• Advanced: Lewis, Chapters ??
Algebraic Constraints on Performance

Goal: keep S & T small
- S small ⇒ low tracking error
- T small ⇒ good noise rejection (and robustness [CDS 110b])

Problem: S + T = 1
- Can’t make both S & T small at the same frequency
- Solution: keep S small at low frequency and T small at high frequency
- Loop gain interpretation: keep L large at low frequency, and small at high frequency
- Transition between large gain and small gain complicated by stability (phase margin)

\[
\begin{align*}
H_{er} &= \frac{1}{1 + PC} =: S \\
H_{yn} &= \frac{PC}{1 + PC} =: T
\end{align*}
\]
Bode’s Integral Formula and the Waterbed Effect

Bode’s integral formula for $S = 1/(1+PC) = 1/(1+L)$:

- Let $p_k$ be the unstable poles of $L(s)$ and assume relative degree of $L(s) \geq 2$
- **Theorem**: the area under the sensitivity function is a conserved quantity:

$$\int_0^\infty \log_e |S(j\omega)|d\omega = \int_0^\infty \log_e \frac{1}{1 + L(j\omega)}d\omega = \pi \sum \text{Re } p_k$$

**Waterbed effect:**

- Making sensitivity smaller over some frequency range requires *increase* in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in $\omega$; Bode plots are logarithmic
Example: Magnetic Levitation

System description
- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States: $z$, $\dot{z}$
- Dynamics: $F = ma$, $F =$ magnetic force generated by wire coil
- See MATLAB handout for details

Controller circuit
- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current
Equations of Motion

Process: actuation, sensing, dynamics

\[
m \ddot{z} = mg - k_m (k_A u)^2 / z^2
\]
\[
v_{ir} = k_T z + v_0
\]

- \(u\) = current to electromagnet
- \(v_{ir}\) = voltage from IR sensor

Linearization:

\[
P(s) = \frac{-k}{s^2 - r^2} \quad k, r > 0
\]

- Poles at \(s = \pm r\) ⇒ open loop unstable

Note: RHP pole in \(L\) ⇒ need one net encirclement (CCW)
Control Design

Need to create encirclement
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator
- Produce phase lead at crossover
- Generates loop in Nyquist plot

\[ C(s) = -k \frac{s + a}{s + b} \]
Performance Limits

Nominal design gives low perf
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement
\[ \int_0^\infty \log |S(j\omega)| d\omega = \pi r \]
- Must increase sensitivity at some point
Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior
- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move opposite from input for a short period of time

Example:
\[
H_1(s) = \frac{s + a}{s^2 + 2\zeta \omega_n s + \omega_n^2} \quad \text{vs} \quad H_2(s) = \frac{s - a}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]
Example: Lateral Control of the Ducted Fan

Source of non-minimum phase behavior

- To move left, need to make $\theta > 0$
- To generate positive $\theta$, need $f_1 > 0$
- Positive $f_1$ causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)

$$H_{xf_1}(s) = \frac{(s^2 - mgl)}{s^2(js^2 + ds + mgl)}$$

- Poles: 0, 0, $-\sigma \pm j \omega_d$
- Zeros: $\pm \sqrt{mgl}$

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<th>Time (sec.)</th>
<th>Amplitude</th>
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<tr>
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<td>0.8</td>
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<tr>
<td>1</td>
<td>-2.5</td>
</tr>
</tbody>
</table>

Fan moves right and then moves to the left
Stability in the Presence of Zeros

Loop gain limitations

- Poles of closed loop = poles of $1 + L$. Suppose $C = k \frac{n_c}{d_c}$, where $k$ is the gain of the controller

$$1 + L = 1 + k \frac{n_c n_p}{d_c d_p} = \frac{d_c d_p + k n_c n_p}{d_c d_p}$$

- For large $k$, closed loop poles approach open loop zeros
- RHP zeros limit maximum gain $\Rightarrow$ serious design constraint!

Root locus interpretation

- Plot location of eigenvalues as a function of the loop gain $k$
- Can show that closed loop poles go from open loop poles ($k = 0$) to open loop zeros ($k = \infty$)
Additional performance limits due to RHP zeros

Another waterbed-like effect: look at maximum of $H_{er}$ over frequency range:

$$M_1 = \max_{\omega_1 \leq \omega \leq \omega_2} |H_{er}(j\omega)|$$

$$M_2 = \max_{0 \leq \omega \leq \infty} |H_{er}(j\omega)|$$

Thm: Suppose that $P$ has a RHP zero at $z$. Then there exist constants $c_1$ and $c_2$ (depending on $\omega_1$, $\omega_2$, $z$) such that $c_1 \log M_1 + c_2 \log M_2 \geq 0$

- $M_1$ typically $<< 1 \Rightarrow M_2$ must be larger than 1 (since sum is positive)
- If we increase performance in active range (make $M_1$ and $H_{er}$ smaller), we must lose performance ($H_{er}$ increases) some place else
- Note that this affects peaks not integrals (different from RHP poles)

$$H(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: 0, 0, $-\sigma \pm j \omega_d$
- Zeros: $\pm \sqrt{mgl}$

Reduced sensitivity $\Rightarrow$ better performance up to higher frequency
Summary: Limits of Performance

Many limits to performance

- Algebraic: $S + T = 1$
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of $S$

Main message: try to avoid RHP poles and zeros whenever possible (eg, re-design)

$$
\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{1 + L(j\omega)} d\omega = \pi \sum \text{Re } p_k
$$

Sensitivity Function

**Frequency (rad/sec)**

**Magnitude (dB)**

-40  -30  -20  -10   0   10  20  30  40

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